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ON DYNAMICS OF CIRCLE MAPS

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Abstract

For self maps on the circle, we study relations between periodic points of the circle map and its lift. Further we give a quite simple number theoretic proof for circle maps of degree 2 or more to have periodic points of all periods. For such maps we study sensitivity, topological entropy, transitivity and especially show that here all forms of transitivity are equivalent. Also, we study some properties of commuting circle maps.

1. Introduction

By a dynamical system, we mean a pair (X, f), where X is a topological space and f is any self map on X. We study the behavior of each point $x \in X$ under the action of f.

A point $x \in X$ is called *periodic* if $f^n(x) = x$ for some positive integer n, where $f^n = f \circ f \circ \cdots \circ f$ (n times). The least such n is called the *period* of the point x. If $n = 1$, then x is called a *fixed point* (for the

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dynamical system (X, f)). A point x is called *eventually periodic* if for some positive integer n, $f^n(x)$ is periodic. For a given point x, the set $\{f^n(x) | n \geq 0\}$ is called the *orbit* of the point x, where f^0 denotes the identity function. Note that a point x is eventually periodic if and only if its orbit is finite. Let $x \in X$, if there exists an $\varepsilon > 0$ such that for each $\delta > 0$ there exist y and a positive integer n such that $d(x, y) < \delta$ and $d(f^n(x), f^n(y)) > \varepsilon$, then f is said to be *sensitive* at x. If f is sensitive at each point $x \in X$, f has *sensitive dependence on initial conditions* or is simply called *sensitive*. A map f is called *transitive* if for any pair of open sets U, V in X, there exists a positive integer n such that $f^n(U) \cap V \neq \emptyset$. If f^n is transitive for each $n \in \mathbb{N}$, then f is called *totally transitive*. If $f \times f : X \times X \to X \times X$ is transitive, then the map f is called *weakly mixing*. f is called *mixing* or *topologically mixing* if for each pair of open sets U, V in X, there exists a positive integer k such that $f^n(U) \cap V \neq \emptyset$ for all $n \geq k$. A map f on a linear continuum is called *turbulent* if there exist compact arcs J and K with atmost one point common such that $J \cup K \subset f(J) \cap f(K)$. If J and K are disjoint, then f is called *strictly turbulent*.

In this article we will mainly study continuous maps on the unit circle to itself, which will be addressed as circle maps (i.e., maps $f : S^1 \to S^1$). Circle maps have been studied earlier in [1, 3, 4, 5, 6, 8, 9, 10, 13]. One of the simplest examples of such a map is the rotation, i.e., translating the angle by a constant say α; called *rational* or *irrational rotation* depending upon whether α is rational or irrational.

In Section 2, we study some properties related to periodic points of circle maps. Efremova [10] and Block et al. [6] independently proved that circle maps of degree atleast 2 have periodic points of all periods (c.f. [1]) and we give a more simple number theoretic proof for the same. In Section 3, we focus on some problems dealing with commuting maps on the circle. In Section 4, we study transitivity, sensitivity, topological entropy and their implications for such maps.

2. Periods and Periodic Points for Circle Maps

The unit circle S^1, can be regarded as $[0, 1]$ with 0 and 1 identified.

Define

$$\pi : \mathbb{R} \to S^1$$

as

$$\pi(x) = (\cos 2\pi x,\ \sin 2\pi x).$$

We observe that

$$\pi(x_1) = \pi(x_2) \Leftrightarrow x_1 - x_2 \in \mathbb{Z}.$$

Also, for a given continuous map f of S^1 to itself, there is a continuous map $F : \mathbb{R} \to \mathbb{R}$ and a unique integer k such that

(1) For each $x \in \mathbb{R}$, $F(x + 1) = F(x) + k$.

(2) $f \circ \pi = \pi \circ F$.

