Far East J. Math. Sci. (FJMS) **20**(2) (2006), 143-149

ON TOTALLY PARANORMAL OPERATORS

JONG-KWANG YOO

(Received November 8, 2005)

Submitted by K. K. Azad

Abstract

If $T \in L(X)$ is a totally paranormal operator on a complex Banach space X, then for all $\lambda \in \mathbb{C}$,

$$X_T(\{\lambda\}) = Ker(T - \lambda) = \{x \in X : \lim_{n\to\infty} \| (T - \lambda)^n x \|^{\frac{1}{n}} = 0\}.$$

In particular, the kernel $KerT$ is the quasi-nilpotent part of T. If λ and μ are distinct eigenvalues of T, then

$$X^* = X_T(\{\lambda\})^{\perp} + X_T(\{\mu\})^{\perp}.$$

Moreover, if $T \in L(H)$ is a totally paranormal operator on a complex Hilbert space H, then $H_T(\{\lambda\}) \perp H_T(\{\mu\})$.

We first recall some basic notions and results from spectral theory; the monographs [4] and [10]. Let X be a complex Banach space, and let $L(X)$ denote the Banach algebra of all bounded linear operators on X. As usual, for $T \in L(X)$ and let $\sigma(T), \sigma_p(T)$ and $\sigma_{sur}(T)$ denote the spectrum, point spectrum and surjectivity spectrum of T, respectively and let $Lat(T)$

2000 Mathematics Subject Classification: Primary 47A11; Secondary 47B40.

Keywords and phrases: algebraic spectral subspace, local spectrum, totally paranormal operator.

stand for the collection of all T-invariant closed linear subspaces of X. Thus $\sigma_{sur}(T)$ consists of all $\lambda \in \mathbb{C}$ for which $(T-\lambda)X \neq X$. It is well known from Proposition 1.3.2 of [8] that $\sigma_{sur}(T)$ is compact with $\partial\sigma(T) \subseteq \sigma_{sur}(T) \subseteq \sigma(T)$. Moreover, if T has the single valued extension property, then $\sigma_{sur}(T) = \sigma(T)$, and if the adjoint T^* has SVEP, then $\sigma_{ap}(T) = \sigma(T)$.

An operator $T \in L(X)$ is said to have the *single-valued extension property*, if for every open $U \subseteq \mathbb{C}$, the only analytic solution $f : U \to X$ of the equation $(T-\lambda)f(\lambda) = 0$ for all $\lambda \in U$ is the constant $f \equiv 0$. Given an arbitrary operator $T \in L(X)$, let $\sigma_T(x) \subseteq \mathbb{C}$ denote the *local spectrum* of T at the point $x \in X$, i.e., complement of the set $\rho_T(x)$ of all $\lambda \in \mathbb{C}$ for which there exist an open neighborhood U of λ in $\mathbb{C}$ and analytic function $f : U \to X$ such that $(T-\mu)f(\mu) = x$ holds for all $\mu \in U$. For every closed subset F of $\mathbb{C}$, let $X_T(F) = \{x \in X : \sigma_T(x) \subseteq F\}$ denote the corresponding *analytic spectral subspace* of T. It is easy to see that $X_T(F)$ is a
T-invariant linear subspace of X and also hyperinvariant for T. An operator $T \in L(X)$ is said to have *Dunford's property* (C) if $X_T(F)$ is closed for every closed $F \subseteq \mathbb{C}$. This condition plays an important role in the theory of spectral operators. It is well known that Dunford's property (C) implies the single-valued extension property.

