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JOINT SPECTRUM OF SEVERAL MATRICES

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Abstract

The joint spectrum $\sigma_J(\mathbf{C})$ of a set of matrices $\mathbf{C} = \{C_1, ..., C_m\}$ is defined. When the matrices in $\mathbf{C}$ can be upper-triangularized by the same similarity transformation, the fan spectrum, $\sigma_F(\mathbf{C})$, is defined and it is shown that $\sigma_J(\mathbf{C}) \subseteq \sigma_F(\mathbf{C})$. If the matrices in $\mathbf{C}$ commute amongst themselves, then $\sigma_J(\mathbf{C}) \subseteq \sigma_F(\mathbf{C})$. A generalization of the fact that an $n \times n$ matrix with n-distinct eigenvalues is diagonalizable is also established.

1. Introduction

Let $T : \mathbb{C}^n \to \mathbb{C}^n$ be a linear transformation. A set $V = \{v_1, ..., v_n\}$ belonging to $\mathbb{C}^n$ is a *fan basis* for T if, for each i in $N(n) = \{1, 2, ..., n\}$,

$$span(v_1, ..., v_i) = \left\{ \sum_{j=1}^{i} \tau_j v_j \mid \tau_j \in \mathbb{C} \right\}$$

is an invariant subspace of T [5, p. 257]. Suppose $\mathbf{C} = \{C_1, ..., C_m\}$ is a set of matrices contained in $\mathbb{C}^{n \times n}$. We say $V = \{v_1, ..., v_n\}$ is a *common fan*

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basis for **C** provided V is a fan basis for each C_i in **C**. This definition is of course equivalent to the statement that each member of **C** can be upper-triangularized by the same similarity transformation. When **C** has a common fan basis, we define the fan spectrum of **C**, $\sigma_F(\mathbf{C})$, to be the set of m-dimensional vectors consisting of the corresponding diagonal elements of the upper-triangularized matrices. See [4] where a joint spectrum for several noncommuting linear operators is defined. The definition of the joint spectrum in [4] is much more involved even in the finite dimensional case and is developed for a functional calculus of noncommuting linear operators. There are also other well-known definitions of joint spectra of several commuting linear operators. See [6]. Our definition of fan spectrum for several matrices with common fan basis appears to be new.

The definition of fan spectrum, though defined in terms of a fixed basis, is, as we show, independent of the fan basis. The proof follows from the fact if a single matrix T is similar to an upper-triangular matrix, then the diagonals of this upper-triangular matrix are the eigenvalues of T and, thus is independent of the similarity transformation.

For a set of matrices $\mathbf{C} = \{C_1, ..., C_m\}$ in $\mathbb{C}^{n\times n}$, we define the *joint spectrum* of **C**, $\sigma_J(\mathbf{C})$, in terms of common eigenvectors of **C**. The vector $(\lambda_1, ..., \lambda_m)$ is in $\sigma_J(\mathbf{C})$, if there exists a non zero vector v in $\mathbb{C}^n$ such that $C_i v = \lambda_i v$ for each i [6]. In this note we show that if **C** has a common fan basis, then we show that $\sigma_J(\mathbf{C}) \subseteq \sigma_F(\mathbf{C})$. We also show that if the matrices in **C** commute amongst themselves, then $\sigma_J(\mathbf{C}) = \sigma_F(\mathbf{C})$. Last of all we show that if $\sigma_J(\mathbf{C})$ consists of n-distinct vectors, then each matrix in **C** is diagonalizable by the same similarity transformation. This provides a generalization of the fact that an $n \times n$ matrix with n-distinct eigenvalues is diagonalizable.

