Advances in Fuzzy Sets and Systems
Volume 3, Number 2, June 2008, Pages 239-254
Published online: July 2, 2008
This paper is available online at http://www.pphmj.com

PERIODIC SOLUTION OF NONLINEAR FUZZY DIFFERENTIAL EQUATIONS

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Abstract

The periodic solutions for a perturbation of autonomous fuzzy differential equations are investigated, by the use of generalized differentiability and the index of fixed point of a compact mapping.

1. Introduction

The fuzzy sets theory has grown considerably during the last decades because of the applicability and the flexibility of the fuzzy sets and fuzzy numbers in many areas. Differential equations in a fuzzy setting constitute a natural way to model uncertainty of dynamical systems. The most used setting is the H-differentiability due to Puri and Radulescu [8]. Many papers have been published in this setting with some shortcomings [7]. The existence of periodic solutions are studied by interpreting the fuzzy differential equation as a system of differential inclusions, but this approach has in turn some shortcomings. Indeed, the solutions obtained are not fuzzy-number-valued functions.

2000 Mathematics Subject Classification: 03E72.

Keywords and phrases: fuzzy differential equations, periodic solutions, absolute retract, fixed point index.

Received January 11, 2008

In this paper, we use a different approach to study the existence of periodic solution of the perturbed fuzzy differential equation

$$x' = f(x) + \beta\psi(t, x),\ t \in [0, T],\ T > 0, \tag{1}$$

where the derivative is in the generalized sense introduced in [1]. We embed the fuzzy numbers set $\mathbf{R}_{\mathcal{F}}$ into an absolute retract (A. R. in short) of a Banach space (Section 2), and we define a complete metric space of continuous function on $[0, T]$ from this A. R., and by using the fixed point properties in A. R., and the fixed point index of compact mapping, we prove the existence of periodic solutions of the equation (1) with respect to a homeomorphism. The advantage of the proposed approach as compared with the differential inclusions' consists of the fact, that it is simple, and the solutions obtained are fuzzy-number-valued functions. Our approach seems to be new in the theory of fuzzy differential equations.

The paper is organized as follows. In Section 2, we give some preliminaries. In Section 3, we present our approach and show our main result. In Section 4, we give an application.

2. Preliminaries

Let X be a non empty set, and $\mathcal{F}(X)$ be the set of all fuzzy subsets of X.

Definition 1. A *fuzzy set* $u \in \mathcal{F}(X)$ is normal if there exists an $x_0 \in X$ such that $u(x_0) = 1$; $u \in \mathcal{F}(X)$ is fuzzy convex if $u(\lambda x + (1 - \lambda)y) \geq \min\{u(x), u(y)\}$, for all $\lambda \in [0, 1]$ and $x, y \in X$.

Let $\mathbf{R}_{\mathcal{F}}$ denote the family of all normal and fuzzy convex sets such that

(a) u is upper semicontinuous

(b) $[u]^0 = cl\{x \in \mathbf{R} \mid u(x) > 0\}$ is compact.

We observe that $\mathbf{R}$ is a subset of $\mathbf{R}_{\mathcal{F}}$, since $\mathbf{R}$ can be considered as the set $\{\chi_x : x \in \mathbf{R}\}$.

Definition 2. The *level set* $[u]^\alpha$ of $u \in \mathbf{R}_{\mathcal{F}}$ is defined by $[u]^\alpha = \{x \in \mathbf{R} \mid u(x) \geq \alpha\}$ for $\alpha \in [0, 1]$. Consequently, $[u]^\alpha$, $\alpha \in [0, 1]$ is a bounded and closed interval of $\mathbf{R}$ represented by $[u]^{\alpha} = [u_{-}^{\alpha}, u_{+}^{\alpha}]$, where $u_{-}, u_{+} : [0, 1] \to \mathbf{R}$ are bounded, left continuous mappings such that $u_{-}(\alpha) = u_{-}^{\alpha}$ is non decreasing in $[0, 1]$ and $u_{+}(\alpha) = u_{+}^{\alpha}$ is non increasing in $[0, 1]$.

The following concepts are well known (see [1]).

Definition 3. (1) For $u, v \in \mathbf{R}_{\mathcal{F}}$, the *sum* $u \oplus v$ and the *scalar multiplicity* $\lambda \odot u$ for all $\lambda \in \mathbf{R}$ are defined by $[u \oplus v]^{\alpha} = [u]^{\alpha} + [v]^{\alpha}$ and $[\lambda \odot u]^{\alpha} = \lambda \cdot [u]^{\alpha}$.