Thus, any continuous function f on S^1 can be identified with a continuous real valued function F via the map π. Any map F satisfying the two conditions is called a *lift* for the map f. It can be seen that if F satisfies the above two conditions, then so does $F + n$ for any integer n. Thus the lift F is not uniquely determined. On the other hand, the integer k is uniquely determined and is called the *degree* of the map f and is denoted by deg f.

It may be noted that if F_1 is a lift for the map f, then F_2 is a lift for f if and only if $F_1 - F_2$ is an integer.

We also observe that for $\tilde{x} \in S^1$ and $x \in \mathbb{R}$ such that $\pi(x) = \tilde{x}$, the following are equivalent:

(1) $f^n(\tilde{x}) = \tilde{x}$.

(2) $F^n(x) = x + r$ for some $r \in \mathbb{Z}$.

Thus it is clear that if x in $\mathbb{R}$ is periodic for a lift F, then the corresponding point $\pi(x)$ on the circle is also periodic for f. However the converse needs not be true, i.e., for a periodic point on the circle, the corresponding point on the real line may not be periodic. It may be seen that, for a periodic point on a circle, we get countably infinite points on the real line and all of them may not be periodic. This gives rise to a few questions.

Question 1. Let F be the lift for the circle map f. For a periodic point of f on the circle, is always some corresponding point periodic for the lift F on the real line?

The answer turns out to be no. It may happen that $\widetilde{x}$ is periodic on circle but none of the corresponding points on the real line is periodic. For example, take f to be the identity map on the circle and $F(x) = x + 1$ as its lift. Then every point on the circle is fixed but none of the corresponding points is fixed for the lift. However if we take the lift to be $F(x) = x$ instead of $F(x) = x + 1$, then every corresponding point is also fixed. Hence whether the corresponding point is periodic or not also depends on the lift chosen.

Suppose f is a circle map and F is one of its lifts. It may happen that for a periodic point $\widetilde{x}$ for f on the circle, its corresponding point x is not periodic for F on the real line. But since there are infinitely many such corresponding points on the real line and infinitely many lifts to choose from, will there be any point x^* corresponding to the periodic point on the circle and a lift F^* such that x^* is periodic for F^*? Thus

Question 2. Given a periodic point $\widetilde{x}$ of the circle map f, does there exist a corresponding point x on the real line and a lift F such that x is periodic for F?

The answer to this question will also vary from case to case. Rational rotations are one of the examples where the above statement fails to hold. We shall try to derive some conditions under which the above question has a positive answer.

We now give a few formulae (c.f. [1]) which give the effect in value when the point or the lift is translated. Let f be a circle map with lift F and degree k. Then,

Formula 1. $(F + n_1)^n(x) = F^n(x) + n_1(1 + k + k^2 + \cdots + k^{n-1}).$

Formula 2. $F^n(x - m) = F^n(x) - k^n m.$

Proposition 2.1. *Let F be a lift for a circle map of degree k. Let $x \in \mathbb{R}$ be such that $F^n(x) = x + r$ for some $r \in \mathbb{Z}$. Then there exist a lift F_1 and $y \in \mathbb{R}$ with $\pi(x) = \pi(y)$ such that $F_1^n(y) = y$ if and only if r is a multiple of $(1 + k + k^2 + \cdots + k^{n-1})$.*

Proof. Let f be a circle map with $\deg f = k$ and let F be a lift for f. Now as $F^n(x) = x + r$, for any $n_1, m \in \mathbb{Z}$, Formulae 1 and 2 give

$$\begin{aligned}(F + n_1)^n(x - m) &= (F + n_1)^n(x) - k^n m\\ &= F^n(x) + n_1(1 + k + k^2 + \cdots + k^{n-1}) - k^n m\\ &= x + r - k^n m + n_1(1 + k + k^2 + \cdots + k^{n-1}),\end{aligned}$$

$x - m$ is periodic for $F + n_1$ if and only if $r + n_1(1 + k + k^2 + \cdots + k^{n-1}) = (k^n - 1)m$. As integral solution to n_1, m_1 is available if and only if r is a multiple of $(1 + k + k^2 + \cdots + k^{n-1})$, the result follows. □