If $A \subseteq \mathbb{C}$, then the *algebraic spectral subspace* $E_T(A)$ is the largest subspace of X on which all the restrictions of $T-\lambda, \lambda \in \mathbb{C}\setminus A$, are surjective. Thus $E_T(A)$ is the largest linear subspace of Y for which

$$(T-\lambda)Y = Y \quad \text{for all } \lambda \in \mathbb{C}\setminus A.$$

It follows from Proposition 1.2.16 in [10] that $X_T(F) \subseteq E_T(F) = E_T(F \cap \sigma(T))$ for every $T \in L(X)$ and for all subsets F of $\mathbb{C}$. The space $E_T(A)$ need not be closed in general. It is clear that $(T-\lambda)E_T(F) = E_T(F)$ for all $\lambda \in \mathbb{C}\setminus F$ as well so that the class which we consider has a maximal element if ordered by inclusion. Algebraic spectral subspace was introduced in [6] in connection with certain problems in automatic continuity. Recall from Corollary 1.3.3 of [3] and [7] that if $T \in L(H)$ is hyponormal, then

$$H_T(F) = E_T(F) = \bigcap_{\lambda \in \mathbb{C} \setminus F} (T - \lambda)H$$

for all closed sets $F \subseteq \mathbb{C}$ and in particular, $\bigcap_{\lambda \in \mathbb{C}} (T - \lambda)H = \{0\}$.

For an arbitrary operator $T \in L(X)$ on a complex Banach space X, the *quasi-nilpotent part* of T is the set

$$H_0(T) := \{x \in X : \lim_{n \to \infty} \| T^n x \|^{\frac{1}{n}} = 0\}.$$

It is easy to check that $H_0(T)$ is a linear hyperinvariant subspace of X, and $Ker(T^n) \subseteq H_0(T) \subseteq \{x \in X : \sigma_T(x) \subseteq \{0\}\}$ for all $n \in \mathbb{N}$. Clearly, T is quasi-nilpotent if and only if $H_0(T) = X$, see Theorem 1.5 in [13]. Moreover, if T is invertible, then $H_0(T) = \{0\}$. In general, $H_0(T)$ need to be closed.

An operator $T \in L(X)$ on the Banach space X is said to be *paranormal* if $\| Tx \|^2 \leq \| T^2 x \| \| x \|$ for all $x \in X$. Paranormality is preserved under restriction to invariant subspaces. It is clear that T is paranormal and invertible, then T^{-1} is also paranormal. If $T - \lambda$ is paranormal for every $\lambda \in \mathbb{C}$, then we say that T is *totally paranormal*, abbreviate it by TPN. The TPN operators form a proper subclass of the paranormal operators. Moreover, all totally paranormal operators have property (C). It is well known that every paranormal operator is isoloid, that is, every isolated point in $\sigma(T)$ is an eigenvalue. Also, every totally paranormal operator is normaloid, that is, the spectral radius $r(T) = \| T \|$.

Lemma 1. *If $T \in L(X)$ is TPN, then T has property (C). Moreover, $\sigma(T \mid E_T(F)) \subseteq F$ for any $F \subseteq \mathbb{C}$, and*

$$E_T(F) = \bigcap_{\lambda \in \mathbb{C} \setminus F,\, n \in \mathbb{N}} (T - \lambda)^n X.$$

Proof. It is easy to check that T has property (C). At first, we show that $Ker(T-\lambda) = Ker(T-\lambda)^n$ for every $\lambda \in \mathbb{C}$ and $n \in \mathbb{N}$. By definition of TPN, $Ker(T-\lambda) = Ker(T-\lambda)^2$ for every $\lambda \in \mathbb{C}$, and hence $Ker(T-\lambda) = Ker(T-\lambda)^n$ for every $\lambda \in \mathbb{C}$ and $n \in \mathbb{N}$. Let $\lambda \in \mathbb{C} \setminus F$. By definition of $E_T(F)$, $(T-\lambda)^n E_T(F) = E_T(F)$ for all $n = 2, 3, \ldots$. If $x \in E_T(F)$, then there is $y \in E_T(F)$ for which $x = (T-\lambda)^n y$. If $x \in Ker(T-\lambda)$, then $y \in Ker(T-\lambda)^2 = Ker(T-\lambda)$ and hence $x = 0$. Since $(T-\lambda)|E_T(F)$ is surjective, $\sigma(T|E_T(F)) \subseteq F$ for any $F \subseteq \mathbb{C}$. This completes the proof.

