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# ANALOGIES IN DIFFERENTIAL EQUATIONS

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## Abstract

This paper describes how the appropriate use of certain analogies enables students to come to deal successfully with different types of problems. In addition, the analogies which form the subject of this work are classified, in order to assist in problem solving activities, and to demonstrate that their use is a very valuable tool in the solution of differential equations that would otherwise be somewhat more complicated or impossible for students to solve using only the knowledge that they possess.

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## Introduction

Problem solving has been recognised as a very important component of the study of mathematical knowledge. Halmos [2] suggested that problem solving is at the heart of mathematics. Kleiner [4] emphasized that the development of mathematical concepts and theories are a result of efforts made to solve a specific problem. Throughout the history of mathematics, it may be seen that mathematical advances are almost always the outcome of trying to solve a specific problem.

It is also a mathematical learning activity that gives rise to the use of heuristic resources and is, at the same time, strongly supported by the use of these resources. Authors have dealt with this subject value the problem solving process as a very important activity in the development of mathematics and pay attention to both the design and the presentation of problems and problem solving procedures. In the words of Campistrous and Rizo [1] "... to solve a problem consists in arriving at the results, that is, in the search for ways by which to arrive at the desired transformation and is not only the solution of the problem in itself." Heuristic resources are used in isolation in the teaching activities and generally as means to optimise communication with the students, never as teaching objectives in themselves. Our experience has also taught us that, in trying to solve a problem, students usually do not set out a plan, that is to say, they do not map out the contours of the problem, nor do they exhaustively analyse the data available to them, and they do not try to relate this data and the unknown factor with previous knowledge that might link both questions together; on many occasions they are not capable of changing the domain in which the problem is presented in such a way as to be able to work in a simpler domain before returning to the original one; neither is the use of heuristic resources such as diagrams, analytical representations, turning to known problems for solutions or working backwards very common. It has been observed that, when it is advisable to build an auxiliary construction to solve a problem, many students in a class are incapable of doing so because, among other reasons, they have become aware of this resource only when they see their professors use it, but have not studied it in itself, which is to say that it has not become part of the students' working approach when searching for ways to solve a problem. Furthermore, it is not common to breakdown a specific problem into simpler ones when looking for the means to solve it, something that would markedly simplify the difficulty with which they are at odds. Finally, to solve a mathematical exercise well not only requires knowledge of the subject matter, but also an ability to pinpoint which techniques can be used, and when and how to use them, in other words, to involve both cognitive and meta-cognitive strategies in the process.

Some considerations that give rise to the use of heuristic strategies:

Heuristic resources, nevertheless, play an important role, as they constitute the basis of the action plan, the decision-making process, and control over the process, when used to establish the guidelines for subsequent action.

We define a heuristic resource as the definition of a moment of thought, such as the materialisation of a sketch, diagram, table, etc. of the mental representation that arises in the process of solving the problem which will be associated with the referential framework of the individual, to his experiences and, ultimately, to his own personality.

Now, we believe that it is not possible to demand that a student uses these resources in the solution of problems if the latter has not been encouraged to develop the abilities required for their use in simpler situations. We understand that we have to convert the whole learning process, at least insofar as mathematics is concerned, into a pedagogic procedure in which aspects of this sort are highlighted, that we have to contribute to the development of creative thinking in the students, and overcome the existing weaknesses to which we referred at the beginning of this article.

We understand that the following constitute didactic methodological suggestions that can contribute to the development of heuristic teaching, which is to say, to enable students to assimilate these strategies consciously for the solution of different problems and, to apply them independently:

1. Select appropriate examples to show the use of heuristic strategies;

2. Explain with precision what the heuristic strategy is and demonstrate when is it advisable to apply it so that its use becomes totally understandable;

3. Highlight the advantages that the use of the heuristic strategy offers in order to encourage generalised usage;

4. Encourage students to grapple with the solution to the problem and demand their active participation in using the heuristic strategies independently; and

5. Promote the conscious application of strategies in the classroom and the discussion of their use in the search for mathematical methods and in the problem solving plan.