Remark 2.2. When we move from the circle to the real line, periodic points are not preserved in general. The problem is to choose a lift F and a point $x \in \mathbb{R}$ (corresponding to the periodic point $\widetilde{x}$ on the circle) such that x is periodic for F. For such a pair to exist as shown in Proposition 2.1, a solution can be obtained for each $m \in \mathbb{Z}$. Thus, the choice of corresponding point is not important and only the lift needs to be chosen suitably. This gives rise to the following result.

Proposition 2.3. *Let F be a lift for a map of degree k. Let $x \in \mathbb{R}$ be such that $F^n(x) = x$. Then for each y such that $\pi(x) = \pi(y)$ there exists a lift F_1 such that $F_1^n(y) = y$.*

Proof. Let f be a degree k map and let F be a lift for f. Now as $F^n(x) = x$, for any n_1, $m \in \mathbb{Z}$,

$$(F + n_1)^n (x - m) = F^n (x - m) + n_1(1 + k + k^2 + \cdots + k^{n-1})$$

$$= x - k^n m + n_1(1 + k + k^2 + \cdots + k^{n-1}),$$

$x - m$ is periodic for $F + n_1$ if and only if $n_1(1 + k + k^2 + \cdots + k^{n-1}) = (k^n - 1)m$. For each $m \in \mathbb{Z}$, integral solution to n_1 is available. Thus for each point of the form $x - m$, there exists a lift for which this point is periodic. But every point y with $\pi(x) = \pi(y)$ is of the form $x - m$ and hence the result follows. □

Proposition 2.4. *Let F_1, F_2 be two lifts for the circle map f of degree $k \neq 1$ and let x_1 be a period n point for F_1. Then there exists a period n point x_2 for F_2 with $\pi(x_1) = \pi(x_2)$ if and only if $F_1 - F_2$ is a multiple of $k - 1$.*

Proof. Let F_1 be a lift with periodic point x_1. Then for any other lift $F_1 + n_1$ and point $x_1 - m_1$, using formulae 1 and 2,

$$(F_1 + n_1)^n (x_1 - m_1) = F_1^n (x_1 - m_1) + n_1(1 + k + k^2 + \cdots + k^{n-1})$$

$$= F_1^n (x_1) - k^n m_1 + n_1(1 + k + k^2 + \cdots + k^{n-1})$$

$$= x_1 - k^n m_1 + n_1(1 + k + k^2 + \cdots + k^{n-1}).$$

Thus $x_1 - m_1$ is periodic for $F_1 + n_1$ if and only if $n_1(1 + k + k^2 + \cdots + k^{n-1}) = (k^n - 1)m_1$. Thus $x_1 - m_1$ is periodic for $F_1 + n_1$ if and only if $n_1 = (k - 1)m_1$. □

Proposition 2.5. *Let F be a lift for a circle map of degree $k \neq 1$. Let $x \in \mathbb{R}$ be such that $F^n(x) = x + r$ for some integer r. Then there exists a period n point x_1 for F with $\pi(x_1) = \pi(x)$ if and only if r is a multiple of $k^n - 1$.*

Proof. Let F be a lift for a map f of degree k. If $F^n(x) = x + r$ for some integer r, then $x - m$ will be a periodic point for F if and only if $r - k^n m = -m$. As any point x_1 with $\pi(x_1) = \pi(x)$ is of the form $x - m$, a periodic point y with $\pi(x) = \pi(y)$ exists if and only if $r = (k^n - 1)m$. □

Remark 2.6. The proof relies on the fact that if x is translated by m, then $F^n(x) - x$ is translated by $(k^n - 1)m$. Thus, if r is not a multiple of $(k^n - 1)$, then by translating the point, the value will never come to zero and the periodic point cannot be obtained.

In order to prove the next result, we shall make use of the lemma given below.

