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ASYMPTOTIC STABILITY OF DELAY-DIFFERENCE SYSTEM OF CELLULAR NEURAL NETWORKS VIA MATRIX INEQUALITIES

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Abstract

In this paper, we obtain some criteria for determining the asymptotic stability of the zero solution of delay-difference system of cellular neural networks in terms of certain matrix inequalities by using a discrete version of the Lyapunov second method.

1. Introduction

In recent decades, cellular neural networks have been extensively studied in many aspects and successfully applied to many fields such as pattern identifying, voice recognizing, system controlling, signal processing systems, static image treatment and solving nonlinear algebraic equations, etc. Such applications are based on the existence of equilibrium points and qualitative properties of systems. In electronic implementation, time delays occur due to some reasons such as circuit integration, switching delays of the amplifiers and communication delays, etc. Therefore, the study of the asymptotic stability of cellular neural networks with delays is of particular importance to manufacturing high quality microelectronic cellular neural networks.

While stability analysis of continuous-time neural networks can employ the stability theory of differential equations [12], it is much

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harder to study the stability of discrete-time neural networks [8] with time delays [3] or impulses [11]. The techniques currently available in the literature for discrete-time systems are mostly based on the construction Lyapunov second method [10]. For Lyapunov second method, it is well known that no general rule exists to guide the construction of a proper Lyapunov function for a given system. In fact, the construction of the Lyapunov function becomes a very difficult task.

In this paper, we consider delay-difference system of cellular neural networks of the form:

$$u(k+1) = -Cu(k) + AS(u(k)) + BS(u(k-h)) + f, \tag{1}$$

where $u \in \Omega \subseteq \mathbb{R}^n$ is the neuron state vector, $h \geq 0$, $C = \mathrm{diag}\{c_1, \ldots, c_n\}$, $c_i \geq 0$, $i = 1, 2, \ldots, n$ is the relaxation matrix, $A = (a_{ij})_{n\times n}$ and $B = (b_{ij})_{n\times n}$ are weight matrices, $f = (f_1, \ldots, f_n) \in \mathbb{R}^n$ is the constant external input vector and $S(z) = [s_1(z_1), \ldots, s_n(z_n)]^T$ with $s_i \in C^1[\mathbb{R}, (-1, 1)]$, where s_i is the neuron activations and monotonically increasing for each $i = 1, 2, \ldots, n$.

The asymptotic stability of the zero solution of the delay-difference system of cellular neural networks has been developed during the past several years. We refer to monographs by Arik [2] and Chua and Yang [6] and the references cited therein. Much less is known regarding the asymptotic stability of the zero solution of the delay-difference system of cellular neural networks. Therefore, the purpose of this paper is to establish sufficient conditions for the asymptotic stability of the zero solution of (1) in terms of certain matrix inequalities.

2. Preliminaries

The following notations will be used throughout the paper. $\mathbb{R}^+$ denotes the set of all non-negative real numbers; $\mathbb{Z}^+$ denotes the set of all non-negative integers; $\mathbb{R}^n$ denotes the n-dimensional Euclidean space with Euclidean norm $\|\cdot\|$ and the scalar product between x and y is defined by $x^T y$; $\mathbb{R}^{n\times m}$ denotes the set of all $(n \times m)$-matrices and A^T denotes the transpose of the matrix A.

We assume that the neuron activation functions are bounded and satisfy the following hypotheses, respectively:

$$|s_i(r_1) - s_i(r_2)| \leq l_i |r_1 - r_2|, \qquad \forall r_1, r_2 \in \mathbb{R} \tag{2}$$

and

$$0 \leq \frac{s_i(r_1) - s_i(r_2)}{r_1 - r_2} \leq l_i, \qquad \forall r_1, r_2 \in \mathbb{R}, \tag{3}$$

where $l_i > 0$ are constants for $i = 1, 2, \ldots, n$.

