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ON PRIME SUBMODULES

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Abstract

In this paper, we study the properties of prime submodules of a finite module over a Noetherian local ring. The results of the paper show that the properties of such prime submodules are similar to that of prime ideals. We will prove a prime avoidance theorem for prime submodules under a slight weaker assumption. Moreover, we will give an upper bound for the length of ascending chains of prime submodules of a finite module.

1. Introduction

In this paper, we will study the properties of prime submodules of a finite generated module over a Noetherian local ring.

As one of the most important notions of commutative rings, prime ideals play an essential role in the classical theory of commutative algebra. A very useful property of them is the prime avoidance theorem. We will prove a similar result for prime submodules under a slight weaker assumption. Another important property of prime ideals is that one can use them to define the Krull dimension for a Noetherian local ring. To be more explicit, the upper bound of the length of every ascending chain of prime ideals is finite. We will extend this to the case of prime submodules.

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Let A be a ring and let N be an A-module. Recall that a submodule $M \subset N$, $M \neq N$, is said to be *prime* if for every $a \in A$, the homothety $h_a : N/M \to N/M$, $h_a(\bar{x}) = a\bar{x}$, $x \in N$, is either injective or null [2], where $\bar{x}$ stands for the image of x in N/M. In particular, it is clear that an ideal P of A is a prime submodule if and only if P is a prime ideal of A. The main results of the paper state as follows they stand cited in the paper:

Theorem 3.4. *Let* (A, m) *be a local ring with infinite residue field. Let* $M \subseteq N$ *be a pair of finitely generated A-module. If* $M_1, M_2, \ldots, M_k$ *are prime submodules of N and* $M \subseteq M_1 \cup M_2 \cup \cdots \cup M_k$, *then there exists some* i $(1 \leq i \leq k)$ *such that* $M \subseteq M_i$.

Theorem 4.5. *Let A be a d-dimensional ring and N be an A-module generated by s elements. Then one upper bound of the length of every ascending chain of prime submodules of N is sd.*

Throughout this paper all rings are commutative Noetherian rings with unit and all notions unexplained are standard, one can find in [3].

2. Basic Facts

In this section, we recall some basic facts about prime submodules and extend some results of [2].

Let A be a ring and let N be an A-module, a submodule $M \subset N$, $M \neq N$, is said to be *prime* if for every $a \in A$, the homothety $h_a : N/M \to N/M$, $h_a(\bar{x}) = a\bar{x}$, for $x \in N$, is either injective or null [2]. It is clear that an ideal P of A is a prime submodule if and only if P is a prime ideal of A. In the following we give some more examples.

Example 2.1. (i) If K is a field, then the prime submodules of a K-vector space V are exactly the vector subspaces $W \subset V$, $W \neq V$.

(ii) If A is a local ring with maximal ideal m, then m^2 is a prime submodule of m.

(iii) If $M = A \oplus A$ is a free-module over domain A, then every direct factor $S \subset M$, $S \neq M$ is a prime submodule of M.

It is easy to see by the definition that if M is a prime submodule of N, then $\operatorname{ann}(N/M)$ is a prime ideal of A, denoted by P_M. We call P_M the *prime ideal* of M.

Clearly, if $M_1 \subseteq M_2$ are prime submodules of N, then $P_{M_1} \subseteq P_{M_2}$. For a prime submodule M of N, we have for $a \in A$ and $n \in N$ with $an \in M$ if and only if either $a \in P_M$ or $n \in M$. In particular, it follows $P_M N \subseteq M$. Thus to every prime submodule, we can associate a prime ideal. Conversely, one can associate a prime submodule to every prime ideal P by the following proposition.

Proposition 2.2. *Let A be a ring and let N be a finitely generated A-module. If P is a prime ideal of A and $P \in \operatorname{supp} N = \{Q \mid Q$ is a prime ideal of A, $N_Q \neq 0\}$. Set $N(P) = \{n \in N \mid$ there exists $a \notin P$, $an \in PN\}$, then*

(i) *$N(P)$ is a prime submodule of N;*

(ii) *$N(P)$ is contained in every prime submodule M with P as its prime ideal, i.e., $N(P_M) \subseteq M$.*

Proof. (i) As P is a prime ideal, it is clear that $N(P)$ is a submodule.