When evaluating the solutions and the means by which these are obtained and when applying the general heuristic programme to problem solving, it is important to state with precision the heuristic strategy that is used, its advantages, limitations and the possibilities of its use in other situations, so that students become aware of its importance.

## The Analogy

Mathematical Didactics is a scientific discipline in full development. Up until now, many concepts were handled in an almost intuitive manner within what is known as "Traditional Didactics", and others are not used in their holistic sense. Amongst the latter, the concept that lies at the core of this work: the analogical transformation. It is as well to state that, although in the so-called "Fundamental Didactics", rooted in the work of the French school, these concepts cease to be transparent and become a subject of study in themselves, as happened with the "function" concept within mathematics, the concept of analogy, from the didactic point of view, has not been studied from all possible perspectives, even though it has been used by mathematicians since ancient times.

Analogy, as a general scientific concept, has been analysed and studied from different perspectives, such as the philosophical, biological, theological, etc. When we use it within the didactic context of mathematics, it does not loose its essence and is understood to be:

- the similarity relationship between mathematical objects,
- the method that allows the understanding and concatenation of a set rule, properties or principles of one field into another,
- in the solution of problems, the use of analogies that assists in the solution process by means of the following:

1. It enables the working method and thought processes to be highlighted, made explicit and understood;
2. It facilitates the formulation of problems, the variation in the conditions, search for relationships and dependencies; and
3. It demonstrates the importance of the idea of a solution and reveals that a determining factor in order to solve a problem is to look for the ways in which to reach the desired transformation.

When analysing the theoretical evolution of mathematics, we note that most of its results were obtained on the basis of using analogies; the discoveries of integral calculus are worthy of mention in which several types of analogies are evident. We can cite the "reduction analogy", as one example, which is identified when the calculation of a definite integral is reduced to the solution of a set limit; another one which we assume to be "structural" in nature can be analysed from the relationship established between integral calculus and the techniques used by Archimedes for the calculation of the area under a curve; the "relational" analogy, which becomes evident when a line integral overlaps a double integral (Green's theorem); and lastly, the "solution" analogy that can follow different directions. We shall look into at least two of these: in the first case, centring on multiple integrals, we examine solutions arrived at through iterated integration and, we examine the use of an indefinite integral for the calculation of the definite integral using Barrow's formula.

This paper features the study of ordinary differential equations in order to arrive at their solutions from specific directions, a situation which allows us to establish the links between the types of analogies to be used and the way to deal with any one specific differential equation.

## 1. Structural Analogy

This analogy is studied from the relationship or external similarity that can be established between a certain type of differential equations (DE) and traditional non-differential equations (TE), such as quadratic, cubic, trigonometric equations, etc. It is important to highlight that, for a better understanding and to save time, we have used differential equations that have already been transformed; in practice, the majority of these equations do not exist in this form, but can easily be transformed into the proposed situations.

1. If we analyse the external structure of this differential equation:

$$(y')^2 + (2x + y)y' + (x^2 + xy) = 0. \tag{1}$$

in relation to the derivative in the equation, we can observe that in the first term, the derivative is squared, in the second, it is in its linear form and in the third the derivative of the function does not exist, thus the relationship can be established by a quadratic equation with the following general form:

$$ax^2 + bx + c = 0.$$

The solution may be approached in two ways: one that uses the cross product (Vieta's theorem), and the other more commonly used one, which uses the discriminant of the formula:

$$x_{1,2} = (-b \pm \sqrt{b^2 - 4ac})/2a.$$

Thus, substituting $x$ by the derivative of $y$, substituting $b$ by $-(2x + y)$ and $c$ by $(x^2 + xy)$, solving this equation gives us:

$$y' = x + y, \qquad y' = x.$$

The first is a first-order differential equation of the first degree, whose solution is immediate, and the second, a variables-separable equation.