By assumptions (2) and (3), we know that the functions $s_i(\cdot)$ satisfy

$$|s_i(x_i)| \leq l_i |x_i|, \quad i = 1, 2, \ldots, n$$

and

$$s_i^2(x_i) \leq l_i x_i s_i(x_i), \quad i = 1, 2, \ldots, n. \tag{4}$$

Matrix $Q \in \mathbb{R}^{n\times n}$ is positive semidefinite $(Q \geq 0)$ if $x^T Qx \geq 0,$ for all $x \in \mathbb{R}^n$. If $x^T Qx > 0$ $(x^T Qx < 0, \text{ resp.})$ for any $x \neq 0,$ then Q is positive (negative, resp.) definite and denoted by $Q > 0, (Q < 0, \text{ resp.})$. It is easy to verify that $Q > 0, (Q < 0, \text{ resp.})$ iff

$$\exists\beta > 0 : x^T Qx \geq \beta \| x \|^2, \ \ \forall x \in \mathbb{R}^n,$$

$$(\exists\beta > 0 : x^T Qx \leq -\beta \| x \|^2, \ \ \forall x \in \mathbb{R}^n, \text{ resp.}).$$

Fact 1. For any positive scalar ε and vectors x and y, the following inequality holds:

$$x^T y + y^T x \leq \varepsilon x^T x + \varepsilon^{-1} y^T y.$$

Lemma 2.1 [4]**.** *The zero solution of difference system is asymptotic stability if there exists a positive definite function* $V(x) : \mathbb{R}^n \to \mathbb{R}^+$ *such* *that*

$$\exists\beta > 0 : \Delta V(x(k)) = V(x(k+1)) - V(x(k)) \le -\beta \| x(k) \|^2,$$

along the solution of the system. In the case the above condition holds for all $x(k) \in V_\delta$, *we say that the zero solution is locally asymptotically stable.*

Lemma 2.2 [5]**.** *For any constant symmetric matrix* $M \in \mathbb{R}^{n\times n}$, $M = M^T > 0$, *scalar* $s \in \mathbb{Z}^+/\{0\}$, *vector function* $W : [0, s] \to \mathbb{R}^n$, *we have*

$$s\sum_{i=0}^{s-1}\left(w^T(i)Mw(i)\right) \ge \left(\sum_{i=0}^{s-1} w(i)\right)^T M \left(\sum_{i=0}^{s-1} w(i)\right).$$

3. Main Results

In this section, we consider the asymptotic stability of the zero solution u^* of (1) in terms of certain matrix inequalities. Without loss of generality, we can assume that $u^* = 0$, $S(0) = 0$ and $f = 0$ (for otherwise, we let $x = u - u^*$ and define $S(x) = S(x + u^*) - S(u^*)$).

The new from of (1) is now given by

$$x(k+1) = -Cx(k) + AS(x(k)) + BS(x(k-h)). \tag{5}$$

Theorem 1. *The zero solution of the discrete-time system* (5) *is asymptotic stable if there exist symmetric positive definite matrices and* P, G, W *and* $L = \mathrm{diag}[l_1, \ldots, l_n] > 0$ *satisfying the following matrix inequalities*:

$$\psi = \begin{pmatrix} (1, 1) & 0 & 0 \\ 0 & (2, 2) & 0 \\ 0 & 0 & (3, 3) \end{pmatrix} < 0, \tag{6}$$

where

$$(1,1) = C^T PC - P + hG + W + \varepsilon A^T PBB^T PA + \varepsilon_1 C^T PBB^T PC$$

$$+ \varepsilon_2 LA^T PBB^T PAL + LA^T PAL + \varepsilon^{-1} LL,$$

$$(2,2) = LB^T PBL + \varepsilon_1^{-1} LL + \varepsilon_2^{-1} LL - W \quad \textit{and}$$

$$(3,3) = -hG.$$

Proof. Consider the Lyapunov function $V = V_1 + V_2 + V_3$, where

$$V_1 = x^T(k) Px(k),$$

$$V_2 = \sum_{i=k-h}^{k-1} (h - k + i) x^T(i) Gx(i),$$

$$V_3 = \sum_{i=k-h}^{k-1} x^T(i) Wx(i),$$

P, G and W being symmetric positive definite solutions of (6), then difference of V along trajectory of solution of (5) is given by $\Delta V = \Delta V_1 + \Delta V_2 + \Delta V_3$, where

