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ON $(2, 3, t)$-GENERATIONS FOR THE JANKO GROUPS J_1 AND J_2

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Abstract

A group G is said to be (r, s, t)-generated if it is a quotient group of the triangle group $T(r, s, t) = \langle x, y, z \mid x^r = y^s = z^t = xyz = 1\rangle$. In Moori [18], the question of finding all possible (r, s, t)-generations for any non-abelian finite simple group was posed. In this article we partially answer this question for the first two Janko groups J_1 and J_2. We compute $(2, 3, t)$-generations for the first two Janko groups J_1 and J_2, where t is any divisor of $|J_i|$ for $i = 1, 2$.

1. Introduction and Preliminaries

Let G be a finite group. G is said to be $(2, 3, t)$-*generated* if it can be generated by two elements x and y such that $o(x) = 2, o(y) = 3$ and $o(xy) = t.$ Recently, there has been some reasonable interest in the $(2, 3, t)$-generations of sporadic simple groups. Ali and Ibrahim in [5, 6]

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investigated the $(2, 3, t)$-generations for the Held's sporadic simple group He and Tits simple group T. Ganief and Moori in [10, 19] established $(2, 3, t)$-generations and $(2, 3, p)$-generations for the sporadic simple groups J_3 and Fi_{22}. For more information regarding the study of $(2, 3, t)$-generations of sporadic simple groups as well as computational techniques, the reader is referred to [1-6, 10, 12, 18, 22].

In the present paper we compute $(2, 3, t)$-generations for the Janko's first two sporadic simple groups J_1 and J_2, where t is any divisor of $|J_i|$ for $i = 1, 2$. For basic properties of the Janko groups J_1 and J_2 and information on its subgroup structure the reader is referred to [14-17]. The $\mathbb{ATLAS}$ of Finite Groups [9] is an important reference and we adopt its notation for subgroups, conjugacy classes, etc. Computations were carried out with the aid of $\mathbb{GAP}$ [21].

In this article, we adopt the same notation as in [1], [2], [10] and [19]. In particular, $\Delta(G) = \Delta_G(C_1, C_2, C_3)$ denotes the structure constant of G for the conjugacy classes C_1, C_2 and C_3, whose value is the cardinality of the set $\Gamma = \{(x, y) | xy = z\}$, where $x \in C_1$, $y \in C_2$ and z is a fixed element of the conjugacy class C_3. It is well known that $\Delta_G(C_1, C_2, C_3)$ can easily be computed from the character table of G by the following formula:

$$\Delta_G(C_1, C_2, C_3) = \frac{|C_1| \cdot |C_2|}{|G|} \times \sum_{i=1}^{m} \frac{\chi_i(x)\chi_i(y)\overline{\chi_i(z)}}{\chi_i(1_G)},$$

where $\chi_1, \chi_2, \ldots, \chi_m$ are the irreducible complex characters of G (see [13, p. 45]). Also, $\Delta^*(G) = \Delta^*_G(C_1, C_2, C_3)$ denotes the number of distinct ordered pairs (x, y) with $x \in C_1$, $y \in C_2$, $xy = z$ and $G = \langle x, y \rangle$. Obviously G is (l, m, n)-generated group if and only if there exist conjugacy classes C_1, C_2, C_3 with representatives x, y, z, respectively, such that $o(x) = l$, $o(y) = m$ and $o(z) = n$, for which $\Delta^*_G(C_1, C_2, C_3) > 0$. In this case we say that G is (C_1, C_2, C_3)-generated. If H is a subgroup of

G containing z and B is a conjugacy class of H such that $z \in B$, then $\Sigma_H(C_1, C_2, B)$ denotes the number of distinct pairs (x, y) such that $x \in C_1$, $y \in C_2$, $xy = z$ and $\langle x, y \rangle$.

For the description of the conjugacy classes, the character tables, permutation characters and for information on the maximal subgroups readers are referred to $\mathbb{ATLAS}$ [9]. A general conjugacy class of elements of order n in G is denoted by nX, e.g., $2A$ represents the first conjugacy class of involutions in a group G and $2AB$ represents the conjugacy classes $2A$ and $2B$ of involutions in a group G. The following results in certain situations are very effective at establishing non-generations.