2. The equation that we will now present for consideration can be analysed as a general Euler equation. Knowing how laborious the method and the finding of the possible combinations are, it becomes important, at

this time, to emphasise the need for the use of analogies and to demonstrate the wealth of mathematical thinking and how useful previously acquired knowledge is when tackling a certain situation.

$$(y')^3 - y(y')^2 - x^2y' - x^2y = 0. \tag{2}$$

This is a first-order homogeneous differential equation of the third degree with variable coefficients. We can relate this to a cubic equation in the derivative of the function. We will use the decomposition of factors to solve:

$$(y')^2(y' - y) - x^2(y' - y) = 0,$$

$$(y' - y)(y' - x)(y' + x) = 0.$$

Following its decomposition, the differential equation is transformed into three differential equations of separable variables.

3. The equation that shall be set out here can be solved by the Clairaut method, however, it can also be solved by using structural analogy, applying fewer content resources and greater creativity.

The equation is as follows:

$$x(y')^2 + 2xy' - y = 0. \tag{3}$$

When analysing it we can see that this is a first-order homogeneous differential equation of the second degree, with variable coefficients. We can relate it to a quadratic equation to solve the derivative of $y$ which, after a set of transformations that we leave for you to perform, will produce the following result:

$$y' = -1 + \sqrt{1 + y/x}.$$

Then, making $y/x = u$, we obtain variables-separable equation whose solutions take us to the calculation of the integral in $u$:

$$\int du/(\sqrt{1+u} - (1+u)),$$

where it is only necessary to perform the transformation to find its primitive.

4. The equation that we present here for consideration is a second-order inhomogeneous equation with variable coefficients:

$$xy'' - 3y' = 4x^2. \tag{4}$$

Even though the structure looks like an equation of the type analysed, one must be careful when applying it as it can lead to a conceptual error. What might it be?

Going back to the equation, what is fundamental in these cases is to lower the order of the differential equation. To achieve this, let us make $y' = v$. By substitution we obtain

$$xv' - 3v = 4x^2.$$

The original equation is reduced to a linear equation. The solution of the homogeneous part is $v = cx^3$ and the solution of the inhomogeneous part is $v = 4x^2 + cx^3$. To find the solution to the equation we need to solve $y' = v$.

We are totally in agreement that it is not always possible to perceive the structural analogy with a known type of equation from the given differential equation, sometimes because of the extent to which it is concealed and because of the type of transformations that are necessary to convert it into a known equation, and at other times simply because the structural relationship "does not exist". But the one thing we are sure about is that the use of this type of analogy makes the work easier. In most cases, we arrive at the explicit solutions of the differential equations and its benefits for a deep analysis of a particular situation may be enjoyed. In other words, it develops mathematical ability and thought processes, as well as leading to meaningful learning of differential equations by relating it to knowledge already held by students. For a certain equation to be solved using the applied techniques, it must firstly be transformed into an equation with the following form:

$$p(x, y)(y')^n + q(x, y)(y')^{n-1} + \cdots + r(x, y) = 0$$

and secondly, the equation must be solved in quadratures.

Following this line of reasoning, we shall now solve the following equation:

$$x\sqrt{1+(y')^2} = 2y'.$$

Using the resources under study, the equation is solved by relating it to a trigonometric identity using the substitution $y' = \tan(t)$.

Heuristic analysis of the analogy of the structure:

From a didactic point of view, it should be highlighted that the solutions of traditional equations (TEs) and the solutions of differential equations (DEs) are problems which, in principle, coincide solely insofar as their objective is to look for the entity or entities that verify a relationship that is expressed by means of an equation. The analogy is the strategy which consists of applying methods used to solve the TEs to the DEs. Anyway, the ensuing change of domain must be done without contradictions: when we look at the equation let us observe that, firstly, both types of problems are of a different nature and therefore a change of domain takes place. In this regard, the use of the analogy and its emphasis, from the didactic point of view, entails a need to explain in great detail the conditions under which this is validly applicable.

Let us examine some examples in which this strategy leads to the solution to certain differential equation.