Theorem 2. *Let $T \in L(X)$ be a totally paranormal operator on a Banach space X. Then the kernel $KerT$ is the quasi-nilpotent part of T. Moreover,*

$$H_0(T) = X_T(\{0\}) = KerT = \{x \in X : \lim_{n\to\infty} \| T^n x \|^{\frac{1}{n}} = 0\}.$$

Proof. It suffices to show that

$$X_T(\{\lambda\}) = Ker(T-\lambda) = \{x \in X : \lim_{n\to\infty} \| (T-\lambda)^n x \|^{\frac{1}{n}} = 0\} \text{ for all } \lambda \in \mathbb{C}.$$

It is clear that $Ker(T-\lambda) \subseteq X_T(\{\lambda\})$ for all $\lambda \in \mathbb{C}$. For the converse, it suffices to show that $X_T(\{\lambda\}) \subseteq Ker(T-\lambda)$ for all $\lambda \in \mathbb{C}$. At first, we show that $\| (T-\lambda)x \|^n \leq \| (T-\lambda)^n x \|$ for any unit vector x and $n \in \mathbb{N}$. Let $x \in X$ be a unit vector. Since T is totally paranormal,

$$\| (T-\lambda)x \|^2 \leq \| (T-\lambda)^2 x \| \| x \|.$$

For any $n = 2, 3, \ldots,$ we obtain

$$\begin{aligned} \| (T-\lambda)^n x \|^2 &= \| (T-\lambda)(T-\lambda)^{n-1} x \|^2 \\ &\leq \| (T-\lambda)^2 (T-\lambda)^{n-1} x \| \| (T-\lambda)^{n-1} x \| \\ &= \| (T-\lambda)^{n+1} x \| \| (T-\lambda)^{n-1} x \|. \end{aligned}$$

It follows from Lemma 1.2 in [7] that

$$\|(T-\lambda)x\|^n \leq \|(T-\lambda)^n x\| \quad \text{for all } n \in \mathbb{N}.$$

This implies that if $\lim_{n\to\infty}\|(T-\lambda)^n x\|^{\frac{1}{n}} = 0$, then $(T-\lambda)x = 0$. Since T has the single-valued extension property,

$$X_T(\{\lambda\}) = X_{T-\lambda}(\{0\}) = \{x \in X : \lim_{n\to\infty}\|(T-\lambda)^n x\|^{\frac{1}{n}} = 0\}.$$

Hence $X_T(\{\lambda\}) \subseteq Ker(T-\lambda)$ for any $\lambda \in \mathbb{C}$. This completes the proof.

It is easy to see that all hyponormal operators are totally paranormal.

Corollary 3. *Let $T \in L(H)$ be a hyponormal operator on a Hilbert space H, and $x_0 \in H$. Then $\lim_{n\to\infty}\|T^n x_0\|^{\frac{1}{n}} = 0$ if and only if $Tx_0 = 0$. Moreover, the kernel $KerT$ is the quasi-nilpotent part of T.*

Theorem 4. *Let $T \in L(X)$ be a totally paranormal operator on a complex Banach space X. If λ and μ are distinct eigenvalues of T, then*

$$X^* = X_T(\{\lambda\})^{\perp} + X_T(\{\mu\})^{\perp}.$$

In particular, if $T \in L(H)$ is a TPN operator on a complex Hilbert space H, then $H_T(\{\lambda\}) \perp H_T(\{\mu\})$.