Theorem 1.1 (Ree [20]). *Suppose G is a group of permutations of a set Ω of size n, and G is generated by $x_1, x_2, \ldots, x_s$, with product $x_1 x_2 \cdots x_s = 1_G$. If the generator x_i has exactly c_i disjoint cycles on Ω (for $1 \le i \le s$) and G is transitive on Ω, then*

$$c_1 + c_2 + \cdots + c_s \le n(s-2) + 2.$$

Lemma 1.2 [8]. *Let G be a finite centerless group and suppose lX, mY, nZ are G-conjugacy classes for which $\Delta^*(G) = \Delta^*_G(lX, mY, nZ) < |C_G(z)|$, $z \in nZ$. Then $\Delta^*(G) = 0$ and therefore G is not (lX, mY, nZ)-generated.*

2. $(2, 3, t)$-generations of J_1

The first Janko group J_1 is a sporadic simple group of order $175560 = 2^3.3.5.7.11.19$ with 7 conjugacy classes of its maximal subgroups and 15 conjugacy classes of its elements. It has a just one conjugacy class of each element of orders 2 and 3, namely, $2A$ and $3A$, respectively. For basic properties of J_1 and computational techniques used in this article, the reader is referred to [1], [2] and [18].

In this section we investigate the $(2, 3, t)$-generations of the smallest Janko group J_1. If the group J_1 is $(2, 3, t)$-generated, then by Conder's result [7], $1/2 + 1/3 + 1/t < 1$. Thus we only need to consider

$t \in \{7, 10, 11, 15, 19\}$.

Lemma 2.1. *The Janko group* J_1 *is* $(2, 3, 7)$*-generated.*

Proof. Woldar [23] proved that J_1 is a Hurwitz group. That is, J_1 is $(2, 3, 7)$-generated group. □

Lemma 2.2. *The sporadic group* J_1 *is* $(2, 3, 10)$*-generated.*

Proof. Using the character table of J_1 we compute in $\mathbb{GAP}$ [21] that $\Delta_{J_1}(2A, 3A, 10AB) = 45$. The only maximal subgroups of J_1 having non-empty intersection with the classes $2A$, $3A$ and $10A$, up to isomorphism, are $2 \times A_5$ and $D_6 \times D_{10}$. We have $\Sigma_{2\times A_5}(2A, 3A, 10AB) = 5$ and $\Sigma_{D_6 \times D_{10}}(2A, 3A, 10AB) = 0$. A fixed element of order 10 in J_1 lies in a unique conjugate copy of each of $2 \times A_5$ and $D_6 \times D_{10}$. This implies that,

$$\begin{aligned}\Delta^*_{J_1}(2A, 3A, 10AB) &\geq \Delta_{J_1}(2A, 3A, 10AB) - \Sigma_{2\times A_5}(2A, 3A, 10AB)\\ &\quad - \Sigma_{D_6\times D_{10}}(2A, 3A, 10AB)\\ &\geq 45 - 5 - 0 > 0.\end{aligned}$$

Hence, J_1 is $(2A, 3A, 10A)$- and $(2A, 3A, 10B)$-generated. This completes the proof. □

Lemma 2.3. *The group* J_1 *is* $(2, 3, 11)$*-generated.*

Proof. This has been proved in Moori [18]. □

Lemma 2.4. *The Janko group* J_1 *is* $(2, 3, 15)$*-generated.*

Proof. Up to isomorphism, $D_6 \times D_{10}$ is the only maximal subgroup of J_1 having elements of order 15. But for this maximal subgroup we have $\Sigma_{D_6\times D_{10}}(2A, 3A, 15AB) = 0$. Since $\Delta_{J_1}(2A, 3A, 15AB) = 60$, the result follows. □

Lemma 2.5. *The group* J_1 *is* $(2, 3, 19)$*-generated.*

Proof. This has been proved in Moori [18] as Lemma 2.4. □

Next, we summarize our results in this section.

Theorem 2.6. *The Janko group* J_1 *is* $(2, 3, t)$*-generated for all* $t \in \{7, 10, 11, 15, 19\}$.

Proof. The proof follows from Lemmas 2.1-2.5. □

3. $(2, 3, t)$-generations of J_2

The Janko's second sporadic simple group J_2 has order 604800 $= 2^7 .\ 3^3 .\ 5^2$ with 21 conjugacy classes of elements. It has two classes of involutions namely, $2A$ and $2B$. The group J_2 acts as a transitive rank-3 group on a set X of 100 points. The point stabilizer of this action is isomorphic to $U_3(3)$ with orbits of lengths 1, 36 and 63. The permutation character of this action on the conjugates of $U_3(3)$ is given by $\chi_{U_3(3)} = \underline{1a} + \underline{36a} + \underline{63a}$, where $\underline{ma}$ is the first irreducible character of degree m in the character table of J_2. For basic properties of J_2 and other related Janko groups and the subgroup structure of their maximal subgroups, the reader is referred to [14-17] and [21].