Looking back at the examples shown in equations (1) and (4). The analogy established with the quadratic model in the case of equation (1) provides a valid solution to the differential equation as the object subjected to the quadrature of this equation is $y'$ such that in $y'$ the relationship expressed by the equation is quadratic, and this relational model permits the discriminant solution. In other words, the analogy applied here consists in solving the second-degree equation where the aim is not to find the value of the function $z = x$ which verifies $az^2 + bz + c = 0$, but to find the value of the function $z = y'$ which verifies the same condition; both problems are structurally the same or, in other words, they represent the same original problem in spite of the differences in expression of the unknown function and therefore, the algebraic solution related to the structure is equally applicable to both of them. Conversely, in the case of equation (4), the virtual analogy of the same with the quadratic equation $xz^2 - 3z - 4x^2 = 0$ implies identifying $y' = z$ and also $y'' = z^2$, which means a conversion to the quadratic form is not viable. How can we identify a valid structural analogy?

## 2. Reduction Analogy

A reduction analogy is known as the transformation process of a certain problem or situation into another known problem or situation. Specifically, we can establish that within differential equations, this analogy is present in the development of the content even if no reference is made to it (which would make the understanding of this subject easier). As an example, we may mention the following:

- the case of a differential equation is being transformed into another known equation, for example, the use of the integral factor to transform the equation into an equation with an exact differential;
- the case of the transformation of a homogeneous equation using a transformation of the type $v = y/x$ into a variables-separate equation.

It may be said that in the development of any theory, the aim is to reduce the problem to another known problem. In this paper we follow the same line of thinking, but we extend its realm of application to other types of equations and we use other types of resources that provide a set of techniques to increase the students' chances of successfully solving equations.

5. The first equation that we set out is a non-linear differential equation with variable coefficients, where the application of the possible traditional methods is clearly cumbersome

$$2xydy/dx + 2y^2 = 3x - 6. \tag{5}$$

Note the second term in the equation. When calculating the derivative of $y^2$, knowing that $y$ is dependant on $x$, and multiplying it by $x$, we obtain the first term of this equation. Thus this relationship allows us to propose the following substitution: $t = y^2$.

When we make this substitution in the original equation, we obtain the following linear equation:

$$xdt/dx + 2t = 3x - 6$$

whose solution, which we shall leave for the reader, is very simple using any of the traditional methods.

6. The equation presented below looks, at first sight, as if it could be tackled by any of the known methods:

$$3x^5dx - y(y^2 - x^3)dy = 0. \tag{6}$$

However, even if it is possible to find its solution using these methods, it is not easy to do so. In relation to what has been explained in the previous exercise, let us try to find the appropriate solution. In that regard, substituting only one variable makes finding the solution almost impossible. Let us try it then with two variables: $t = x^3$, $r = y^2$, where $dt = 3x^2dx$, $dr = 2ydy$. When we substitute these values in the original equation, we obtain

$$tdt - 1/2(r - t)dr = 0.$$

This is an equation that is easier to solve with two simple algebraic transformations, giving:

$$(y^2 - 2x^3)(y^2 + x^3)^2 = c.$$

7. In the equation that we are proposing below, the function that we are looking for might appear implicit, a situation that makes its structure and solution difficult. The equation is

$$y' + y \ln y = ye^x. \tag{7}$$

Using what we have learnt working with the structural analogies, we shall transform this equation. Let us divide the equation by $y$, whence

$$y'/y + \ln y = e^x.$$

Then, making $u = \ln y$, after deriving gives us $du/dx = 1/y(y')$, the first term of the differential equation. After that, making the substitution gives us: $du/dx + u = e^x$. The result is a linear equation whose solution method is known.

