## A COUNTEREXAMPLE CONCERNING

## A SERIES OF T. L. LAI

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#### Abstract

We prove that the i.i.d. hypothesis in Lai's theorem cannot be relaxed to bounded sequences, not even to bounded martingales.


Let us recall the following classical result (cf. Lai [1]). Let $X, X_{1}, X_{2}, \ldots$ be i.i.d. random variables with $E X=0, E X^{2}=\sigma^{2}$, and partial sums $S_{n}=X_{1}+\cdots+X_{n}$. Let $p>1$, and assume that $E\left[|X|^{2 p}\left(\log ^{+}|X|\right)^{-p}\right]<\infty$. Then, for any $\varepsilon>\sigma \sqrt{2 p-2}$,

$$
\sum_{n=2}^{\infty} n^{p-2} P\left[\left|S_{n}\right|>\varepsilon(n \log n)^{1 / 2}\right]<\infty
$$

Conversely, if the sum is finite for some $\varepsilon>0$, then

$$
E\left[|X|^{2 p}\left(\log ^{+}|X|\right)^{-p}\right]<\infty
$$

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Our main result below shows that Lai's theorem cannot be extended to bounded sequences, not even to bounded martingales. This is in sharp contrast with the limiting case of Lai's theorem (i.e., $p=1$ ); the i.i.d. case was proved in [2] assuming more than a second moment and in [3]-[4] we proved that the corresponding series converges for any $L^{q}$-martingale difference and along a subsequence of any $L^{q}$ - bounded sequence, for $q>2$.

Throughout the paper, $C>0$ denotes a generic numerical constant. Let $p \geq 1$; we say that the sequence $\left(X_{n}\right)_{n \geq 1}$ is $L^{p}$ - bounded if it has finite $p$ th moments, i.e., $\left\|X_{n}\right\|_{p} \leq C$ for some $C>0$ and any $n \geq 1$. We say that $\left(X_{n}\right)_{n \geq 1}$ is a martingale difference if $E\left[S_{n} \mid \mathcal{F}_{n-1}\right]=S_{n-1}$ a.s. for $n \geq 1$, where $\mathcal{F}_{n}$ is the $\sigma$-algebra generated by $X_{1}, \ldots, X_{n}$ (here $S_{0}=0$ and $\mathcal{F}_{0}$ is the trivial $\sigma$-algebra).

Theorem. Let $p>1$. Then there exists a martingale difference $\left(X_{n}\right)_{n \geq 1}$ that satisfies

$$
\begin{equation*}
\sup _{n \geq 1} E\left[\left|X_{n}\right|^{2 p}\left(\log ^{+}\left|X_{n}\right|\right)^{-p}\right]<\infty \tag{1}
\end{equation*}
$$

(in particular $\left(X_{n}\right)_{n \geq 1}$ is $L^{p}$ - bounded), and such that the series

$$
\begin{equation*}
\sum_{N=2}^{\infty} N^{p-2} P\left[\left|\sum_{k=1}^{N} X_{n_{k}}\right|>\varepsilon(N \log N)^{1 / 2}\right] \tag{2}
\end{equation*}
$$

diverges for any subsequence $1 \leq n_{1}<n_{2}<\cdots$ of natural numbers and $\varepsilon>0$.

Proof. Consider $X_{n}=Z \cdot Y_{n}$, where $\left(Y_{n}\right)_{n \geq 1}$ is an i.i.d. bounded sequence with $E\left(Y_{1}\right)=0$, and $Z$ is independent of $\left(Y_{n}\right)_{n \geq 1}$ with

$$
\begin{equation*}
P[|Z|>n]=C n^{-\gamma} \tag{3}
\end{equation*}
$$

for $n \geq 1$. Here $C>0$ is a normalization factor and $\gamma>2 p$.

Obviously $\left(X_{n}\right)_{n \geq 1}$ is a martingale difference and has finite $(2 p)$ th moments, as

$$
\begin{align*}
E\left|X_{n}\right|^{2 p} \leq C E|Z|^{2 p} & \leq C \sum_{n=1}^{\infty} n^{2 p}(P[|Z|>n]-P[|Z|>n+1]) \\
& \sim \sum_{n=1}^{\infty} n^{2 p-\gamma-1}<\infty \tag{4}
\end{align*}
$$

As the sequence $\left(Y_{n}\right)_{n \geq 1}$ is bounded, we have

$$
\left(\log ^{+}\left|X_{n}\right|\right)^{-p} \leq\left(\log ^{+}|Z|+C\right)^{-p} \leq C \quad \text { a.s. }
$$

and, together with (4), implies Condition (1).
As $Z$ is independent of $\left(Y_{n}\right)_{n \geq 1}$, and applying the central limit theorem to $\left(Y_{n}\right)_{n \geq 1}$, we obtain

$$
\begin{aligned}
& P\left[\left|\sum_{k=1}^{N} X_{n_{k}}\right|>\varepsilon(N \log N)^{1 / 2}\right] \\
\geq & P\left[\left|\sum_{k=1}^{N} Y_{n_{k}}\right|>N^{1 / 2}\right] \cdot P\left[|Z|>\varepsilon(\log N)^{1 / 2}\right] \\
\geq & C \cdot P\left[|Z|>\varepsilon(\log N)^{1 / 2}\right]
\end{aligned}
$$

Hence, applying (3) to (5), we obtain that series (2) is greater than or equal to

$$
C \cdot \sum_{N=2}^{\infty} N^{p-2}(\log N)^{-\gamma / 2}
$$

which diverges if $p>1$.

## References

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