(2) The set $\widetilde{0} = \chi_0 \in \mathbf{R}_{\mathcal{F}}$ and it is the neutral element for the addition $\oplus$ in $\mathbf{R}_{\mathcal{F}}$.

(3) No $u \in \mathbf{R}_{\mathcal{F}} \setminus \mathbf{R}$ has a symmetric with respect to $\oplus$ and $\widetilde{0}$.

(4) For all $u \in \mathbf{R}_{\mathcal{F}}$ and $a, b \in \mathbf{R}_{+}$ or $a, b \in \mathbf{R}_{-}$, $(a + b) \odot u = a \odot u \oplus b \odot u$. The case of general $a, b \in \mathbf{R}$ does not hold.

(5) For any $\lambda \in \mathbf{R}$ and $u, v \in \mathbf{R}_{\mathcal{F}}$, $\lambda \odot (u \odot v) = \lambda \odot u \oplus \lambda \odot v$.

(6) For $\lambda, \mu \in \mathbf{R}$ and $u \in \mathbf{R}_{\mathcal{F}}$, $\lambda \odot (\mu \odot u) = (\lambda \cdot \mu) \odot u$.

We define the following Hausdorff metric in $\mathbf{R}_{\mathcal{F}}$ (see [1]).

Definition 4. Define $d : \mathbf{R}_{\mathcal{F}} \times \mathbf{R}_{\mathcal{F}} \to \mathbf{R}_{+} \cup \{0\}$ by the equation

$$d(u, v) = \sup_{0 \le \alpha \le 1} \max(|u_{-}^{\alpha} - v_{-}^{\alpha}|, |u_{+}^{\alpha} - v_{+}^{\alpha}|).$$

Using results in [1], we know that $(\mathbf{R}_{\mathcal{F}}, d)$ is a complete metric space and d satisfies

(a) $d(u \oplus w, v \oplus w) = d(u, v)$ for all $u, v, w \in \mathbf{R}_{\mathcal{F}}$;

(b) $d(\lambda \odot u, \lambda \odot v) = |\lambda| d(u, v)$ for all $u, v \in \mathbf{R}_{\mathcal{F}}, \lambda \in \mathbf{R}$;

(c) $d(u \oplus w_1, v \oplus w_2) \le d(u, v) + d(w_1, w_2)$ for all $u, v, w_1, w_2 \in \mathbf{R}_{\mathcal{F}}$.

We recall some differentiability properties for the fuzzy-number-valued mappings in [1].

Definition 5. A *mapping* $f:(a, b) \subset \mathbf{R} \to \mathbf{R}_{\mathcal{F}}$ is strongly differentiable in the generalized sense at $t_0 \in (a, b)$, if for all $h > 0$ sufficiently small, there exists a $f'(t_0) \in \mathbf{R}_{\mathcal{F}}$ such that

(i) there exist $f(t_0 + h) - f(t_0)$ and $f(t_0) - f(t_0 + h)$ such that the limits (in the metric d)

$$\lim_{h \to 0^+} \frac{f(t_0 + h) - f(t_0)}{h}$$

and

$$\lim_{h \to 0^+} \frac{f(t_0 - f(t_0 - h))}{h}$$

exist and are equal to $f'(t_0)$ or

(ii) there exist $f(t_0) - f(t_0 + h)$ and $f(t_0 - h) - f(t_0)$ such that the limits in the metric d

$$\lim_{h \to 0^+} \frac{f(t_0) - f(t_0 + h)}{-h}$$

and

$$\lim_{h \to 0^+} \frac{f(t_0 - h) - f(t_0)}{-h}$$

exist and are equal to $f'(t_0)$ or

(iii) there exist $f(t_0) - f(t_0 + h)$ and $f(t_0) - f(t_0 - h)$ such that the limits in the metric d

$$\lim_{h \to 0^+} \frac{f(t_0) - f(t_0 + h)}{-h}$$

and

$$\lim_{h \to 0^+} \frac{f(t_0) - f(t_0 - h)}{h}$$

exist and are equal to $f'(t_0)$ or

(iv) there exist $f(t_0 + h) - f(t_0)$ and $f(t_0 - h) - f(t_0)$ such that the limits in the metric d

$$\lim_{h \to 0^+} \frac{f(t_0 + h) - f(t_0)}{h}$$

and

$$\lim_{h \to 0^+} \frac{f(t_0 - h) - f(t_0)}{-h}$$

exist and are equal to $f'(t_0)$.