2. Fan Spectrum

Let $T : \mathbb{C}^n \to \mathbb{C}^n$ be a linear transformation. We recall that the set $V = \{v_1, ..., v_n\}$ contained in $\mathbb{C}^n$ is a fan basis for T if, for each i in $N(n)$,

$$span(v_1, \ldots, v_i) = \left\{ \sum_{j=1}^{i} \tau_j v_j \mid \tau_j \in \mathbb{C} \right\}$$

is an invariant subspace of T. The matrix representation of T relative to V, $[T]_V$, takes the form

$$[T]_V = \begin{bmatrix} t_{11} & \cdots & t_{1n} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & t_{nn} \end{bmatrix}.$$

Since the matrix is upper-triangular, the fan spectrum of T is defined to be the diagonal entries of $[T]_V$, $\{t_{11}, \ldots, t_{nn}\}$, which is precisely the set of eigenvalues of T.

Let $\mathbf{A} = \{A_1, \ldots, A_m\}$ be a set of matrices contained in $\mathbb{C}^{n \times n}$. We now state the definition of the fan spectrum of $\mathbf{A}$ which we denote by $\sigma_F(\mathbf{A})$. Suppose that $\mathbf{A}$ has a common fan basis $U = \{u_1, \ldots, u_n\}$. For i in $N(m)$, set

$$A_i(u_j) = \sum_{l=1}^{j} a(i, j, l) u_l.$$

Define vectors $a(j)$ in C^m by

$$a(j) = [a(1, j, j), \ldots, a(m, j, j)]^T.$$

Finally, define the fan spectrum of $\mathbf{A}$, $\sigma_F(\mathbf{A})$ by

$$\sigma_F(\mathbf{A}) = \{a(1), \ldots, a(n)\}.$$

Example 1. If

$$A_1 = \begin{bmatrix} 36 & 21 & 48 \\ -3 & 6 & -6 \\ -18 & -9 & -24 \end{bmatrix} \text{ and } A_2 = \begin{bmatrix} -18 & -19 & -22 \\ 1 & -2 & 2 \\ 9 & 9 & 11 \end{bmatrix}.$$

We show that $\mathbf{A} = \{A_1, A_2\}$ has a common fan basis and we determine $\sigma_F(\mathbf{A})$. Set

$$S = \begin{bmatrix} -5 & 4 & 2 \\ 1 & -1 & 0 \\ 3 & -2 & -1 \end{bmatrix}.$$

Then

$$S^{-1}A_1S = \begin{bmatrix} 1 & 0 & 2 \\ 1 & -1 & 2 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 36 & 21 & 48 \\ -3 & 6 & -6 \\ -18 & -9 & -24 \end{bmatrix} \begin{bmatrix} -5 & 4 & 2 \\ 1 & -1 & 0 \\ 3 & -2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -3 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 12 \end{bmatrix},$$

$$S^{-1}A_2S = \begin{bmatrix} 1 & 0 & 2 \\ 1 & -1 & 2 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} -18 & -19 & -22 \\ 1 & -2 & 2 \\ 9 & 9 & 11 \end{bmatrix} \begin{bmatrix} -5 & 4 & 2 \\ 1 & -1 & 0 \\ 3 & -2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -7 \end{bmatrix}.$$

Thus $\sigma(A_1) = \{12, 3, 3\}$, $\sigma(A_2) = \{-7, -1, -1\}$ and

$$\sigma_F(\mathbf{A}) = \left\{ \begin{bmatrix} 12 \\ -7 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \end{bmatrix} \right\}.$$

We now show that the definition of the fan spectrum is independent of the fan basis. To be precise, suppose that $\mathbf{A}$ has two common fan bases $V = \{v_1, \dots, v_n\}$ and $W = \{w_1, \dots, w_n\}$. For i in $N(m)$ and j in $N(n)$, set

$$A_i(v_j) = \sum_{l=1}^{j} a(i, j, l)v_l \text{ and } A_i(w_j) = \sum_{l=1}^{j} p(i, j, l)w_l.$$

Define vectors $a(j)$ and $p(j)$ in C^m by

$$a(j) = [a(1, j, j), \dots, a(m, j, j)]^T$$

and

$$p(j) = [p(1, j, j), \dots, p(m, j, j)]^T \in \mathbb{C}^m.$$

The fan spectrum of $\mathbf{A}$ with respect to V, $\sigma_{F,V}(\mathbf{A})$, and the fan spectrum of $\mathbf{A}$ with respect to W, $\sigma_{F,W}(\mathbf{A})$, are given by

$$\sigma_{F,V}(\mathbf{A}) = \{a(1), \ldots, a(n)\} \text{ and } \sigma_{F,W}(\mathbf{A}) = \{p(1), \ldots, p(n)\}.$$