Lemma 2.7. *Let f be a circle map of degree $k \geq 2$. If $m, n \in \mathbb{Z}$ such that m divides n and if $F^m(x) - x \in \mathbb{Z}$, then*

$$F^n(x) - x = (F^m(x) - x)\,(1 + k^m + k^{2m} + \cdots + k^{(s-1)m}),$$

where

$$s = \frac{n}{m}.$$

The proof is evident from the fact that if m divides n, then F^n can be written as $s - 1$ compositions of F^m with itself, where $s = \frac{n}{m}$. Now proof follows by repeatedly applying F^m on x.

Proposition 2.8. *Let f be a circle map with degree ≥ 2. Then, f has periodic points of all periods.*

Proof. Let F be one of the lifts for the circle map f, such that $\deg f = k \geq 2$. By formula 2, for any integer n, $F^n(1) - 1 = F^n(0) + (k^n - 1)$. Thus, $F^n(x) - x$ assumes atleast $(k^n - 1)$ integer values in $[0, 1]$. This means there are atleast $(k^n - 1)$ fixed points for f^n. We claim that atleast one among them is a point of prime period n for f.

Since $k \geq 2,$ the existence of a fixed point for f is always guaranteed and hence we assume $n \geq 2.$ If m divides n and $F^m(x) - x$ is an integer, then by Lemma 2.7, $F^n(x) - x$ is also an integer. Let $n = ms.$ Then if $F^m(x) - x = r \in \mathbb{Z},$ then $F^n(x) - x = r(1 + k^m + k^{2m} + \cdots + k^{(s-1)m}).$ This means, for any two distinct consecutive integers values a and $a + 1$ assumed by $F^m(x) - x,$ $F^n(x) - x$ assumes all integer values between $a(1 + k^m + k^{2m} + \cdots + k^{(s-1)m})$ and $(a + 1)(1 + k^m + k^{2m} + \cdots + k^{(s-1)m}).$ Even if $F^m(x) - x$ assumes only one integer value, $F^n(x) - x$ assumes atleast three values. In both the cases, we get an $x \in \mathbb{R}$ such that $F^n(x) - x$ is an integer but $F^m(x) - x$ is not an integer.

If $n = p^r$ for some prime p and natural number $r \geq 1.$ Let $m = p^{r-1}.$ Then as above, we get a point $x \in \mathbb{R}$ such that $F^n(x) - x$ is an integer but $F^m(x) - x$ is not an integer. For this x, the corresponding point on the circle is a point of period n and we are done in this case.

Let $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r},$ where p_i are distinct primes in increasing order and $k_i > 0$ for every choice of i and $r \geq 2.$ Let $m_i = \dfrac{n}{p_i}.$ Then for each m such that m divides n, there exists an i, $1 \leq i \leq r$ such that m divides $m_i.$ If each $F^{m_i}(x) - x$ assumes a single value, then as $F^n(x) - x$ assumes more values, we get a point $x \in \mathbb{R}$ such that $F^n(x) - x$ is an integer but $F^{m_i}(x) - x$ is not an integer for any i. The corresponding point on the circle is of period n for f in this case. If some $F^{m_i}(x) - x$ assumes two integer values, a and $a + 1,$ then $F^n(x) - x$ assumes all integer values between $a(1 + k^{m_i} + k^{2m_i} + \cdots + k^{(p_i-1)m_i})$ and $(a + 1)(1 + k^{m_i} + k^{2m_i} + \cdots + k^{(p_i-1)m_i}).$ As the number of intermediate integer values assumed is more than r and also $(1 + k^{m_i} + k^{2m_i} + \cdots + k^{(p_i-1)m_i}) > r$ for all i, we get a period n point in this case also. Thus, in any case a periodic point of prime period n is guaranteed and hence the proposition. $\square$