Definition 6. Consider $f : (a, b) \subset \mathbf{R} \to \mathbf{R}_{\mathcal{F}}$, $t_0 \in (a, b)$ and $(h_n)_{n \in \mathbf{N}}$ a positive sequence in $\mathbf{R}$ such that $h_n \to 0$ as $n \to \infty$ and $n_0 \in \mathbf{N}$. Denote by

$$A_n^1 = \{n \geq n_0 \mid \exists E_n^1 := f(t_0 + h_n) - f(t_0)\}$$

$$A_n^2 = \{n \geq n_0 \mid \exists E_n^2 := f(t_0) - f(t_0 - h_n)\}$$

$$A_n^3 = \{n \geq n_0 \mid \exists E_n^3 := f(t_0) - f(t_0 + h_n)\}$$

$$A_n^4 = \{n \geq n_0 \mid \exists E_n^4 := f(t_0 - h_n) - f(t_0)\}$$

f is said *weakly differentiable* in the generalized sense, if and only if, $\bigcup_{i=1}^4 A_{n_0}^i = \{n \in \mathbf{N} : n \geq n_0\}$ and there exists an element $f'(t_0) \in \mathbf{R}_{\mathcal{F}}$ such that, if for some $i \in \{1, 2, 3, 4\}$, $\operatorname{Card}(A_n^i) = +\infty$ then,

$$\lim_{\substack{n \to \infty \\ n \in A_{n_0}^i}} d\left(\frac{E_n^i}{(-1)^{i+1} h_n}, f'(x_0) \right) = 0.$$

Let $\overline{C}[0, 1] = \{\varphi : [0, 1] \to \mathbf{R} \mid \varphi$ is bounded on $[0, 1]$, left continuous at 0 and has right limit for any $t \in [0, 1[\}$.

$\overline{C}[0, 1]$ is a Banach space with the norm $\| \varphi \| = \sup\{| \varphi(t) | : t \in [0, 1]\}$, so is $\mathbf{E} = \overline{\mathbf{C}}[\mathbf{0}, \mathbf{1}] \times \overline{\mathbf{C}}[\mathbf{0}, \mathbf{1}]$ with the norm $\| (\varphi, \psi) \| = \max\{\| \varphi \|, \| \psi \|\}$. Let $j : \mathbf{R}_{\mathcal{F}} \to \mathbf{E}$ be a mapping defined by $j(u) = (u_-, u_+)$, where $u_-, u_+ : [0, 1] \to \mathbf{R}$ are the mappings defined above by $u_-(\alpha) = u_-^{\alpha}$ and $u_+(\alpha) = u_+^{\alpha}$.

We shall use the following embedding theorem [5]:

Theorem 1. *The mapping j embeds* $\mathbf{R}_{\mathcal{F}}$ *into the Banach space* $\mathbf{E}$ *as a closed and convex cone with vertex at* 0. *Moreover, j satisfies the following properties*:

(1) $j(\lambda \odot u \oplus \mu \odot v) = \lambda j(u) + \mu j(v),$ *for all* $u, v \in \mathbf{R}, \lambda, \mu \geq 0;$

(2) $d(u, v) = \| j(u) - j(v) \|.$

J and its inverse are clearly continuous.

Theorem 2 [1]. *Let* $f : (a, b) \subset \mathbf{R} \to \mathbf{R}_{\mathcal{F}}$ *and* $t_0 \in (a, b)$.

(i) *If f is strongly differentiable in the generalized sense* (i) (*resp.* (ii)) *at* t_0, *then* $j \circ f$ *is Fréchet differentiable at* t_0, *and* $(j \circ f)'(t_0) = j(f'(t_0));$

(ii) *If f is strongly differentiable at* t_0 *in the generalized sense* (iii) (*resp.* (iv)), *then* $j \circ f$ *is Fréchet differentiable at left and at right* (*resp. at right and at left*) *at* t_0, *and* $(j \circ f)'_r(t_0 = j(f'(t_0)), (j \circ f)'_l(t_0) = -\widetilde{j}(f'(t_0)))$ (*resp.* $(j \circ f)'_r = -\widetilde{j}(f'(t_0))$ *and* $(j \circ f)'_l(t_0) = j(f'(t_0))$).

Theorem 3 [1]. *Let* $g : (a, b) \subset \mathbf{R} \to \mathbf{R}$ *be differentiable and* $c \in \mathbf{R}_{\mathcal{F}}$.

(1) *If* $g'(t)$ *has at most a finite number of roots in* (a, b), *then* $f = c \odot g : \mathbf{R} \to \mathbf{R}_{\mathcal{F}}$ *is strongly differentiable in the generalized sense on* (a, b) *and* $f'(t) = c \odot g'(t)$ *for all* $t \in (a, b)$;

(2) *For all* $t \in (a, b)$, $f = c \odot g$ *is weakly differentiable in the generalized sense and* $f'(t) = c \odot g'(t)$.