We will show that $\sigma_{F,V}(\mathbf{A}) = \sigma_{F,W}(\mathbf{A})$. Consider the linear transformation $B : \mathbb{C}^n \to \mathbb{C}^n$ defined by

$$B = \sum_{i=1}^{m} \alpha_i A_i, \text{ where } \alpha = [\alpha_1, \ldots, \alpha_m] \in \mathbb{C}^m.$$

It is clear that

$$B(v_j) = \sum_{i=1}^{m} \alpha_i A_i(v_j) = \sum_{i=1}^{m} \sum_{l=1}^{j} \alpha_i a(i, j, l) v_l = \sum_{l=1}^{j} \left[\sum_{i=1}^{m} \alpha_i a(i, j, l) \right] v_l,$$

$$B(w_j) = \sum_{i=1}^{m} \alpha_i A_i(w_j) = \sum_{i=1}^{m} \sum_{l=1}^{j} \alpha_i p(i, j, l) w_l = \sum_{l=1}^{j} \left[\sum_{i=1}^{m} \alpha_i p(i, j, l) \right] w_l,$$

that is, both $V = \{v_1, \ldots, v_n\}$ and $W = \{w_1, \ldots, w_n\}$ are fan bases of B and the set

$$\left\{ \left[\sum_{i=1}^{m} \alpha_i a(i, j, j) \right], \, j = 1, \ldots, m \right\} = \{a(j) \cdot \alpha, \, j = 1, \ldots, m\}$$

and the set

$$\left\{ \left[\sum_{i=1}^{m} \alpha_i p(i, j, j) \right], \, j = 1, \ldots, m \right\} = \{p(j) \cdot \alpha, \, j = 1, \ldots, m\}$$

are both the set of the eigenvalues of B. Thus they are equal for any α. It is easy to see that this can only happen when

$$\{a(1), \ldots, a(n)\} = \{p(1), \ldots, p(n)\}.$$

Therefore $\sigma_{F,V}(\mathbf{A}) = \sigma_{F,W}(\mathbf{A})$.

3. Commuting Matrices and the Joint Spectrum

Suppose that $\mathbf{A} = \{A_1, ..., A_m\}$ is a set of matrices. We define $\sigma_J(\mathbf{A})$, the joint spectrum of **A**. The vector $[\lambda_1, ..., \lambda_m]^T$ is in $\sigma_J(\mathbf{A})$ if and only if there exists a vector $v \in C^m$ such that

$$A_i v = \lambda_i v \text{ for } i \text{ in } N(m).$$

In other words, v is an eigenvector for each A_i, and so will be referred to as a common eigenvector (of **A**). It is clear that in general $\sigma_J(\mathbf{A})$ can be empty. But if **A** has a common fan basis, we will show that $\sigma_J(\mathbf{A})$ is not empty and $\sigma_J(\mathbf{A})$ is a subset of $\sigma_F(\mathbf{A})$.

The definition of the fan spectrum of $\mathbf{A} = \{A_1, ..., A_m\}$ is an extrinsic definition in that it is defined in terms of a particular basis for the matrices in **A**. The joint spectrum, which does not rely on a particular basis, is an intrinsic definition. We say that $\mathbf{A} = \{A_1, ..., A_m\}$ is *commuting* if for i, j in $N(m)$, $A_i A_j = A_j A_i$. It is well known that commuting matrices have a common fan basis [7, p. 2]. For a commuting set of matrices we will prove that $\sigma_J(\mathbf{A}) = \sigma_F(\mathbf{A})$.