$$\Delta V_1 = V_1(x(k+1)) - V_1(x(k))$$

$$= [-Cx(k) + AS(x(k)) + BS(x(k-h))]^T P$$

$$\times [-Cx(k) + AS(x(k)) + BS(x(k-h))] - x^T(k) Px(k)$$

$$= x^T(k)[C^T PC - P] x(k)$$

$$- x^T(k) C^T PAS(x(k)) - S^T(x(k)) A^T PCx(k)$$

$$- x^T(k) C^T PBS(x(k-h)) - S^T(x(k-h)) B^T PCx(k)$$

$$+ S^T(x(k)) A^T PBS(x(k-h)) + S^T(x(k-h)) B^T PAS(x(k))$$

$$+ S^T(x(k)) A^T PAS(x(k)) + S^T(x(k-h)) B^T PBS(x(k-h)),$$

$$\Delta V_2 = \Delta\left(\sum_{i=k-h}^{k-1}(h-k+i)x^T(i)Gx(i)\right) = hx^T(k)Gx(k) - \sum_{i=k-h}^{k-1} x^T(i)Gx(i)$$

and

$$\Delta V_3 = \Delta\left(\sum_{i=k-h}^{k-1} x^T(i)Wx(i)\right) = x^T(k)Wx(k) - x^T(k-h)Wx(k-h),$$

where (4) and Fact 1 are utilized in (7), respectively.

Note that

$$-x^T(k)C^TPAS(x(k)) - S^T(x(k))A^TPCx(k)$$

$$\leq \varepsilon x^T(k)C^TPAA^TPCx(k) + \varepsilon^{-1}S^T(x(k))S(x(k)),$$

$$-x^T(k)C^TPBS(x(k-h)) - S^T(x(k-h))B^TPCx(k)$$

$$\leq \varepsilon_1 x^T(k)C^TPBB^TPCx(k) + \varepsilon_1{}^{-1}S^T(x(k-h))S(x(k-h)),$$

$$+ S^T(x(k))A^TPBS(x(k-h)) + S^T(x(k-h))B^TPAS(x(k))$$

$$\leq \varepsilon_2 S^T(k)A^TPBB^TPAS(k) + \varepsilon_2{}^{-1}S^T(x(k-h))S(x(k-h)),$$

$$S^T(x(k-h))B^TPBS(x(k-h)) \leq x^T(k-h)LB^TPBLx(k-h),$$

$$S^T(x(k))A^TPAS(x(k)) \leq x^T(k)LA^TPALx(k),$$

$$\varepsilon_2 S^T(k)A^TPBB^TPAS(k) \leq \varepsilon_2 x^T(k)LA^TPBB^TPALx(k)$$

and

$$\varepsilon_1{}^{-1}S^T(x(k-h))S(x(k-h)) \leq \varepsilon_1{}^{-1}x^T(k-h)LLx(k-h),$$

$$\varepsilon_2{}^{-1}S^T(x(k-h))S(x(k-h)) \leq \varepsilon_2{}^{-1}x^T(k-h)LLx(k-h),$$

$$\varepsilon^{-1}S^T(x(k))S(x(k)) \leq \varepsilon^{-1}x^T(k)LLx(k),$$

hence

$$
\begin{aligned}
\Delta V_1 \le\; & x^T(k)\left[C^T PC - P\right]x(k) + \varepsilon x^T(k)A^T PBB^T PAx(k) \\
& + \varepsilon_1 x^T(k)C^T PBB^T PCx(k) + x^T(k-h)LB^T PBLx(k-h) \\
& + x^T(k)LA^T PALx(k) + \varepsilon_2 x^T(k)LA^T PBB^T PALx(k) \\
& + \varepsilon_1{}^{-1} x^T(k-h)LLx(k-h) + \varepsilon_2{}^{-1} x^T(k-h)LLx(k-h) \\
& + \varepsilon^{-1} x^T(k)LLx(k),
\end{aligned}
$$