3. Main Result

Consider the space Λ of all functions $u : [0, T] \to \mathbf{R}_{\mathcal{F}}$ continuously differentiable on $[0, T]$, $T > 0$ and define on Λ the distance $D(u, v) = \sup_{t \in [0, T]} d(u(t), v(t))$. Then (Λ, D) is clearly a complete metric space. Consider $j(\Lambda)$ and the distance $D^*(j(u), j(v)) = \sup_{t \in [0, T]} \| j(u(t)) - j(v(t)) \|$. Then, by the definition of j, (Λ, D) and $(j(\Lambda), D^*)$ are isometric. Therefore, $(j(\Lambda), D^*)$ is a complete metric space. Since $j(\mathbf{R}_{\mathcal{F}})$ is a non empty closed and convex subset of a Banach space, it is an absolute retract (AR) of this space, and since $(j(\Lambda), D^*)$ is a complete metric space defined on $j(\mathbf{R}_{\mathcal{F}})$, it is an absolute retract of the same Banach space (see e.g. [2] p. 65). Therefore, $(j(\Lambda), D^*)$ has the fixed point property.

We consider the following assumptions for the functions f and ψ in equation (1).

Let $f : \mathbf{R}_{\mathcal{F}} \to \mathbf{R}_{\mathcal{F}}$ be C^1 and $\psi : \mathbf{R}_+ \times \mathbf{R}_{\mathcal{F}} \to \mathbf{R}_{\mathcal{F}}$ be continuous and T-periodic in t for some real number $T > 0$. Define $F(t, x; \beta) = f(x) + \beta \odot \psi(t, x)$, then

(A1) $j \circ F(t, x; \beta) \in \mathbf{T}(y, j(\mathbf{R}_{\mathcal{F}}))$ for all $y \in \partial j(\mathbf{R}_{\mathcal{F}})$, and $\beta \geq 0$, where

$$\mathbf{T}(y, j(\mathbf{R}_{\mathcal{F}})) = \left\{ v : \lim_{h \to 0^+} \frac{1}{h} \| (y + hv) - j(\mathbf{R}_{\mathcal{F}}) \| = 0, \ y \in j(\mathbf{R}_{\mathcal{F}}) \right\}$$

is the Bouligand tangent cone to $j(\mathbf{R})$ at y.

(A2) There exists $\Omega \subset j(\Lambda)$ an open and bounded set, such that $f(x) \neq 0$ for all $x \in j^{-1}(\partial\Omega)$.

We observe that, solutions of

$$x' = f(x) \tag{2}$$

correspond to the solutions of

$$j(x') = j(f(x)). \tag{3}$$

Since f is C^1, $j \circ f$ is C^1 too, and there exists $\varepsilon_0 > 0$ such that (3) induces a semi dynamical system $\pi(\varepsilon, \cdot) = \pi_\varepsilon(\cdot)$. By (A2), $\pi_\varepsilon(\cdot)$ has no rest point on $\partial\Omega$. Therefore, $i(\pi_\varepsilon, \Omega, j(\Lambda))$ is well defined and constant for all $0 < \varepsilon \leq \varepsilon_0$ whenever ε_0 is small enough ([8] Proposition 4.2).

Define

$$i_*(\pi_0, \Omega, j(\Lambda)) = \lim_{\varepsilon \to 0^+} i(\pi_\varepsilon, \Omega, j(\Lambda)),$$

let $x(t) = \omega(t, 0, x_0)$ be the unique and maximal solution of the Cauchy problem $x' = f(x)$, $x(0) = x_0$, and assume that

(A3), : $w'_{(3)}(t, 0, x_0) = \frac{\partial}{\partial x}\omega(t, 0, x_0)$ exists and is continuously invertible.

Define the compact operator $\chi : j(\Lambda) \to j(\Lambda)$ by

$$\chi(j(x)) = j \circ \omega(\cdot, 0, x(T)).$$

Then our main result can be formulated as follows:

Theorem 4. *Assume that* $(A1)$, $(A2)$, $(A3)$ *hold and*

(A4) $\chi(\zeta) \neq \zeta$ *for all* $\zeta \in \partial\Omega$

(A5) $i_*(\pi_0, \Omega, j(\Lambda)) \neq 0.$

Then there exists $\beta_0 > 0$ *such that for all* $0 < \beta \leq \beta_0$, *the problem* (1) *has at least one T-periodic solution in* Λ.

To prove Theorem 4, we need some additional and technical results.