Now, we claim $N(P) \neq N$. In fact, by the assumption $P \in \operatorname{supp} N$, so $N_P \neq 0$. It follows that $N_P \neq PN_P$, we conclude that $N(P) \neq N$. If on the contrary $N(P) = N$, then for an arbitrary element $n \in N$, there exists $a \notin P$ such that $an \in PN$. It implies for any $\frac{n}{s} \in N_P$ with $s \in A \setminus P$, there exists $a \notin P$ such that $\frac{n}{s} = \frac{an}{as} \in PN_P$. Hence $N_P \subseteq PN_P$. Clearly, $PN_P \subseteq N_P$. It shows that $N_P = PN_P$, a contradiction. This proves $N(P) \neq N$.

Moreover, if $a \in P$, then h_a vanishes on $N/N(P)$, because $PN \subseteq N(P)$. Assume $a \notin P$. If $h_a : N/N(P) \to N/N(P)$ is not injective, there exists $n \in N$, $n \notin N(P)$, such that $an \in N(P)$, and this means that for some $b \notin P$, we have $b(an) \in PN$. Hence $n \in N(P)$, by the fact $ba \notin P$. This is a contradiction and the proof completes.

(ii) Let M be a prime submodule having P as its prime ideal. Assume $n \in M(P_M)$. Then there exists $a \notin P$ such that $an \in P_M N \subseteq M$, and thus $n \in M$, i.e., $N(P_M) \subseteq M$.

Remark 2.3. (i) $N(P)$ is called the *minimal prime submodule* associated to P.

(ii) Proposition 2.2 is an improvement of [2, Proposition 1.5], where A is a domain and N is torsion-free.

(iii) It is easy to see that if $P_1 \subset P_2$, $P_1 \neq P_2$ are prime ideals of A and $P_i \in \operatorname{supp} N$ $(i = 1, 2)$, then $N(P_1)$ is a proper submodule of $N(P_2)$.

3. Prime Avoidance

It is well known that many important properties of a commutative ring can be characterized by prime ideals. One important reason is that there is a prime avoidance theorem for prime ideals. In this section we deal with such properties for prime submodules. Our results show that the prime avoidance theorem is still valid for prime submodules under a slight weaker assumption.

Let $P_1, P_2, \ldots, P_n$ be prime ideals of ring A and let I be an ideal of A. The prime avoidance theorem of prime ideals states that if $I \subseteq P_1 \cup P_2 \cup \cdots \cup P_n$, then $I \subseteq P_i$ for some i $(1 \leq i \leq n)$. This result is not true for general prime submodules. For instance, let (A, m) be a local ring with finite residue field, set $M = A/m \oplus A/m$. Then M is contained in the union of finite one-dimensional A/m-subspaces, and all one-dimensional subspaces are prime submodules. However, we can still obtain some results with some mild additional assumptions.

Proposition 3.1. *Let A be a ring and let N be a finitely generated A-module. Let* $M \subseteq N$ *is a submodule of N and* $M_1, M_2, \ldots, M_k$ *are prime submodules of N. If* $P_{M_1}, P_{M_2}, \ldots, P_{M_k}$ *are distinct and* $M \not\subseteq M_i$ $(1 \leq i \leq k)$, *then* $M \not\subseteq M_1 \cup M_2 \cup \cdots \cup M_k$.

Proof. If necessary, one can reorder $M_1, M_2, \ldots, M_k$. So we may assume that P_{M_k} is the minimal prime ideal of $P_{M_1}, P_{M_2}, \ldots, P_{M_k}$.

Now we prove by induction on k. For $k = 1$, the result is trivial.

Assume $k > 1$. By induction there exists $x \in M \setminus M_1 \cup \cdots \cup M_{k-1}$. If $x \notin M_k$, there is nothing to prove.