Lemma 1. *Suppose that* $\mathbf{A} = \{A_1, ..., A_m\}$ *belongs to* $\mathbb{C}^{n \times n}$ *and has a common fan basis. Then*

$$\sigma_J(\mathbf{A}) \textit{ is a subset of } \sigma_F(\mathbf{A}).$$

Proof. Let $\{v_1, ..., v_n\}$ be a common fan basis for **A**. For i in $N(m)$ and j in $N(n)$, set

$$A_i(v_j) = \sum_{l=1}^{j} a(i, j, l) v_l. \tag{3.1}$$

Let

$$V_k = Span(v_1, ..., v_k)$$

and set

$$\mathbf{A} | V_k = \{A_1 | V_k, ..., A_m | V_k\}.$$

We establish by induction that for k in $N(n)$, $\sigma_J(\mathbf{A} | V_k) \subseteq \sigma_F(\mathbf{A} | V_k)$.

The case $k = 1$, is obvious. So assume the result is true for the $k-1$ case. The fan spectrum $\sigma_F(\mathbf{A} | V_k)$ is given by

$$\sigma_F(\mathbf{A} | V_k) = \{a(1), \ldots, a(k)\},$$

where

$$a(j) = [a(1, j, j), \ldots, a(m, j, j)]^T \text{ is a vector in } \mathbb{C}^m.$$

Let $(\lambda_1, \ldots, \lambda_m)^T$ be a vector in $\sigma_J(\mathbf{A} | V_k)$ with common eigenvector v. If $v \in V_{k-1}$, then

$$(\lambda_1, \ldots, \lambda_m)^T \in \sigma_J(\mathbf{A} | V_{k-1}).$$

So by the induction assumption, $(\lambda_1, \ldots, \lambda_m)^T$ belongs to $\sigma_F(\mathbf{A} | V_{k-1})$ and hence belongs to $\sigma_F(\mathbf{A} | V_k)$.

Now assume that v is not in V_{k-1}. Then v takes the form

$$v = \sum_{l=1}^{k} c_l v_l, \text{ and } c_k \neq 0.$$

Let i be in $N(m)$. On the one hand,

$$A_i(v) = \lambda_i \sum_{j=1}^{k} c_j v_j = \lambda_i \sum_{j=1}^{k-1} c_j v_j + \lambda_i c_k v_k, \tag{3.2}$$

and on the other hand, by (3.1),

$$\begin{aligned} A_i(v) &= \sum_{j=1}^{k} c_j A_i v_j = \sum_{j=1}^{k-1} c_j A_i v_j + c_k A_i v_k \\ &= \sum_{j=1}^{k-1} c_j \sum_{l=1}^{j} a(i, j, l) v_l + c_k \sum_{l=1}^{k} a(i, k, l) v_l \\ &= \sum_{j=1}^{k-1} \sum_{l=1}^{j} c_j a(i, j, l) v_l + \sum_{l=1}^{k-1} c_k a(i, k, l) v_l + c_k a(i, k, k) v_k. \end{aligned} \tag{3.3}$$

Equating the v_k coefficients in (3.2) and (3.3), we have

$$\lambda_i c_k = c_k a(i, k, k).$$

Since $c_k \neq 0$, it follows that $\lambda_i = a(i, k, k)$. Therefore $(\lambda_1, ..., \lambda_m)$ must equal $a(k)$ which belongs to $\sigma_F(\mathbf{A} | V_k)$. □

In the lemmas that follow, generalized eigenspaces will play a significant role. If T is a matrix and t is an eigenvalue of T, then (T, t) will denote the generalized eigenspace of T corresponding to the eigenvalue t and $m(T, t)$ will denote the algebraic multiplicity of the eigenvalue t. The dimension of a generalized eigenspace (T, t) equals the multiplicity $m(T, t)$ of the eigenvalue t, [2, p. 312], an important fact that we put to use in the next lemma.