Remark 2.9. The above proof heavily relies on the fact that $(1 + k^m + k^{2m} + \cdots + k^{(s-1)m})$ is not of absolute value 1. As long as $\deg f \geq 2$, this is guaranteed. But with $k = -2$, $n = 2$, $m = 1$, $s = 2$, the value for the expression $(1 + k^m + k^{2m} + \cdots + k^{(s-1)m})$ is -1 and thus the above reasoning fails. This means, in a map of degree -2, there may not exist a period 2 point. Hence for all other negative values of k, we have

Corollary 1. *Let f be a circle map with $\deg f = k$ such that $k \leq -3$. Then, f has periodic points of all periods.*

3. Commuting Circle Maps

For a very long time, the composition of two dynamical systems was not well-known, i.e., given (X, f) and (X, g) as two dynamical systems, how does $(X, f \circ g)$ behave? What properties of individual dynamical systems are carried to the composition? Answers to these questions are very important in understanding the combined effect of different dynamical systems. For example, composition of two sensitive maps needs not be sensitive. In fact, it is difficult to analyze the problem if the systems are not commuting and also such systems have very less practical significance. Hence, such questions have been studied for commuting dynamical systems only.

One of the problems that was unsolved for a very long time was whether two commuting maps of the interval necessarily have a common fixed point. This problem was later solved by showing the existence of two commuting interval maps without common fixed point (c.f. [7]).

We wish to study the same problem for circle maps. Before discussing the problem, we give one of the basic results which gives the criteria for commuting maps to have a common fixed point.

Proposition 3.1. *Let f and g be two commuting self maps on any compact space X. Then a periodic point of f is a periodic point of g if and only if it is a periodic point of $f \circ g$.*

Proof. For the forward part, if x is periodic point for f of degree n and periodic point of g of degree m, then $(f \circ g)^{nm} = x$ and thus x is periodic for the composition. For the converse part, if x is periodic point for f of degree n and periodic point of $f \circ g$ of degree m, then, $(f \circ g)^{nm} = x \Rightarrow (g \circ f)^{nm} = x \Rightarrow g^{nm}(f^{nm}(x)) = x$ or $g^{nm}(x) = x$ as $f^{nm}(x) = x$. □

It can be seen that every two rational rotations commute, but do not have a common fixed point. Thus any two commuting circle maps may not have a common fixed point. However, in some cases we wish to derive conditions under which the commuting maps necessarily have a common fixed point. For circle maps of degree 1, it may be difficult to derive such conditions. Thus, throughout this section, we shall consider circle maps with degree not equal to 1 whenever needed.

Observation. If f and g are two commuting circle maps, then for given lifts F and G of f and g, respectively, there exists an integer n such that $F \circ G - G \circ F = n$.

It can be seen that if f and g commute, then the corresponding lifts need not commute. Thus a question arises that given two commuting circle maps f and g, do there exist lifts which commute. The next result characterizes the above statement.

Proposition 3.2. *Let f and g be two commuting circle maps with degree k_1 and k_2, respectively and let F and G be two lifts for f and g respectively. Then there exist commuting lifts for f and g if and only if* $\gcd(k_1 - 1, k_2 - 1)$ *divides* $F(G(0)) - G(F(0))$.

Proof. For any $n, m \in \mathbb{Z}$,

$$(F + n)(G + m)(x) = (F + n)(G(x) + m) = F(G(x) + m) + n$$
$$= F(G(x)) + k_1 m + n.$$

Similarly, $(G + m)(F + n)(x) = (G + m)(F(x) + n) = G(F(x) + n) + m = G(F(x)) + k_2 n + m$. There will exist commuting lifts if and only if $(F + n)$ and $(G + m)$ commute for some n and m which will happen if and only if

$$F(G(x)) + k_1 m + n = G(F(x)) + k_2 n + m$$
$$\Leftrightarrow (k_2 - 1)n - (k_1 - 1)m = F(G(x)) - G(F(x)).$$

Now if f and g commute, then $F \circ G$ and $G \circ F$ are lifts of same map and thus their difference is a constant and thus $F(G(0)) - G(F(0)) = F(G(x)) - G(F(x))\ \forall x.$ Thus a solution to the above equality exists if and only if $\gcd(k_1 - 1, k_2 - 1)$ divides $F(G(0)) - G(F(0)).$ Hence the result follows. □

However, it is evident that any pair of lifts do not commute for commuting maps. Thus we now try to characterize those lifts of commuting maps which themselves commute.