If $x \in M_k$, then we can choose $y \in M \setminus M_k$ and an element p such that

$$p \in P_{M_1} \cap \cdots \cap P_{M_{k-1}} \setminus P_{M_k}.$$

Since M_i are all prime submodules, we have

$$py \in M_i \ (1 \leq i \leq k-1) \quad \text{and} \quad py \notin M_k.$$

Set $z = x + py$. It is clear $z \notin M_i$ $(1 \leq i \leq k)$, and $z \in M$. Hence $M \not\subseteq M_1 \cup \cdots \cup M_k$. Equivalently, we have the following.

Proposition 3.2. *Let A be a ring and let N be a finitely generated A-module. Let $M \subseteq N$ be a submodule of N and let $M_1, M_2, \ldots, M_k$ be prime submodules of N. If $P_{M_1}, P_{M_2}, \ldots, P_{M_k}$ are distinct and $M \subseteq M_1 \cup \cdots \cup M_k$, then $M \subseteq M_i$ for some i $(1 \leq i \leq k)$.*

To prove the main result of the section, we need the following lemma.

Lemma 3.3. *Let (A, m) be a Noetherian local ring with infinite residue field. Let N be a finitely generated A-module and let $M \subseteq N$ be a submodule of N. If $M_1, \ldots, M_k$ are prime submodules of N, $P_{M_1} = P_{M_2} = \cdots = P_{M_k} = P$ and $M \subseteq M_1 \cup M_2 \cup \cdots \cup M_k$, then $M \subseteq M_i$ for some i $(1 \leq i \leq k)$.*

Proof. As $N(P)$ is the minimal prime submodule having P as its prime ideal, $N(P) \subseteq M_i$ $(1 \leq i \leq k)$. Since $M \subseteq M_1 \cup M_2 \cup \cdots \cup M_k$, we have

$$(M + N(P))/N(P) \subseteq M_1/N(P) \cup \cdots \cup M_k/N(P).$$

Note that

$$(M_P + N(P)_P)/N(P)_P \subseteq (M_1)_P/N(P)_P \cup \cdots \cup (M_k)_P/N(P)_P.$$

As A/m is an infinite field, it implies $K = A_P/PA_P$ is an infinite field. It follows that

$$(M_P + N(P)_P)/N(P)_P\ ,\ (M_j)_P/N(P)_P\ \ (1 \leq j \leq k),$$

are all finite vector spaces over K. Hence there exists i $(1 \leq i \leq k)$ such that

$$(M_P + N(P)_P)/N(P)_P \subseteq (M_i)_P/N(P)_P.$$

Thus $M_P \subseteq (M_i)_P$, that is, $((M + M_i)/M_i)_P = 0$. Since M_i is a prime submodule having P as its prime ideal, we have $(M + M_i)/M_i = 0$, and thus $M \subseteq M_i$.

We now come to the main result of this section.

Theorem 3.4. *Let (A, m) be a local ring with the infinite residue field. Let N be a finitely generated A-module and M be a submodule of N. If $M_1, M_2, \ldots, M_k$ are prime submodules of N and $M \subseteq M_1 \cup M_1 \cup \cdots \cup M_k$, then there exists some i $(1 \leq i \leq k)$ such that $M \subseteq M_i$.*

Proof. If $k = 1$, then there is nothing to prove.

Assume $k > 1$. By induction we can assume that the conclusion holds for the number of the prime submodules less than k. If $P_{M_1}, P_{M_2}, \ldots, P_{M_k}$ are all the same as a prime ideal P, then the result is true according to Lemma 3.3.

If $P_{M_1}, P_{M_2}, \ldots, P_{M_k}$ are not the same, without loss of generality, we can assume $P = P_{M_k}$ is a minimal prime ideal among $P_{M_1}, P_{M_2}, \ldots, P_{M_k}$ such that

(1) $P_{M_1}, P_{M_2}, \ldots, P_{M_s}$ are not equal to P,

(2) $P_{M_{s+1}} = \cdots = P_{M_k} = P$.