In this section we investigate the $(2, 3, t)$-generations of the second Janko group J_2. It is well known that if the group J_2 is $(2, 3, t)$-generated, then $1/2 + 1/3 + 1/t < 1$. Thus we only need to consider $t \in \{7, 8, 10, 12, 15\}$.

Lemma 3.1. *The Janko group* J_2 *is* $(2X, 3Y, 7A)$*-generated, where* $X \in \{A, B\}$ *and* $Y \in \{A, B\}$, *if and only if* $X = Y = B$.

Proof. See [18]. □

Lemma 3.2. *The sporadic group* J_2 *is* $(2, 3, 8)$*-generated.*

Proof. The non-generation of J_2 by the triple $(2A, 3A, 8A)$ follows immediately since $\Delta_{J_2}(2A, 3A, 8A) = 0$. The group J_2 is also not $(2B, 3A, 8A)$-generated by Lemma 1.2 as $\Delta_{J_2}(2B, 3A, 8A) < |C_{J_2}(8A)|$.

Next we show that J_2 is not $(2A, 3B, 8A)$- and $(2B, 3B, 8A)$-generated. The group J_2 acts as a transitive rank-3 group on a set X of 100 elements. The point stabilizer of this action is isomorphic to $U_3(3)$ and the permutation character of this action on the conjugates of $U_3(3)$ is given by $\chi_{U_3(3)} = \underline{1a} + \underline{36a} + \underline{63a}$, where $\underline{ma}$ is the first irreducible character of degree m in the character table of J_2. This implies that, in the action of J_2 on the set X, the elements in the classes $2A$, $2B$, $3B$ and $8A$ have cycle types $1^{20}2^{40}$, 2^{50}, $1^4 3^{32}$ and $1^2 2^3 4^3 8^{10}$, respectively and we obtain that number of cycles of representatives in the classes $2A$, $2B$, $3B$ and $8A$ are $60, 50, 36$ and 18, respectively. Since, $60 + 36 + 18 > 102$ and $50 + 36 + 18 > 102$. Ree's transitivity condition Lemma 1.1 shows that $(2A, 3B, 8A)$ and $(2B, 3B, 8A)$ are not generating triples for J_2. This completes the proof. □

Lemma 3.3. *The group J_2 is $(2X, 3Y, 10Z)$-generated, where $X, Y \in \{A, B\}$ and $Z \in \{A, B, C, D\}$, if and only if $X = Y = B$ and $Z = C, D$.*

Proof. For the triples $(2A, 3A, 10A)$, $(2A, 3A, 10B)$, $(2A, 3B, 10A)$, $(2A, 3B, 10B)$, $(2B, 3A, 10A)$, $(2B, 3A, 10B)$, $(2A, 3A, 10C)$ and $(2A, 3A, 10D)$, non-generation follows immediately as the structure constant on J_2 of each triple is zero.

For the triple $(2B, 3A, 10C)$ and $(2B, 3A, 10D)$ we obtain that $\Delta_{J_2}(2B, 3A, 10CD) = 5 < |C_{J_2}(10CD)| = 10$. Thus $(2B, 3A, 10C)$ and $(2B, 3A, 10D)$ are not generating triples of J_2.

Next, consider the case $X = A$, $Y = B$ and $Z = C$ or D. Using the character table of the group J_2, we calculate $\Delta_{J_2}(2A, 3B, 10CD) = 10$. The only maximal subgroup of J_2 that has non-empty intersection with the classes in the triples $(2A, 3B, 10C)$ and $(2A, 3B, 10D)$ is, up to isomorphism, $H_1 \cong 5^2 : D_{12}$. We obtain that $\Sigma_{H_1}(2A, 3B, 10CD) = 10$.

Further, a fixed element of order 10 in J_2 lies in a unique conjugate of H_1. Since $\Delta^*_{J_2}(2A, 3B, 10CD) = \Delta_{J_2}(2A, 3B, 10CD) - \Sigma_{H_1}(2A, 3B, 10CD) = 10 - 1(10) = 0$. Therefore, $(2A, 3B, 10C)$ and $(2A, 3B, 10D)$ are not generating triples for J_2. Also, we can apply a similar method to show that J_2 is not $(2B, 3B, 10A)$- and $(2B, 3B, 10B)$-generated.