8. The use of the transformation of the co-ordinate systems is a well-known method for the solution of multiple mathematical situations. It is, however, hardly used in the solution of differential equations. To explain the advantages of its use is unnecessary as they are the same as those provided in other cases in which they are used. In order to solve the following equation:

$$(3x^3 + xy^2 - y^2x^2 + y^4)dx + (3x^2y + y^3 + yx^3 + xy^3)dy = 0 \tag{8}$$

when we factor the terms that are linked to $dx$ and $dy$, and after applying a set of elementary transformations, we get

$$(3x^2 + y^2)(xdx + ydy) + (x^2 + y^2)(-y^2dx + xydy) = 0.$$

Let us make the transformation into polar co-ordinates where $x = r\cos\theta$; $y = r\sin\theta$. In finding $dx$ and $dy$, making a substitution in the differential equation and applying some elementary transformations, we get

$$r^3(2\cos^2\theta + 1)dr + (r^5 \sin\theta)d\theta = 0.$$

Thus, it becomes an easy equation with separable variables. Knowing that $r = \sqrt{x^2 + y^2}$ and $\theta = \arctan(y/x)$, the solution is transformed into the new system.

To find types of equations that can be solved using these techniques under study becomes a very interesting activity. In this respect, we can establish that the equation $y' = f(xy)y/x$ is transformed into a variables-separate equation using the substitution: $u = xy$.

## 3. Solution Analogies

These analogies are analogies that allow a relationship between the resources used to tackle different mathematical situations to be established and their applications within a set content, in our case, differential equations.

It is pertinent to remark that these analogies, with respect to the solutions that we will analyse below, firstly allow us to establish a natural link between the different mathematical contents and the solutions to differential equations. Secondly, they provide an integrated vision of the traditional solving methods.

9. The differential equation that we have set out below is a second-order equation with variable coefficients (Euler equation):

$$x^2 y'' - xy' + y = 0 \tag{9}$$

whose traditional solving method is to make the substitution $x = e^t$ or $t = \ln x$. But in this case we shall assume its solution: let us assume that the solution is a power function of the form $y = x^m$. Then, when we calculate the first and second derivative, and substitute them in the equation, we obtain: $(m-1)^2 = 0$, which has a double solution $m = 1$. One solution to the equation is $y_1 = cx$.

The question is how to obtain a generic solution. If we were to use the method applied to the linear differential equation with constant coefficients, we would have to multiply the solution $y_1$ by $(ax + b)$ (a situation that does not appear in an explicit way in the textbooks, but in the majority of cases they are limited to their practical repercussion). In this case it is not feasible to apply the latter resource. The way in which the practical results are obtained for differential equations with constant coefficients does not create a basis upon which we can tackle the latter problem. We are in a position to answer the two questions: the answers will be given through a general demonstration that lays the foundations for the theoretical elements and thus, the practical elements necessary to tackle the situation.

Let us analyse the following homogeneous differential equation of the second degree with variable coefficients:

$$y'' + p(x)y' + q(x)y = 0.$$

It is clear that any equation of this type can be placed in the given equation. We have already explained that one way of obtaining a particular solution of the equation is $y_1(x)$. Let us assume that the general solution is $y = t(x)y_1(x)$. Finding the first and second derivatives and making the substitution in the equation, gives us:

$$y_1 t'' + (2y_1' + py_1)t' = 0.$$

This is obtained after conveniently structuring the equation and also bearing in mind that $y_1$ is the solution to the given equation. This equation is of the same class of equations that we have solved before, therefore, making $u = t'$, the order is reduced and the equation is transformed as follows:

$$y_1 u' + (2y_1' + py_1)u = 0$$

which is a variables-separate equation:

$$uy_1^2 = ce^{-\int pdx}, \qquad t' = u = \left(ce^{-\int pdx}\right)\Big/ y_1^2.$$

All that remains is to calculate the value of the integral, therefore

$$t = c\int\left(\left(e^{-\int pdx}\right)\Big/ y_1^2\right)dx + c_1.$$

The desired solution is

$$y_2 = y_1(x)\int\left(\left(e^{-\int p(x)dx}\right)\Big/ y_1^2\right)dx.$$

Returning to the previous problem we find that $y_2 = x\ln(x)$. Therefore, the generic solution is

$$y = cx + c_1 x\ln(x).$$

As a way of proving this, we suggest that the justification for multiplying by $x$ in differential equations with constant coefficients is fully explained.