Lemma 2. *Let* $V = \{v_1, ..., v_n\}$ *contained in* $\mathbb{C}^n$ *be a fan basis for a linear transformation* $T : \mathbb{C}^n \to \mathbb{C}^n$. *Assume*

$$T(v_i) = \sum_{j=1}^{i} t_{ij} v_j \text{ with } t_{ij} \in \mathbb{C}.$$

Let v *be a generalized eigenvector. If* v *does not belong to* $V' =$ *span*$(v_1, ..., v_{n-1})$, *then* v *is in* (T, t_{nn}), *the generalized eigenspace of* T *corresponding to the eigenvalue* t_{nn}.

Proof. Suppose that v is in (T, t_{ii}), the generalized eigenspace of T corresponding to the eigenvalue t_{ii}. Since v is not in V' and V' is an invariant subspace of T,

$$\dim(T | V', t_{ii}) + 1 = \dim(T, t_{ii}).$$

So

$$m(T | V', t_{ii}) + 1 = m(T, t_{ii}).$$

It is clear, from the matrix representation of T, that $i = n$. Thus $t_{ii} = t_{nn}$. □

Lemma 3. *Suppose* $\mathbf{A} = \{A_1, ..., A_m\}$ *is a set of commuting matrices belonging to* $\mathbb{C}^{n\times n}$. *Let* $\{v_1, ..., v_n\}$ *be a common fan basis for* $\mathbf{A}$. *For i in* $N(m)$ *and j in* $N(n)$, *set*

$$A_i(v_j) = \sum_{l=1}^{j} a(i, j, l)v_l.$$

Then

$$\{0\} \neq \cap_{i=1}^{m}(A_i, a(i, n, n)) \subsetneq V' = span(v_1, ..., v_{n-1}).$$

Proof. We argue by induction on m, the number of matrices in $\mathbf{A}$. For $m = 1$, it is immediate that $\{0\} \neq (A_1, a(1, n, n))$. For the second part, observe that $\mathbb{C}^n$ is a direct sum of the generalized eigenspaces of A_1. So there is a generalized eigenvector v not in V'. By Lemma 2, v is a vector in $(A_1, a(1, n, n))$, so $(A_1, a(1, n, n))$ is not contained in V'. Now we consider the case when there are m linear transformations A_i, i in $N(m)$. Set

$$J = \cap_{i=1}^{m-1}(A_i, a(i, n, n)).$$

By the induction assumption, we have $\{0\} \neq J$ and J is not a subset of V'. Since $\mathbf{A}$ is commutative, A_m restricted to J, maps J into itself, $A_m | J : J \to J$. The vector space J is the direct sum of the generalized spaces of $A_m | J$. Since J is not a subset of V', there is a generalize eigenvector v of $A_m | J$ not in V'. But the v is a generalized eigenvector of A_m. By Lemma 2, v is in $(A_m, a(m, n, n))$. So

$$\{0\} \neq \cap_{i=1}^{m}(A_i, a(i, n, n)) \subsetneq V'.$$

This completes the proof. □

The following corollary is an immediate result of the above lemma.

Corollary 1. *Suppose* $\mathbf{A} = \{A_1, ..., A_m\}$ *is a set of commuting matrices belonging to* $\mathbb{C}^{n\times n}$. *Then* $\mathbf{A}$ *has a common basis of generalized eigenvectors.*

Lemma 4. *Suppose* $\mathbf{A} = \{A_1, ..., A_m\}$ *is a set of commuting matrices belonging to* $\mathbb{C}^{n\times n}$. *For* i *in* $N(m)$, *let* λ_i *be an eigenvalue of* A_i. *If* $\bigcap_{i=1}^{m}(A_i, \lambda_i) \neq \{0\}$, *then the linear transformations* $\mathbf{A}$ *have a common eigenvector, i.e., there exists a* v *in* $\bigcap_{i=1}^{m}(A_i, \lambda_i)$ *such that* $A_i v = \lambda_i v$ *for* i *in* $N(m)$.