Proof. Let $M = X_T(\{\lambda\})$ and $N = X_T(\{\mu\})$. Then $M \cap N = \{0\}$, and $M + N = X_T(\mathbb{C}) = X$. Since we know from Lemma 1 that T has SVEP. It follows from (f) and (g) of Proposition 1.2.6 in [10] that

$$X = X_T(\{\lambda\} \cup \{\mu\}) = X_T(\{\lambda\}) \oplus X_T(\{\mu\}).$$

Finally, we show that $H_T(\{\lambda\}) \perp H_T(\{\mu\})$. It follows from Theorem 2 that $H_T(\{\zeta\}) = Ker(T-\zeta)$ for all $\zeta \in \mathbb{C}$. Thus $H_T(\{\zeta\})^{\perp} = Ker(T-\zeta)^{\perp} = \overline{ran(T-\zeta)^*}$ for all $\zeta \in \mathbb{C}$. Without loss of generality, we suppose that $|\lambda| \leq 1$ and $\mu = 1$. Let $T := \begin{pmatrix} \lambda & A \\ 0 & B \end{pmatrix}$ on $H_T(\{\lambda\}) \oplus H_T(\{\lambda\})^{\perp}$. If $x = y \oplus z \in H$

is an arbitrary eigenvector of T with respect to 1, then $\lambda y + Az = y$ and $Bz = z$. If $z = 0$, then $y = 0$, and so $x = 0$. This contradiction shows that $z \neq 0$. Let $u := z/\| z \|$ and $w = 0 \oplus u \in H$. Then $T^n w = (1 - \lambda^n) y \oplus u$. By totally paranormality of T, $\| Tw \|^n \leq \| T^n w \|$ for all $n \in \mathbb{N}$. Thus we have

$$(\sqrt{| 1 - \lambda |^2 \| y \|^2 + 1})^n \leq \sqrt{| 1 - \lambda^n |^2 \| y \|^2 + 1} \leq \sqrt{1 + 4\| y \|^2}. \tag{1}$$

If $y \neq 0$, then the left side of (1) tends to ∞ as $n \to \infty$. This is impossible unless $y = 0$. Hence $x = 0 \oplus z \in H_T(\{\lambda\})^{\perp}$.

References

[1] E. Albrecht and J. Eschmeier, Functional models and local spectral theory, University of Saarbrucken and University of Munster, 1988, preprint.

[2] E. Bishop, A duality theory for arbitrary operators, Pacific J. Math. 9 (1959), 379-397.

[3] K. Clancey, Seminormal operators, Lecture Note in Math., 742, Springer, Berlin, Heidelberg, New York, 1979.

[4] I. Colojoarva and C. Foiás, Theory of Generalized Spectral Operators, Gordon & Breach, New York, 1968.

[5] J. Eschmeier and B. Prunaru, Invariant subspaces for operators with Bishop's property (β) and thick spectrum, J. Funct. Anal. 94 (1990), 196-222.

[6] B. E. Johnshon and A. M. Sinclair, Continuity of linear operators commuting with linear operators II, Trans. Amer. Math. Soc. 146 (1969), 533-540.

[7] K. B. Laursen, Operators with finite ascent, Pacific J. Math. 152 (1992), 326-336.

[8] K. B. Laursen and M. M. Neumann, Automatic continuity of intertwining linear operators on Banach spaces, Rendiconti Del circolo Matematico di Palermo, Serie II, Tomo XL (1991), 325-341.

[9] K. B. Laursen and M. M. Neumann, Asymptotic intertwining and spectral inclusions on Banach spaces, Czechoslovak Math. J. 43(118) (1993), 483-497.

[10] K. B. Laursen and M. M. Neumann, An Introduction to Local Spectral Theory, Clarendon Press, Oxford, New York, 2000.

[11] K. B. Laursen and M. M. Neumann, Local spectral theory and spectral inclusions, Glasg. Math. J. 36 (1994), 331-343.

[12] V. G. Miller and M. M. Neumann, Local spectral theory for multipliers and convolution operators, Algebraic Methods in Operator Theory, pp. 25-36, Birkhäuser, Boston, 1994.

[13] P. Vrbová, On local spectral properties of operators in Banach spaces, Czechoslovak Math. J. 23(98) (1973), 483-492.

Department of Liberal Arts and Science
Chodang University
419 Sungnamri Muangun 534-701
Chonnam, Korea
e-mail: jkyoo@chodang.ac.kr