Consider the homotopy $H_\varepsilon : j(\Lambda) \times [0, 1] \to j(\Lambda)$ defined as follows:

$$H_\varepsilon(y, \lambda) = (1 - \lambda)\pi_\varepsilon(y) + \lambda\chi(y),$$

then for all $0 < \varepsilon \leq \varepsilon_0$ and $\varepsilon_0 > 0$ sufficiently small, $H_\varepsilon(\cdot, \lambda)$ is an admissible homotopy by (A2) and (A4). Hence, $i(H_\varepsilon(\cdot, \lambda), \Omega, j(\Lambda))$ is well defined and constant for all $\lambda \in [0, 1]$.

Therefore,

$$i(H_\varepsilon(\cdot, 0), \Omega, j(\Lambda)) = i(H_\varepsilon(\cdot, 1), j(\Lambda)),$$

that is

$$i(\pi_\varepsilon, \Omega, j(\Lambda)) = i(\chi, \Omega, j(\Lambda)). \tag{4}$$

Then, taking limits as $\varepsilon \to 0^+$, we obtain the following Lemma:

Lemma 1. *Assume that* (A1), (A2) *and* (A4) *hold. Then*

$$i_*(\pi_0, \Omega, j(\Lambda)) = i(\chi, \Omega, j(\Lambda)).$$

We are now in a position to prove Theorem 4.

3.1. Proof of Theorem 4

Consider the following change of variable in (1)

$$x(t) = \omega(t, 0, z(t)), \tag{5}$$

for every $z(t) \in \Lambda$. Then (5) defines x uniquely such that $x(0) = z(0)$, and x is a solution of (1) in Λ, if and only if, z satisfies the equation

$$\begin{aligned} &\omega'_{(1)}(t, 0, z(t)) \oplus \omega'_{(3)}(t, 0, z(t)) \cdot z'(t) \\ &= f(\omega(t, 0, z(t)) \oplus (\beta \odot \psi(t, \omega(t, 0, z(t))))), \end{aligned} \tag{6}$$

where $\omega'_{(j)} = \dfrac{\partial}{\partial x_j} \omega(x_1, x_2, x_3)$, $j = 1, 2, 3.$ By the definition of $\omega(t, 0, z(t))$, we know that

$$\omega'_{(1)}(t, 0, z(t)) = f(\omega(t, 0, z(t))).$$

Hence, by (A3), one can rewrite (6) as

$$\begin{aligned} z'(t) &= \beta \odot (\omega'_{(3)}(t, 0, z(t)))^{-1} \psi(t, \omega(t, 0, z(t))) \\ &= \beta \odot \Psi(t, z(t)). \end{aligned} \tag{7}$$

By integrating (7), we obtain

$$z(t) = z(0) \oplus \left(\beta \odot \int_0^t \Psi(s, z(s)) ds\right). \tag{8}$$

If x is a T-periodic solution of (1), then

$$j(z(0)) = j(x(0)) = j(x(T)) = j(\omega(T, 0, z(T))).$$

Hence, the problem of existence of T-periodic solutions of equation (1) is equivalent to the problem of existence of solutions of the integral equation

$$j(z(t)) = j(\omega(T, 0, z(T))) + \beta \int_0^t j(\Psi(s, z(s))) ds. \tag{9}$$

Define the constants

$$M_0 = \max\{\| j(\xi) - \chi(j(\xi)) \|, \xi = x(0)\}$$

$$M_1 = \min\{\| y(t) - \chi(y(t)) \| : y(t) \in \partial\Omega, t \in [0, T]\}$$

$$M_2 = \max\left\{\left\| \frac{d}{dt}\chi(y(t)) \right\| : y(t) \in \overline{\Omega}, t \in [0, T]\right\}$$

$$M_3 = \max\{\| j(\Psi(t, z(t))) \| : (t, z(t)) \in [0, T] \times j^{-1}(\overline{\Omega})\},$$

and define

$$\beta_0 = \min\left(\frac{M_1}{TM_3(M_2 + 1) + M_0}, \frac{M_1}{TM_3(M_2 + 1) + M_0 + K_0}\right),$$

where $K_0 = \max\{\| j(\omega(T, 0, z(T))) \| : z \in \overline{\Omega}\}$.