If $M \not\subseteq M_i$ for all i $(1 \leq i \leq k)$, then by induction we can assume there exist $x \in M \setminus M_1 \cup \cdots \cup M_s$ and $y \in M \setminus M_{s+1} \cup \cdots \cup M_k$. Hence, one can choose $p \in P_{M_1} \cap P_{M_2} \cap \cdots \cap P_{M_s} \setminus P$ such that for any positive integer n,

$$p^n y + x \notin M_1 \cup \cdots \cup M_s.$$

For each i $(s+1 \leq i \leq k)$, there is at most one n_i such that $p^{n_i} y + x \in M_i$. If on the contrary we have

$$p^{n_i} y + x \in M_i, \quad p^{n_i'} y + x \in M_i \quad (n_i' > n_i),$$

then it implies

$$(1 - p^{n_i' - n_i}) p^{n_i} y \in M_i.$$

Since A is a local ring, we have $p^{n_i} y \in M_i$. This contradicts to the fact $p \notin P_{M_i}$, $y \notin M_i$ $(s+1 \leq i \leq k)$.

So one can choose n large enough such that $p^n y + x \notin M_i$ for all i $(1 \leq i \leq k)$. On the other hand, $p^n y + x \in M$. This contradicts the fact $M \subseteq M_1 \cup M_2 \cup \cdots \cup M_k$. The proof is complete.

Next we wish to prove a result which can be considered as a natural extension of a result of Davis [1].

Theorem 3.5. *Let* (A, m) *be a local ring with the infinite residue field and let* $M \subset N$ *be a pair of finitely generated A-modules. Let* $M_1, M_2, \ldots, M_k$ *be prime submodules of N and* $t \in N$. *If* $At + M \not\subseteq M_i$ $(1 \leq i \leq k)$, *then there exists* $y \in M$ *such that* $t + y \notin M_i$ $(1 \leq i \leq k)$.

Proof. First we assume $P_{M_1} = P_{M_2} = \cdots = P_{M_k} = P$. Following three cases will be discussed:

Case (i). If $t \notin M_i$ for all i $(1 \leq i \leq k)$, then there is nothing to prove.

Case (ii). If $t \in M_i$ for all i $(1 \leq i \leq k)$, then $M \not\subseteq M_i$ $(1 \leq i \leq k)$. According to Theorem 3.4, there exists $y \in M \setminus M_1 \cup \cdots \cup M_k$. It is clear that $t + y$ is the required element.

Case (iii). If $t \in M_i$ $(1 \le i \le s)$ and $t \notin M_i$ $(s+1 \le i \le k)$, then by Case (ii), there exists

$$y \in M \setminus M_1 \cup \cdots \cup M_s$$

such that $t + y \notin M_i$ $(1 \le i \le s)$. For each i $(s+1 \le i \le k)$, M_i contains at most one of the elements of the form

$$(1 - p^{n_i})t + y,$$

where p is fixed and $p \notin P$, n_i is integer. Otherwise we have

$$(1 - p^{n_i})t + y \in M_i, \quad (1 - p^{n_i'})t + y \in M_i \quad (n_i' > n_i).$$

Then $p^{n_i}t \in M_i$. Since A is a local ring and M_i is a prime submodule, we have $t \in M_i$. This contradicts to $t \notin M_i$. Hence we can choose n large enough such that $(1 - p^n)t + y \notin M_i$ $(i \ge s+1)$. It is clear that $(1 - p^n)t + y \notin M_i$ $(1 \le i \le s)$. Hence $t + (1 - p^n)^{-1} y \notin M_i$ $(1 \le i \le k)$.

Secondly, we assume $P_{M_1}, P_{M_2}, \ldots, P_{M_k}$ are not the same. Without loss of generality, we assume that $Q = P_{M_k}$ is a minimal prime ideal among $P_{M_1}, P_{M_2}, \ldots, P_{M_k}$ such that the following two conditions hold:

(1) $P_{M_1}, P_{M_2}, \ldots, P_{M_r}$ are not equal to Q;

(2) $P_{M_{r+1}} = \cdots = P_{M_k} = Q$.