then we have

$$
\begin{aligned}
\Delta V \le\; & x^T(k)\left[C^T PC - P + hG + W + \varepsilon A^T PBB^T PA + \varepsilon_1 C^T PBB^T PC \right. \\
& \left. + \varepsilon_2 LA^T PBB^T PAL + LA^T PAL + \varepsilon^{-1} LL\right]x(k) \\
& + x^T(k-h)\left[LB^T PBL + \varepsilon_1{}^{-1} LL + \varepsilon_2{}^{-1} LL - W\right]x(k-h) \\
& - \sum_{i=k-h}^{k-1} x^T(i)Gx(i).
\end{aligned}
$$

Using Lemma 2.2, we obtain

$$
\sum_{i=k-h}^{k-1} x^T(i)Gx(i) \ge \left(\frac{1}{h}\sum_{i=k-h}^{k-1} x(i)\right)^T (hG)\left(\frac{1}{h}\sum_{i=k-h}^{k-1} x(i)\right).
$$

From the above inequality it follows that

$$
\begin{aligned}
\Delta V \le\; & x^T(k)\left[C^T PC - P + hG + W + \varepsilon A^T PBB^T PA + \varepsilon_1 C^T PBB^T PC \right. \\
& \left. + \varepsilon_2 LA^T PBB^T PAL + LA^T PAL + \varepsilon^{-1} LL\right]x(k) \\
& + x^T(k-h)\left[LB^T PBL + \varepsilon_1{}^{-1} LL + \varepsilon_2{}^{-1} LL - W\right]x(k-h) \\
& - \left(\frac{1}{h}\sum_{i=k-h}^{k-1} x(i)\right)^T (hG)\left(\frac{1}{h}\sum_{i=k-h}^{k-1} x(i)\right)
\end{aligned}
$$

$$= \left(x^T(k), x^T(k-h), (\frac{1}{h}\sum_{i=k-h}^{k-1} x(i))^T \right) \begin{pmatrix} (1,1) & 0 & 0 \\ 0 & (2,2) & 0 \\ 0 & 0 & (3,3) \end{pmatrix}$$

$$\times \begin{pmatrix} x(k) \\ x(k-h) \\ \left(\frac{1}{h} \sum_{i=k-h}^{k-1} x(i) \right) \end{pmatrix}$$

$$= y^T(k) \begin{pmatrix} (1,1) & 0 & 0 \\ 0 & (2,2) & 0 \\ 0 & 0 & (3,3) \end{pmatrix} y(k)$$

$$= y^T(k)\psi y(k),$$

where

$$(1,1) = C^T PC - P + hG + W + \varepsilon A^T PBB^T PA + \varepsilon_1 C^T PBB^T PC$$

$$+ \varepsilon_2 LA^T PBB^T PAL + LA^T PAL + \varepsilon^{-1} LL,$$

$$(2,2) = LB^T PBL + \varepsilon_1^{-1} LL + \varepsilon_2^{-1} LL - W,$$

$$(3,3) = -hG,$$

and $y(k) = \begin{pmatrix} x(k) \\ x(k-h) \\ \left(\frac{1}{h} \sum_{i=k-h}^{k-1} x(i) \right) \end{pmatrix}.$

By the condition (6), ΔV is negative definite, namely there is a number $\beta > 0$ such that $\Delta V(y(k)) \leq -\beta \| y(k) \|^2$, and hence the asymptotic stability of the system immediately follows from Lemma 2.1. This completes the proof.

5. Conclusions

In this paper based on a discrete analog of the Lyapunov second method, we have established a sufficient condition for the asymptotic stability of delay-difference system of cellular neural networks in terms of certain matrix inequalities. The result has been applied to obtain new stability conditions for some classes of delay-difference equation such as delay-difference system of cellular neural networks with multiple delays in the terms of certain matrix inequalities.

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