For every $0 < \beta \leq \beta_0$, define the mapping

$$\Phi_\beta : j(\Lambda) \to j(\Lambda)$$

by

$$\Phi_\beta(j(z))(t) = j(\omega(T, 0, (T))) + \beta\int_0^t j(\Psi(s, z(s)))ds. \tag{10}$$

Then, for all $0 < \beta \leq \beta_0$, the problem of existence of solutions of equation (9) in $j(\Lambda)$ is equivalent to the problem of finding the fixed points of the mapping Φ_β in $j(\Lambda)$. Since $j(\Lambda)$ is an absolute retract, it will suffice to find an open and bounded set $\mathcal{O} \subset j(\Lambda)$ such that (see [4]),

(1) $\Phi_\beta : \overline{\mathcal{O}} \to j(\Lambda)$ is compact;

(2) $Fix(\Phi_\beta) \cap \partial\mathcal{O} = \varnothing$;

(3) $i(\Phi_\beta, \mathcal{O}, j(\Lambda)) \neq 0$.

Let us choose $\mathcal{O} = \Omega$ and prove (1), (2), (3) for Ω.

(1) Since $j(\Lambda)$ is a complete metric space of continuous functions defined on $[0, T]$, it will suffice to prove that Φ_β is completely continuous, i.e., equicontinuous and uniformly bounded. Let $t_1, t_2 \in [0, T]$ and

$z \in j^{-1}(\overline{\Omega})$. Then

$$\begin{aligned}
&\|\Phi_\beta(j(z))(t_2) - \Phi_\beta(j(z))(t_1)\| \\
&= \beta \left\| \int_0^{t_2} j(\Psi(s,\ z(s)))ds - \int_0^{t_1} j(\Psi(s,\ z(s)))ds \right\| \\
&= \beta \left\| j\left(\int_0^{t_2} \Psi(s,\ z(s))ds\right) - j\left(\int_0^{t_1} \Psi(s,\ z(s))\right)ds \right\| \\
&= \beta d\left(\int_0^{t_2} \Psi(s,\ z(s))ds,\ \int_0^{t_1} \Psi(s,\ z(s))ds\right) \\
&\leq \beta \int_{t_1}^{t_2} d(\Psi(s,\ z(s)),\ \widetilde{0})ds \\
&= \beta \int_{t_1}^{t_2} \| j(\Psi(s,\ z(s))) \| ds \\
&\leq \beta M_3 |\, t_2 - t_1 \,|,
\end{aligned} \tag{11}$$

and for every $t \in [0, 1]$, and every $z \in j^{-1}(\overline{\Omega})$, we have

$$\begin{aligned}
\|\Phi_\beta(j(z))(t)\| &\leq \| j(\omega(T,\ 0,\ z(T))) \| + \beta \int_0^T \| j(\Psi(s,\ z(s))) \| ds \\
&\leq K_0 + \beta T M_3.
\end{aligned} \tag{12}$$

Hence by (11) and (12) and the Arzela-Ascoli theorem, it results that Φ_β is compact on $\overline{\Omega}$.

(2) Let us assume by contradiction that there exist a $\beta \in (0,\ \beta_0]$ and $z_\beta(\cdot) \in j^{-1}(\partial\Omega)$ such that

$$j(z_\beta(t)) = \Phi_\beta(j(z_\beta))(t)$$

for all $t \in [0, 1]$. Then using the constants M_j, $j = 0, 1, 2, 3,$ and the mean value theorem, we get

$$\|j(z_\beta(t)) - \chi(j(z_\beta(t)))\| = \|j(z_\beta(t)) - j(\omega, (T, 0, z_\beta(T)))$$

$$+ j(\omega, (T, 0, z_\beta(T))) - \chi(j(z_\beta(t)))\|$$

$$= \left\| \beta\int_0^t j(\Psi(s, z_\beta(s)))ds + j(\omega(T, 0, z_\beta(T))) - \chi(j(z_\beta(t))) \right\|$$

$$\leq \beta_0\int_0^T \|j(\Psi(s, z_\beta(s)))\|ds + \|j(\omega(T, 0, z_\beta(T))) - \chi(j(z_\beta(0)))\|$$

$$+ \|\chi(j(z_\beta(0))) - \chi(j(z_\beta(t)))\|$$

$$\leq \beta_0 TM_3 + \|j(z_\beta(0)) - \chi(j(z_\beta(0)))\| + \|\chi(j(z_\beta(0))) - \chi(j(z_\beta(t)))\|$$

$$\leq \beta_0 TM_3 + M_0 + \beta_0 M_2 M_3 T < M_1,$$

a contradiction with the definition of M_1. Hence, $Fix(\Phi_\beta) \cap \partial\Omega = \emptyset$, for all $0 < \beta \leq \beta_0$.