Finally, we consider the case $X = Y = B$ and $Z = C$ or D. As in the above case, the only maximal subgroup of J_2 which may contribute to the structure constants $\Delta_{J_2}(2B, 3B, 10CD) = 60$, up to isomorphism, is the group $H_1 \cong 5^2 : D_{12}$. However we compute that $\Sigma_{H_1}(2B, 3B, 10CD) = 0$. Hence $\Delta^*_{J_2}(2B, 3B, 10CD) = \Delta_{J_2}(2B, 3B, 10CD) = 60$, proving that $(2B, 3B, 10C)$ and $(2B, 3B, 10D)$ are generating triples of J_2. This completes the proof. □

Lemma 3.4. *The Janko group* J_2 *is* $(2X, 3Y, 12A)$*-generated, where* $X, Y \in \{A, B\}$ *if and only if* $X = Y = B$.

Proof. Since $\Delta_{J_2}(2B, 3A, 12A) = 0$, the group J_2 is not $(2B, 3A, 12A)$-generated. Consider the triples $(2A, 3A, 12A)$ and $(2A, 3B, 12A)$. By the algebra constants of the group J_2, we have $\Delta_{J_2}(2A, 3A, 12AB) = 3 < 12 = |C_{J_2}(12AB)|$. Hence, by Lemma 1.2, the Janko group J_2 is not $(2A, 3A, 12A)$- and $(2A, 3B, 12A)$-generated.

Finally, we consider the case $X = Y = B$. The maximal subgroups of J_2 that may contain $(2B, 3B, 12A)$-generated proper subgroups are isomorphic to $3.A_6.2_2$ and $2^{2+4}.3.S_3$. We calculate that

$$\Delta_{J_2}(2B, 3B, 12A) = 96,$$

$$\Sigma_{3.A_6.2_2}(2B, 3B, 12A) = 0 = \Sigma_{2^{2+4}.3.S_3}(2B, 3B, 12A).$$

Thus, $\Delta^*_{J_2}(2B, 3B, 12A) = \Delta_{J_2}(2B, 3B, 12A) = 96 > 0$, and so the group J_2 is $(2B, 3B, 12A)$-generated. This completes the proof. □

Lemma 3.5. *The Janko group* J_2 *is* $(2X, 3Y, 15Z)$*-generated, where* $X, Y, Z \in \{A, B\}$ *if and only if* $X = Y = B$.

Proof. Since $(15A)^{-1} = 15B$, the results obtained by replacing one of these classes with the other are the same. So, let $15Z$ denote the class $15A$ or $15B$.

First consider the case $Y = A$. The non-generation of J_2 by the triples $(2A, 3A, 15Z)$ and $(2B, 3A, 15Z)$ follows immediately since $\Delta_{J_2}(2A, 3A, 15Z) = 0 = \Delta_{J_2}(2B, 3A, 15Z)$. Next, since $\Delta_{J_2}(2A, 3B, 15Z) = 10 < 15 = |C_{J_2}(15Z)|$, by Lemma 1.2, the Janko group J_2 is not $(2A, 3B, 15Z)$-generated.

Finally, consider the case $X = Y = B$. We compute that $\Delta_{J_2}(2B, 3B, 15Z) = 75$. The maximal subgroups of J_2 that have non-empty intersection with the classes $2B$, $3B$ and $15Z$, up to isomorphism, are $H_2 \cong 3.A_6.2_2$ and $H_3 \cong A_4 \times A_5$. We calculate that $\Sigma_{H_2}(2B, 3B, 15Z) = 0$ and $\Sigma_{H_3}(2B, 3B, 15Z) = 15$. Further as an element of order 15 in J_2 is contained in two conjugates of H_3, we obtain

$$\begin{aligned}\Delta^*_{J_2}(2B, 3B, 15Z) &\geq \Delta_{J_2}(2B, 3B, 15Z) - \Sigma_{J_2}(2B, 3B, 15Z)\\ &\quad - 2\Sigma_{J_3}(2B, 3B, 15Z)\\ &\geq 75 - 2(15) > 0,\end{aligned}$$

proving that $(2B, 3B, 15Z)$ is a generating triple of J_2. This completes the proof. □

We now summarize the above results of this section in the following theorem.

Theorem 3.6. *The Janko group* J_2 *is* $(2X, 3Y, tZ)$*-generated if and only if* $X = Y = B$ *and* $tZ \in \{10CD, 12A, 15AB\}$.

Proof. This follows from Lemmas 3.1-3.5. □

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