By induction there exist elements $(t + y_1) \notin M_i$ $(1 \le i \le r)$ and $(t + y_2) \notin M_i$ $(r+1 \le i \le k)$. Set $q \in P_{M_1} \cap P_{M_2} \cap \cdots \cap P_{M_t} \setminus Q$. According to previous discussion in the proof of Theorem 3.4, there are at most one q^{n_i} such that $t + y_1 + q^{n_i}(t + y_2) \in M_i$ $(r+1 \le i \le k)$. So we can choose n large enough such that $(1 + q^n)t + y_1 + q^n y_2 \notin M_i$ $(1 \le i \le k)$. Set

$$y = (1 + q^n)^{-1}(y_1 + q^n y_2).$$

Then $t + y$ is the desired element, and this completes the proof.

Remark 3.6. Readers might notice that Proposition 3.1 is true for all Noetherian rings. However, the proofs of Theorems 3.4 and 3.5 depend on the property of any element which is not in the maximal ideal of a local ring has an inverse. We do not know if there is a positive answer to the following question, where the ring A need not be a local ring.

Question 3.7. Let A be a ring with infinite residue fields. Let N be an A-module and let M be a submodule of N. If $M_1, M_2, ..., M_k$ are prime submodules of N and $M \subseteq M_1 \cup \cdots \cup M_k$, then $M \subseteq M_i$ for some i $(1 \leq i \leq k)$.

In the following we deal with the intersection of prime submodules. It is easy to see that if M_1 and M_2 are prime submodules of module N with prime ideal P, then $M_1 \cap M_2$ is a prime submodule having P as its prime ideal.

Proposition 3.8. *Let A be a ring and let N be a finitely generated A-module. If $M, M_1, M_2, ..., M_k$ are prime submodules of N and $M \supseteq M_1 \cap \cdots \cap M_k$, then $M \supseteq N(P_{M_i})$ for some i $(1 \leq i \leq k)$.*

Proof. Since $M \supset M_1 \cap \cdots \cap M_k \supseteq N(P_{M_1}) \cap \cdots \cap N(P_{M_k})$, we can assume that their prime ideals $P_{M_1}, ..., P_{M_K}$ are distinct and P_{M_k} is a minimal prime ideal among $P_{M_1}, ..., P_{M_K}$.

Assume $M \not\supseteq N(P_{M_i})$ $(1 \leq i \leq k)$, then $P_M \not\supseteq P_{M_i}$. If on the contrary $P_M \supseteq P_{M_i}$, then $N(P_M) \supseteq N(P_{M_i})$. It shows that $M \supseteq N(P_M) \supseteq N(P_{M_i})$, a contradiction. Hence $P_M \not\supseteq P_{M_i}$ $(1 \leq i \leq k)$. We choose

$$x \in N(P_{M_k}) \setminus M, \quad p \in P_{M_1} \cap \cdots \cap P_{M_{k-1}} \setminus P_{M_k} \cup P_M,$$

then $px \in N(P_{M_1}) \cap \cdots \cap N(P_{M_k})$. Notice that $px \notin M$. So $M \not\supseteq N(P_{M_1}) \cap \cdots \cap N(P_{M_k})$. This leads us to a contradiction and the proof completes.

4. Prime Dimension of Modules

It is known that one can define the Krull dimension of a Noetherian ring A by means of prime ideals of ring A, i.e., the maximal length n of a prime ideals chain of A, $P_0 \subset P_1 \subset \cdots \subset P_n$. The Krull dimension of a module N is defined by $\dim N = \dim(A/\operatorname{ann} N)$. In this section, we discuss the length of prime submodules chains. Naturally, we can define prime dimension of a module by means of prime submodules of a module. The main result of the section shows that there is a bound for such length. Let us begin with a definition.

Definition 4.1. Let A be a ring and let N be an A-module. Set $D(N) = \sup\{n \mid M_0 \subset M_1 \subset \cdots \subset M_n,\ M_i \neq M_{i+1}\ (0 \leq i \leq n-1),\ M_i$ is a prime submodule of N, $0 \leq i \leq n\}$. We call $D(N)$ the *prime dimension* of N.

Remark 4.2. (i) If $D(N)$ does not exist, then we set $D(N) = +\infty$;

(ii) If N has no prime submodules, then we set $D(N) = -1$.

Example 4.3. (i) If $N = A$, then $D(N)$ is just equal to the Krull dimension;

(ii) If V is an n-dimension k-vector space, where k is a field, then $N(V) = n - 1$.