Proof. Let $W = \bigcap_{i=1}^{m}(A_i, \lambda_i)$. Let E_i be the eigenspace of A_i corresponding to λ_i and let $F_i = W \cap E_i$. Since $\mathbf{A}$ is commutative, for each i and j in $N(m)$, F_j is an invariant subspace of A_i. We need to show that $\bigcap_{i=1}^{m} F_i \neq \{0\}$. We prove this by induction on m. The case $m = 1$ is obvious. So assume that $J = \bigcap_{i=1}^{m-1} F_i \neq \{0\}$. It follows that J is invariant for A_m. So A_m has an eigenvector v in J. It is clear that v belongs to E_m since v is an eigenvector, v is in W and W is a subset of (A_m, λ_m). Therefore v belongs to $\bigcap_{i=1}^{m} F_i$. □

Theorem 1. *Suppose* $\mathbf{A} = \{A_1, ..., A_m\}$ *is a set of commuting matrices belonging to* $\mathbb{C}^{n\times n}$. *Then*

$$\sigma_J(\mathbf{A}) = \sigma_F(\mathbf{A}).$$

Proof. By Lemma 1, $\sigma_J(\mathbf{A})$ is contained in $\sigma_F(\mathbf{A})$. We need to prove $\sigma_F(\mathbf{A})$ is contained in $\sigma_J(\mathbf{A})$. Let $\{v_1, ..., v_n\}$ be a common fan basis for $\mathbf{A}$. With i in $N(m)$ and j in $N(n)$, set

$$A_i(v_j) = \sum_{l=1}^{j} a(i, j, l) v_l.$$

Then the fan spectrum of $\mathbf{A}$, $\sigma_F(\mathbf{A})$, is

$$\sigma_F(\mathbf{A}) = \{a(1), ..., a(n)\},$$

where

$$a(j) = [a(1, j, j), ..., a(m, j, j)]^T \text{ is a vector in } \mathbb{C}^m.$$

Pick $a(j)$ in $\sigma_F(\mathbf{A})$ and set $V_j = \{v_1, \ldots, v_j\}$. By Lemma 3, $\bigcap_{i=1}^{m}(A_i|V_j, a(i, j, j)) \neq \{0\}$. By Lemma 4, there is a v in $\bigcap_{i=1}^{m}(A_i|V_j, a(j, i, i))$ such that for i in $N(m)$, $A_i v = a(i, j, j)v$. Therefore $a(j)$ belongs to $\sigma_J(\mathbf{A})$. □

The fan spectrum need not be the same as the joint spectrum as the next example illustrates.

Example 2. Let $\mathbf{A} = \{A_1, A_2\}$, where

$$A_1 = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix}.$$

Calculating the joint spectrum, we find

$$\sigma_J(\mathbf{A}) = \left\{ \begin{pmatrix} 2 \\ 3 \end{pmatrix} \right\}.$$

Let

$$P = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}.$$

Then

$$P^{-1}A_1P = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}, \quad P^{-1}A_2P = \begin{bmatrix} 2 & -2 \\ 0 & 3 \end{bmatrix}.$$

So

$$\sigma_F(\mathbf{A}) = \left\{ \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix} \right\}.$$

It is easy to see that if $\mathbf{A} = \{A_1, \ldots, A_m\} \subset \mathbb{C}^{2\times 2}$ and $\sigma_J(\mathbf{A}) \neq \phi$, then the matrices in $\mathbf{A}$ have a common fan basis. Next we provide an example of two matrices $\mathbf{A} = \{A_1, A_2\} \subset \mathbb{C}^{3\times 3}$, where $\sigma_J(\mathbf{A}) \neq \phi$, but the matrices in $\mathbf{A}$ do not have a common fan basis.

Example 3. Let $\mathbf{A} = \{A_1, A_2\} \subset \mathbb{C}^{3\times 3}$, where

$$A_1 = \begin{bmatrix} 1 & -4 & 0 \\ 1 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } A_2 = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

A straightforward computation shows that

$$\begin{bmatrix} 1 & -4 \\ 1 & -3 \end{bmatrix} \text{ and } \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$$

do not have a common fan basis. It follows then from the block structure of A_1 and A_2 that A_1 and A_2 do not have a common fan basis. Let $v = [0 \quad 0 \quad 1]^T$. Then

$$A_1 v = v \text{ and } A_2 v = 2v,$$

so $(1 \quad 2)^T \in \sigma_J(\mathbf{A})$. In fact $\sigma_J(\mathbf{A}) = \{(1 \quad 2)^T\}$.