10. Most of the time, students find it hard to understand why is it necessary to find the generic solution to homogeneous equations and in particular for inhomogeneous ones in the solution of linear differential equations. The explanation for this can be given based on the name "linear" equations. It is no less true that many authors associate the name with the use of the operators and that situation is not evident in the solution plan.

We are of the opinion that this denomination can be analysed from the relationship that exists between the solution of a system of linear equations and the solution of this type of equation. We will explain the existing analogy below.

Given a system of linear equations:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_m = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_m = b_2$$
$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nm}x_m = b_n. \qquad (10)$$

Expressing the system in the form of a matrix, gives us: $AX = B$, where $X$ is a vector of magnitude $m$ and $B$ is a vector of magnitude $n$, $A$ is a matrix that takes the role of a linear operator transforming the essential elements of $R^m$ into $R^n$. The solution to the system will be all the values of $x$ in $R^m$ which have the vector $B$ as an image, through the function represented by $A$. The solution to an inhomogeneous system, which is to say the generic solution (if the system is compatible), is given by adding a particular solution to the general solution of the inhomogeneous system.

If an analogy between the solution of the differential equation and the system's solution is established, the equation can be represented by means of an operator in the following way: $U(D)y = F(x)$, where $y$ is a function that can be derived $n$ times and $F(x)$ is a continuous function. In this way, the operator transforms the space of the $n$ times derivable functions into the space of the continuous functions. The solution consists of all the $y(x)$ that have $F(x)$ as an image through the operator. As may be observed, it is more of a natural than a direct relation. We could produce an example, but there would be little sense to it at this point.

11. The equation that we set out below for your consideration may be transformed into a linear equation of the first order in the traditional way. We shall use another method to solve a particular class of equations of this type:

$$xy' + y = 3x^2. \qquad (11)$$

Observe that in the equation, the function $p(x)$ that accompanies the derivative of the unknown function is that which is derived from the function that accompanies the unknown function. We then make $(p(x)y)' = 3x^2$. Integrating both members, gives us, as in this case, $p(x) = x$. Thus $y = x^2 + c/x$. In this regard, we can state that every equation with the following form, or that can be transformed into this form:

$$p(x)y' + p(x)'y = q(x)$$

can be solved using the latter method, which we shall now demonstrate. Given the expression:

$$(p(x)y)' = p(x)'y + p(x)y',$$

which is in direct relationship with the proposed equations, and making $q(x) = (p(x)y)'$, on integrating both members we obtain

$$p(x)y = \int q(x)dx + c.$$

Then, dividing this by $p(x) \neq 0$ the result is $y$, the unknown function that we were looking for. Apply this method to the following equation:

$$y + x/2y' = e^x/2x.$$

12. Let us examine a method for the solution of a Bernoulli type equation. Given the equation:

$$y/x - y' \ln(x) = \ln^{-3}(x). \tag{12}$$

Let us see if it meets the conditions of the case analysed beforehand, but bear in mind that there is a negative sign. Thinking of the derivate of a quotient, we carry out the same analysis. Thus, as $-\ln^2(x)(y/\ln(x))'$ $= \ln^3(x)$ on solving we obtain the following solution: $y = \ln(x)(x - x\ln(x) + c)$. The demonstration of the method is based on the previous resource.

## Conclusions

The execution of this investigation and the analysis of its constituent aspects lead us to the following conclusions:

The work carried out around the use of the analogies in mathematics demonstrates its use in each era of the historical development of our science, though it should be said that from a didactic perspective, in many cases, it is not made explicit and, in others, not even considered. With regard to its use as a didactic resource in differential equations, while avoiding any absolute statements, we can affirm that it is very poorly used.

The analogies examined in this paper stimulate logical thinking processes and ensure meaningful learning by applying different techniques based on previously acquired knowledge. Moreover, they make up essential elements within the theoretical and methodological aspects of differential equations as they allow different types of equations to be generalised and methods for their solution to be found. The treatment of differential equations from this point of view prepares the student for the fundamental ability that mathematics must develop: the solution of problems, at a fundamental level, by being able to tackle differential equations from an active standpoint, with an integrated scientific vision and with specific problem solving techniques.

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