(3) Consider the homotopy $F : j(\Lambda) \times [0, 1] \to j(\Lambda)$ defined by

$$F(j(z(t)), \lambda) = (1 - \lambda)\chi(j(z(t))) + \lambda\Phi_\beta(j(z(t))),$$

and assume by contradiction that there exist $\beta \in (0, \beta_0]$, and $z_\beta(\cdot) \in j^{-1}(\partial\Omega)$ such that $j(z_\beta(t)) = F(j(z_\beta(t)), \lambda)$, then

$$\|j(z_\beta(t)) - \chi(j(z_\beta(t)))\|$$

$$= \lambda\left\| \chi(j(z_\beta(t))) - j(\omega(T, 0, z_\beta(T))) - \beta\int_0^t j(\Psi(s, z_\beta(s)))ds \right\|$$

$$\leq \beta_0\int_0^T \|j(\Psi(s, z_\beta(s)))\|ds + \|j(\omega(T, 0, z_\beta(T))) - \chi(j(z_\beta(0)))\|$$

$$+ \|\chi(j(z_\beta(0))) - \chi(j(z_\beta(0)))\|$$

$$\leq \beta_0 TM_3 + M_0 + \beta_0 TM_2 M_3 < M_1,$$

which contradicts the definition of M_1. Therefore, $F(\cdot, \lambda)$ is an admissible homotopy and by the homotopy invariance property, we have

$$i(F(\cdot, 0), \Omega, j(\Lambda)) = i(F(\cdot, 1), \Omega, j(\Lambda)),$$

that is

$$i(\chi, \Omega, j(\Lambda)) = i(\Phi_\beta, \Omega, j(\Lambda)).$$

By using Lemma 1 and (A5), we have

$$i(\Phi_\beta, \Omega, j(\Lambda)) \neq 0,$$

for all $\beta \in (0, \beta_0]$.

4. Application

Let $g : j(\mathbf{R}_{\mathcal{F}}) \to \mathbf{R}$ be continuously differentiable such that $dg(y)$ has only a finite number of roots in $B(0, \varrho)$, $\varrho > 0$. Let $c \in \mathbf{R}_{\mathcal{F}}$ and $f : \mathbf{R}_{\mathcal{F}} \to \mathbf{R}_{\mathcal{F}}$ such that $f(x) = c \odot g(j(x))$. As g is C^1, we have f is C^1. Since (A1) and (A2) are standard and can easily be satisfied by $f = c \odot g \circ j$, we shall only prove that (A3) is satisfied. We need the following hypotheses:

(H1) There exists a function $V : j(\Lambda) \to \mathbf{R}_+$ continuous such that

(1) $V(0) = 0$ and $V(y) \neq 0$ for all $y \neq 0$;

(2) if there exists $(y_n)_n \in \mathbf{N} \subset j(\Lambda)$ such that $\lim_{n\to\infty} V(y_n) = 0$ then $\lim_{n\to\infty} \| u_n \| = 0$.

(H2) There exists a function $M_+ : j(\Lambda) \times j(\Lambda) \to \mathbf{R}$ such that $M_+(u, y)$ is continuous in y uniformly for all $u \in B(u_0, R_0) \subset j(\Lambda)$ and satisfies

(i) $V(u + y) - V(u) \leq M_+(u, y) + o(\| y \|)$;

(ii) $M_+(u, \lambda y) = \lambda M_+(u, y)$, for all $\lambda \geq 0$;

(iii) $M_+(u, y_1 + y_2) \leq M_+(u, y_1) + M_+(u, y_2)$;

(iv) There exists a function $g_0 : \mathbf{R}_+ \to \mathbf{R}_+$ continuous such that

$$M_+(y,\ j(f'(x))y) \leq g_0(V(y))$$

for all $y \in j(\Lambda)$ and $x \in B(x_0,\ \varrho) \subset \Lambda.$

Theorem 5. *Assume that* (A1), (A2), (H1) *and* (H2) *hold and*

(H3) *the maximal solution of*

$$\omega' = g_0(\omega),\ \omega(0) = 0$$

is $\omega(t) \equiv 0$ *for all* $t \geq 0.$

Then the Fréchet derivative of $j(x(t)) = j(\omega(t,\ 0,\ x_0))$ *exists and satisfies the equation*

$$y' = j(f'(\omega(t,\ 0,\ x_0)))y,\ (y(0)) = I. \tag{13}$$

Proof. By (A1) if $\beta = 0,$ then the Cauchy problem

$$x' = f(x),\ x(0) = x_0 \tag{14}$$

has a unique maximal solution $x(t) = \omega(t,\ 0,\ x_0)$ for all $t \geq 0,$ and clearly (13) has a unique solution. Let us denote this solution by $y(t) = j(U(t))$ for $U(t) \in \Lambda,$ and define the function

$$j(z(t)) = j(\omega(t,\ 0,\ x_0 + h)) - j(\omega(t,\ 0,\ x_0)) - j(U(t)) \cdot h$$