If $\mathbf{A} = \{A_1, \ldots, A_m\}$ is a set of commuting matrices in $\mathbb{C}^{n\times n}$, then, as a consequence of Theorem 1, every eigenvalue of every matrix A_i is a coordinate in at least one vector in $\sigma_J(\mathbf{A})$.

If an $n \times n$ matrix T has n distinct eigenvalues, then it is diagonalizable. For a set of commuting matrices $\mathbf{A} = \{A_1, \ldots, A_m\}$, there is a similar result; to establish it we find it helpful to introduce the following notation.

We associate with m vectors $w_1, \ldots, w_m$ in $\mathbb{C}^n$ an $n \times m$ matrix $[w_1, \ldots, w_m]$, such that, as suggested by the notation, the (i, j) entry of the matrix is the i-th coordinate of w_j. For i in $N(m)$, let λ_i be a complex number. For a vector w in $\mathbb{C}^n$, we write

$$(\lambda_1, \ldots, \lambda_m)w = [\lambda_1 w, \ldots, \lambda_m w].$$

For a set of matrices $\mathbf{A} = \{A_1, \ldots, A_m\}$ belonging to $\mathbb{C}^{n\times n}$ and a vector v in $\mathbb{C}^n$, we write

$$\mathbf{A}v = [A_1 v, \ldots, A_m v].$$

Proposition 1. *Let* $\mathbf{A} = \{A_1, \ldots, A_m\}$ *be a set of matrices in* $\mathbb{C}^{n\times n}$. *If the joint spectrum of* $\mathbf{A}$, $\sigma_J(\mathbf{A})$, *consists of n distinct vectors, then every matrix in* $\mathbf{A}$ *is diagonalizable by the same similarity matrix.*

Proof. Set

$$\sigma_J(\mathbf{A}) = \{a(1), \ldots, a(n)\},$$

where

$$a(i) = (a(i, 1), \ldots, a(i, m)),$$

and let v_i be a common eigenvector corresponding to $a(i)$. We prove by induction that $V = \{v_1, \ldots, v_n\}$ is a linearly independent set. For k in $N(n-1)$, we assume that $\{v_1, \ldots, v_k\}$ is a linearly independent set and prove, from this assumption, that $\{v_1, \ldots, v_k, v_{k+1}\}$ is too. To the contrary, assume that v_{k+1} can be written in the form

$$v_{k+1} = \sum_{i=1}^{k} c_i v_i. \tag{3.4}$$

Then

$$\mathbf{A}v_{k+1} = \sum_{i=1}^{k} c_i \mathbf{A} v_i.$$

So

$$a(k+1)v_{k+1} = \sum_{i=1}^{k} c_i a(i) v_i. \tag{3.5}$$

It follows from (3.4) that

$$a(k+1)v_{k+1} = \sum_{i=1}^{k} c_i a(k+1) v_i. \tag{3.6}$$

Subtracting (3.5) from (3.6), we have that

$$\sum_{i=1}^{k} c_i (a(k+1) - a(i)) v_i = 0.$$

Choose i in $N(k)$. We claim that $c_i = 0$. Since $a(k+1) - a(i)$ is not zero, there is a coordinate of this vector, say $a(k+1, l) - a(i, l)$, not equal to zero. Consider

$$\sum_{i'=1}^{k} c_{i'}(a(k+1, l) - a(i, l))v_{i'} = 0.$$

Since $\{v_1, ..., v_k\}$ is a linearly independent set and $a(k+1) - a(i)$ is not zero, we have that c_i is zero. Therefore, for i in $N(k)$, c_i is zero and this contradicts the fact that v_{k+1} is a common eigenvector.

It follows that $V = \{v_1, ..., v_n\}$ is a linearly independent set. Consequently, every matrix in **A** has a basis of eigenvectors and is, therefore, diagonalizable. □

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