Proposition 3.3. *Let f and g be commuting maps and let F and G be their commuting lifts. Then any other lifts F_1 and G_1 of f and g commute if and only if* $\dfrac{F_1 - F}{G_1 - G} = \dfrac{k_1 - 1}{k_2 - 1}.$

Proof. If F and G are commuting lifts, then any other lifts F_1 and G_1 of f and g are of the form $F + n$ and $G + m,$ respectively. Now, $(F + n)(G + m)(x) = F(G(x)) + k_1 m + n.$ Similarly, $(G + m)(F + n)(x) = G(F(x)) + k_2 n + m.$ Now $F(G(x)) = G(F(x)).$ Thus the new lifts commute if and only if $k_1 m + n = k_2 n + m$ which will happen if and only if $\dfrac{n}{m} = \dfrac{k_1 - 1}{k_2 - 1}.$ □

We now wish to derive conditions under which two commuting circle maps have a common fixed point.

Proposition 3.4. *Let f and g be two commuting circle maps with lifts F and G, respectively. If there exists a common fixed point, then there exist commuting lifts.*

Proof. Let f and g have a common fixed point, i.e., there exists a point $\tilde{x} \in S^1$ such that $f(\tilde{x}) = g(\tilde{x}) = \tilde{x}.$ Let $x \in \mathbb{R}$ such that $\pi(x) = \tilde{x}.$ Then $F(x) = x + r$ and $G(x) = x + s,$ where $r, s \in \mathbb{Z}.$ Then x is a fixed point of $F^* = F - r$ and $G^* = G - s.$ Then $F^* \circ G^* - G^* \circ F^*$ assumes value 0 at $x.$ Also as f and g commute, $F^* \circ G^* - G^* \circ F^*$ is a constant. Hence the proposition holds. □

In the above result, we have shown the existence of two commuting lifts. However, two commuting lifts with a common fixed point can also be obtained. Conversely, if we have commuting lifts with a common fixed point, then common fixed point for the commuting circle maps is guaranteed.

We feel that for commuting maps of the circle, there exists a common fixed point if and only if there exist commuting lifts for them. However, we provide the proof for the case when the lifts are linear. We believe that the result holds good even when the lifts are not linear, but we have no proof for this case and leave it open.

Proposition 3.5. *Let f and g be two commuting circle maps with linear lifts F and G, respectively. Then there exists a common fixed point for f and g if and only if there exist commuting lifts.*

Proof. Let F and G be linear lifts for f and g, respectively. Let $F(x) = m_1 x + c_1$ and $G(x) = m_2 x + c_2$. If $m_1 = m_2$, then lifts commute if and only if $c_1 = c_2$ making the function same and thus every fixed point is common. If $m_1 \neq m_2$, then F and G commute if and only if $m_1 c_2 + c_1 = m_2 c_1 + c_2$ implying that $\frac{-c_1}{m_1 - 1}$ is the common fixed point. □

Though, we have derived some conditions for commuting maps with common fixed point; similar questions can be raised for circle maps with common periodic points. We feel that in such cases, some stronger forms of the above conditions will be obtained.

4. Transitivity and Related Properties for Circle Maps

We first state some of the known results, for the sake of completion.