for all $t \in [0,\ T],\ x_0 + h$ and $x_0 \in B(x_0,\ \varrho) \subset \Lambda,\ \varrho > 0,\ h \neq 0.$ Since $\frac{1}{\| h \|} j(z(t)) = j\left(\frac{z(t)}{\| h \|}\right),$ we have by (H2) (i),

$$D^+V\left(j\left(\frac{z(t)}{\| h \|}\right)\right) = \lim_{h' \to 0^+} \sup \frac{1}{h}\left(V\left(j\left(\frac{z(t)}{\| h \|}\right) + h'\right) - V\left(j\left(\frac{z(t)}{\| h \|}\right)\right)\right)$$

$$\leq M_+\left(j\left(\frac{z(t)}{\| h \|}\right),\ \left(\frac{z'(t)}{\| h \|}\right)\right).$$

We observe that

$$j\left(\frac{(t)}{\| h \|}\right) \leq \frac{j(f(\omega(t,\ 0,\ x_0 + h))) - j(f(\omega(t,\ 0,\ x_0)))}{\| h \|}$$

$$- j(f'(\omega(t,\ 0,\ x_0)))j(U(t))\frac{h}{\| h \|},$$

and from the Fréchet differentiability of $j \circ f$ and the definition of f, we have

$$j(f(\omega(t, 0, x_0 + h))) - j(f(\omega(t, 0, x_0)))$$

$$= j(c \odot g(j(\omega(t, 0, x_0 + h)))) - j(c \odot g(j(\omega(t, 0, x_0))))$$

$$= j(c \odot g'(j(\omega(t, 0, x_0))))[j(\omega(t, 0, x_0 + h)) - j(\omega(t, 0, x_0))]$$

$$+ O(\| j(\omega(t, 0, x_0 + h)) - j(\omega(t, 0, x_0)) \|).$$

By setting $\pi(h) = O(\| j(\omega(t, 0, x_0 + h)) - j(\omega(t, 0, x_0)) \|)$, and using the definition of $j(z(t))$ and (H1) (ii), we get

$$D^+V\left(j\left(\frac{z(t)}{\| h \|}\right)\right) \leq M_+\left(\left(j\frac{z(t)}{\| h \|}\right), j(f'(\omega(t, 0, x_0)))j\left(\frac{z(t)}{\| h \|}\right) + \frac{\pi(h)}{\| h \|}\right)$$

$$\leq M_+\left(j\left(\frac{z(t)}{\| h \|}\right), j(f'(\omega(t, 0, x_0)))j\left(\frac{z(t)}{\| h \|}\right)\right)$$

$$+ M_+\left(j\left(\frac{z(t)}{\| h \|}\right), \frac{\pi(h)}{\| h \|}\right). \tag{15}$$

Therefore, by using (H2) (iv), we can write

$$D^+V\left(j\left(\frac{z(t)}{\| h \|}\right)\right) \leq g_0\left(V\left(j\left(\frac{z(t)}{\| h \|}\right)\right)\right) + O(1), \tag{16}$$

since it can be shown that for every $t \in [0, T]$,

$$\lim_{\| h \| \to 0} M^+\left(j\left(\frac{z(t)}{\| h \|}\right), \frac{\pi(h)}{\| h \|}\right) = 0 \tag{17}$$

see [6] for more detail. Since $j(U(0)) = y(0) = I$ and $\omega(0, 0, x_0) = x_0$, we have $j(z(0)) = 0$. Thus, by (H1) (i), $V\left(j\left(\frac{z(0)}{\| h \|}\right)\right) = 0$. Therefore, using the Lemma 1.1 of [6], we get $\lim_{\| h \| \to 0} V\left(j\left(\frac{z(t)}{\| h \|}\right)\right) = 0$, and hence $\lim_{\| h \| \to 0} j\left(\frac{z(t)}{\| h \|}\right) = 0$. Thus, $j(\omega(t, 0, x_0))$ is Fréchet differentiable relative to x_0 and its Fréchet derivative is $\frac{\partial}{\partial x} j(\omega(t, 0, x_0)) = j(U(t))$.

Since $j(U(t))$ is invertible with continuous inverse, an j^{-1} is continuous, we get U^{-1} exists, is continuous and equal to $\left(\frac{\partial}{\partial x}\omega(t, 0, x_0)\right)^{-1}$. Thus $f = c \odot g$ satisfies (A3).

5. Acknowledgment

The author thanks the Department of Mathematics of the University of Notre Dame (IN. USA), where this work was initiated, for the hospitality and the grant.

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