According to the theory of the Krull dimension, the Krull dimension of a Noetherian local ring is finite. Similarly, we wish to prove that prime dimension of a finitely generated module over a Noetherian ring is finite. Before proving this result, we need the following notion. Let A be a ring. Then a submodule $M \subset N$, $M \neq N$ of an A-module N is said to be an *O-submodule* if N/M is torsion-free or, equivalently, if zero is the unique non-injective homothety on N/M. Clearly, an O-submodule is a prime submodule. Moreover, we need the following proposition which one can refer to [2, Proposition 2.8]:

Proposition 4.4. *Let A be a domain and let N be a torsion-free finitely generated A-module. Assume that $M \subseteq M'$ with $M' \neq N$ are two submodules of N such that*:

(i) *M is an O-submodule of N*;

(ii) $\mathrm{rk}(M) = \mathrm{rk}(M')$, *where* $\mathrm{rk}(M)$ *stands for the rank of M.*

Then $M = M'$.

Now we can prove the main result of this section.

Theorem 4.5. *Let A be a d-dimensional Noetherian ring and let N be a finitely generated A-module. Then* $D(N) \leq sd$, *where s is the smallest number of generators of N.*

Proof. First we consider the length of prime submodules of N with fixed prime ideal P. Let

$$M_0 \subset M_2 \subset \cdots \subset M_k, \quad M_i \neq M_{i+1} \ (0 \leq i \leq k-1)$$

be a chain of prime submodules of N such that $P_{M_i} = P \ (0 \leq i \leq k)$. We will show $k \leq s$.

For each $i \ (0 \leq i \leq k)$, let us consider the exact sequence of torsion-free A/P-module,

$$0 \to M_{i+1}/M_i \to N/M_i \to N/M_{i+1} \to 0,$$

where $M_{i+1}/M_i \to N/M_i$ is an embedded homomorphism, $N/M_i \to N/M_{i+1}$ is a natural homomorphism. It is clear that M_{i+1}/M_i is an O-submodule of A/P-module N/M_i.

For any chain of O-submodules of N/M_i,

$$L_1/M_i \subset \cdots \subset L_t/M_i$$

with $L_j/M_i \neq L_{j+1}/M_i \ (1 \leq j \leq t-1)$, we have

$$(L_1/M_i)_P \subset \cdots \subset (L_t/M_i)_P, \quad (L_j/M_i)_P \neq (L_{j+1}/M_i)_P \quad (1 \leq j \leq t-1).$$

Otherwise there exists $1 \leq j < j+1 \leq t$ such that $(L_j/M_i)_P = (L_{j+1}/M_i)_P$, according to Proposition 4.4, $L_j/M_i = L_{j+1}/M_i$, a contradiction.

Since $(N/M_i)_P$ is a finite dimensional A_P/PA_P-vector space and s is the smallest number of generators, it shows that the dimension of the A_P/PA_P-vector space $(N/M_i)_P$ is at most s. Hence we have $t \leq s$.

Now, the following

$$M_1/M_0 \subset M_2/M_0 \subset \cdots \subset M_k/M_0, \quad M_i/M_0 \neq M_{i+1}/M_0$$

$$(1 \leq i \leq k-1)$$

is a chain of O-submodules of N/M_0 which satisfies previous condition just discussed above. Hence $k \leq s$.

For any chain of prime submodules $N_0 \subset N_1 \subset \cdots \subset N_n$ of N such that $N_i \neq N_{i+1} \neq N$ $(0 \leq i \leq n-1)$, we have a chain of prime ideals:

$$P_{N_0} \subseteq P_{N_1} \subseteq \cdots \subseteq P_{N_n}.$$

Note that $\dim A = d$, we have $n \leq d$. Hence $n \leq sd$, i.e., $D(N) \leq sd$.

Corollary 4.6. *Let* (A, m) *be a local ring and let* N *be a finitely generated* A*-module. Then* $s-1 \leq D(N) \leq sd$, *where* $d = \dim(A)$, *s is the smallest number of generators.*

The proof follows from Theorem 4.5 and [2, Proposition 3.5].

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