Theorem 4.1 [3]**.** *For a given circle map f, the following are equivalent*:

(1) f^n *is transitive for each positive integer n and f has a fixed point.*

(2) *f is topologically mixing.*

Theorem 4.2 [3, 8]**.** *For a given circle map f, the following are equivalent*:

(1) *f has positive topological entropy.*

(2) f^n *is strictly turbulent for some positive integer n.*

Theorem 4.3 [13]. *Let* $f : S^1 \to S^1$ *have a dense orbit. If* $(a, b) \subset S^1$ *is free of periodic points, so is* $f^j(a, b)$ *for all positive integers j.*

Theorem 4.4 [13]. *Let* $f : S^1 \to S^1$ *have a dense orbit. Then the following are equivalent.*

(1) *f has a periodic point.*

(2) *f is sensitive to initial conditions.*

Theorem 4.5 [9]. *Let* $n \geq 2$. *If* $f : X \to X$ *is transitive but* f^n *is not, then there is a closed set* $K \neq X$ *with nonempty interior and a divisor* $m > 1$ *of n such that:*

For two different indices $i, j,\ 1 \leq i, j \leq m - 1,\ f^i(K) \cap f^j(K)$ *has an empty interior.*

An arc $J \subseteq S^1$ is one-way (with respect to $f : S^1 \to S^1$) if whenever $x, y, f^m(x), f^n(y) \in J$ and the orientation of the circle is taken so that if the arc $[x, f^m(x)] \subseteq J$, then arc $[y, f^n(y)] \subseteq J$.

Theorem 4.6 [9]. *Let J be an arc which contains no periodic point. If* $\bigcup_{k \geq 0} f^k(J)$ *is not the whole circle, then J is one-way.*

We observe by Theorem 4.4 that if $\deg f \geq 2$ and *f* is transitive, then *f* has sensitive dependence on initial conditions as degree atleast 2 ensures the existence of a periodic point. We now give a result for sensitive maps on the circle.

Proposition 4.7. *If a continuous map* $f : S^1 \to S^1$ *is sensitive and has a periodic point, then the eventually periodic points are dense in* S^1.

Proof. We shall show that any arc $J \subset S^1$ contains an eventually periodic point. Since *f* is sensitive, there exist $\delta > 0$ and infinitely many *n* such that $diam(f^n(J)) > \delta$. Since each of the $f^n(J)$ is an arc, we can find $r, s \in \mathbb{Z}$ such that $f^r(J) \cap f^{r+s}(J) \neq \varnothing$. Therefore, by induction, we have, $f^{(k-1)s}(f^r(J)) \cap f^{ks}(f^r(J)) \neq \varnothing$ for $k \in \mathbb{Z}$. Put $g = f^s$ and let $L = \bigcup_{k=0}^{\infty} g^k(f^r(J))$. Then L is an arc invariant under g. Also g is sensitive.

We claim that $g^n(y) = y$ for some $y \in L$ and $n \in \mathbb{N}$. Then since $y = f^{ks+r}(z)$ for some $z \in J$, J contains the eventually periodic point z and hence we are done.

Suppose $g^n(y) \neq y$ for every $y \in L$ and every $n \in \mathbb{N}$. Then as f has a periodic point, $\bigcup_{k \geq 0} f^k(L) \neq S^1$ and so L is one way by Theorem 4.6. Hence every g orbit of L is monotone and converges to some point of $\overline{L} \setminus L$, which must be a fixed point of g. This contradicts the sensitivity of g and hence proves the claim. □

Thus transitive maps of the circle with $\deg f \geq 2$ are sensitive by Theorem 4.4, and hence the set of eventually periodic points in this case will be dense in S^1. In fact, this result can be strengthened as follows.

Proposition 4.8. *Let f be a map with $\deg f \geq 2$. If f is transitive, then the set of periodic points is dense.*

Proof. We have to show that set of periodic points is dense. It is evident that arc (a, b) is free of periodic points implies $f^j(a, b)$ is also free of periodic points by Theorem 4.3. Thus $\bigcup_{j=1}^{\infty} f^j(a, b)$ is free of periodic points. But as f is transitive the above set leaves atmost one point. Thus atmost one point is periodic. But since $\deg f \geq 2$, it has periodic points of all periods. Hence the proposition follows. □

Proposition 4.9. *Let f be a circle map with $\deg f \geq 2$. Then it has positive topological entropy.*

Proof. If f is a circle map of degree atleast two, then f^2 is of degree atleast four. Thus F^2 assumes values r, $r+1$, $r+2$, $r+3$, $r+4$, where F is some lift of f and $r = F^2(0)$. Thus there exist two disjoint intervals

(a, b) and (c, d) such that $F^2((a, b)) = [r, r+1]$ and $F^2((c, d)) = [r+2, r+3]$. Thus we get two disjoint arcs on the circle which are mapped to whole circle under f^2. This implies that f^2 is strictly turbulent. Hence *f* has a positive topological entropy by Theorem 4.2. □

Proposition 4.10. *For any circle map f with* $\deg f \geq 2$. *The following are equivalent.*

(1) *f is transitive.*

(2) *f is totally transitive.*

(3) *f is weakly mixing.*

(4) *f is topologically mixing.*

Proof. If *f* is transitive, then each f^n is transitive using Theorem 4.5. Further, totally transitive implies weakly mixing and weakly mixing implies transitivity. Finally topologically mixing is equivalent to transitivity via Theorem 4.1 and thus the proposition holds. □

The simplest condition for a piecewise monotone circle map with $\deg f \geq 1$ to be transitive is

Result 4.11. Let *f* be a piecewise monotone circle map *f* such that $\deg f \geq 1$ with $|f'| \geq 2$ whenever it is defined and having critical points c_i and lift *F*. If $F(\pi^{-1}(c_i)) - F(\pi^{-1}(c_j)) \in \mathbb{Z}$, whenever c_i and c_j are two adjacent critical points, then *f* is transitive.

Proof. Any arc expands under the action of *f* to cover two critical points and subsequently to cover the whole of S^1. □

The result for interval maps in [11] can be easily modified for circle maps of degree atleast one to yield:

Result 4.12. For $n \geq 2$, if $f : S^1 \to S^1$ is a piecewise monotone map with $\deg f \geq 1$ such that $|f'(x)| \geq n$ at all points where *f* is differentiable and $S^1 \setminus f(J)$ is finite for any arc *J* containing atleast *n* critical points, then *f* is transitive.

Result 4.13. Let $f : S^1 \to S^1$ be a piecewise monotone map with $\deg f \geq 1$ and let C be the set of all critical points of *f*. Let $c \in S^1$ such that $C \subset f^{-1}(C) \cup \{c\}$. If $|f'(x)| \geq 2$ whenever f is differentiable, then exactly one of the following happens.

(1) *f* is transitive.

(2) Either $f(c, c_*)$ or $f(c, c^*)$ or their union is invariant, where c_* and c^* are critical points of *f* adjacent to *c* one on each side.

Recently, the study of dynamics on hyperspace of a metric space has attracted attention as a result of the work on 'fractals'. It is not certain that all dynamic properties can be lifted to the hyperspace. The possible implications on both sides can help in analyzing such properties which can be lifted. Also, the analysis of properties that fail to be lifted can throw light on the changes that take place, when the dynamics has to depend on combined effect. Recently, several questions have been raised regarding the relation between the dynamics of a mapping on space X and its induced counterpart on the hyperspace $\mathcal{K}(X)$. Román-Flores [12] proved that the induced map on the hyperspace if transitive guarantees that the original map is transitive, although the converse may not be true. An interesting observation here is that the irrational rotations on the circle are the simplest examples of transitivity in the individual case failing to extend to the same property on its hyperspace. In the same direction, Banks [2] observed that the property of topological mixing is equivalent both on the base space X as well as the hyperspace $\mathcal{K}(X)$. To this, we simply note that the induced maps of transitive circle maps of degree 2 or higher will also be transitive on its hyperspace since in such cases they are topological mixing.

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