Far East Journal of Theoretical Statistics
Volume 25, Number 2, 2008, Pages 239-249
Published Online: July 17, 2008
This paper is available online at http://www.pphmj.com
© 2008 Pushpa Publishing House

CHARACTERIZATIONS OF A LOCATION-SCALE FAMILY OF DISTRIBUTIONS DEFINED BY A SYMMETRIC DISTRIBUTION

TRUC T. NGUYEN, ARJUN K. GUPTA, LEANDRO PARDO* and KHOAN T. DINH†

Department of Mathematics and Statistics
Bowling Green State University
Bowling Green, OH 43403, U. S. A.

*Department of Statistics and O.R.
Complutense University of Madrid
Spain

†US Environmental Protection Agency
Washington, DC 20460, U. S. A.

Abstract

Characterizations of a family of distributions with location and scale parameters are given. These characterizations can be used as transformations in the procedure to construct an EDF (empirical distribution function) goodness-of-fit test for this family.

1. Introduction

Let $\{f(x; \theta) : \theta = (\theta_1, ..., \theta_k) \in \Theta\}$ be a given family of distributions, and let $X_1, ..., X_n$, $n > k$, be a given random sample. To construct an EDF (empirical distribution function) goodness-of-fit test for checking whether a density $f(x; \theta)$ of the above family of distributions fits well to

2000 Mathematics Subject Classification: Primary 62E10; Secondary 62F03.

Keywords and phrases: conditional density function, characteristic function, goodness-of-fit test, Laplace distribution.

Received October 20, 2007

the given sample, we construct a transformation $Y_j = h_j(X_1, \ldots, X_n)$, $j = 1, \ldots, n-k$ such that the joint density $g(y_1, \ldots, y_{n-k})$ of $Y_1, \ldots, Y_{n-k}$ does not depend on the parameter θ. This transformation defines a characterization that $X_1, \ldots, X_n$ are i.i.d. according to $f(x; \theta)$ if and only if $Y_1, \ldots, Y_{n-k}$ are jointly distributed according to $g(y_1, \ldots, y_{n-k})$. Then by Rosenblatt [7], this is equivalent to $U_1 = F_{Y_1}(Y_1)$, $U_2 = F_{Y_2 \mid Y_1}(Y_2), \ldots$, $U_j = F_{Y_j \mid Y_1, \ldots, Y_{j-1}}(Y_j), \ldots$, $U_{n-k} = F_{Y_{n-k} \mid Y_1, \ldots, Y_{n-k-1}}(Y_{n-k})$ are i.i.d. according to a uniform distribution over the interval $(0, 1)$. To test whether $X_1, \ldots, X_n$ are i.i.d. according to $f(x; \theta)$ amounts to testing the equivalent hypothesis that $U_1, \ldots, U_{n-k}$ are i.i.d. according to a uniform distribution over the interval $(0, 1)$. In this case the null hypothesis is simple and an EDF goodness-of-fit test can be used (see D'Agostino and Stephens [1]).

The two principal families of models which play important role in the theory of statistical inference are exponential family and group family. In group family, the most popular family is the location-scale family of distributions. Let F_0 be a specified distribution. The location family of distributions defined by F_0 is the family $F_0(x - \mu)$, $-\infty < \mu < \infty$. The scale family of distributions defined by F_0 is the family $F_0(x/\sigma)$, $\sigma > 0$, and the family of distributions $F_0((x - \mu)/\sigma)$, $-\infty < \mu < \infty$, $\sigma > 0$, is the location-scale family of distributions defined by F_0. If F_0 has a density f_0, then $f_0(x - \mu)$, $(1/\sigma) f_0(x/\sigma)$, $(1/\sigma) f_0((x - \mu)/\sigma)$ are the densities of these families, respectively (Lehmann and Casella [5]). The purpose of this paper is to find characterizations that can be used in the procedure to fit a family of distributions with location and scale parameters defined by a given distribution F_0 to a given set of data using an EDF goodness-of-fit test. A transformation for location families of distributions and for scale families of distributions over the set of positive real numbers was given by Kotlarski [3]. He also gave transformations for normal distributions to independent Student's t distributions.

In this paper, first the results of Kotlarski are used to determine the population density of a random sample $X_1, ..., X_n$ based on the joint density of $Y_1, ..., Y_{n-k}$. In some cases a close form for the joint density of $Y_1, ..., Y_{n-k}$ is difficult if not impossible to be obtained such as in the case the density f_0 of F_0 is a function of absolute values of x_i, $i = 1, ..., n$. One example of this case is a random sample $X_1, ..., X_n$ from a Laplace distribution with location and scale parameters. To study these cases we propose modifications for the results of Kotlarski's to the results related to the joint density of some type of order statistics, then apply these results to the case of Laplace distribution with location and scale parameters. Nguyen et al. [6] use this technique to study the case of Laplace distribution with location and scale parameters. If a random variable X has the density $f(x)$, then the density function of $|X|$ is $g(x) = f(x) + f(-x)$ for all $x > 0$. Conversely, if $|X|$ has the density $g(x)$, $x > 0$, then X has the density $f(x) = g(|x|h(x))$ for all x, where $h(x) \geq 0$ with $h(x) + h(-x) = 1$.

2. Characterizations of a Location Family of Distributions

The following result of Kotlarski [3] plays an important role in our study. It characterizes scale families of distributions defined by distributions on the set of positive real numbers.

Lemma 2.1. *Let* $U_0, U_1, ..., U_n$, $n \geq 2$, *be independent positive random variables. Denote* $Z_1 = \frac{U_1}{U_0}, ..., Z_n = \frac{U_n}{U_0}$. *If the joint characteristic function of* $\ln Z_1, ..., \ln Z_n$ *does not vanish, then the joint distribution of* $Z_1, ..., Z_n$ *determines all the distributions of* $U_0, U_1, ..., U_n$ *up to a change of scale.*

This result of Kotlarski can be paraphrased under an alternative form. Under this form it characterizes a location family of distributions defined by a distribution F_0 with a non-vanishing characteristic function.

Let $U_0, U_1, ..., U_n$, $n \geq 2$, be independent random variables. Denote $Y_1 = U_1 - U_0, ...,$ $Y_n = U_n - U_0$. If the joint characteristic function of $Y_1, ..., Y_n$ does not vanish, then the joint distribution of $Y_1, ..., Y_n$

determines all the distributions of $U_0, U_1, \ldots, U_n$ up to a change of location.

Note that in Lemma 2.1 the condition that the joint characteristic function of $Z_1, \ldots, Z_n$ is $\prod_{j=1}^{n} \phi_j(t_j)\phi_0\left(-\sum_{j=1}^{n} t_j\right)$, where $\phi_j(t_j)$ is the characteristic function of $\ln U_j$, $j = 0, \ldots, n$, does not vanish is equivalent to the condition that the characteristic function of each Y_j, $j = 0, \ldots, n-1$ does not vanish (see Corollary 2.1).

Let $X_1, \ldots, X_n$, $n \geq 3$, be a random sample from a location family of distributions with the density $f_0(x - \mu)$, where $-\infty < \mu < \infty$ and f_0 is a specified density. By a transformation technique, the joint density of $Y_1 = X_1 - X_n, \ldots, Y_{n-1} = X_{n-1} - X_n$ is

$$f_{Y_1, \ldots, Y_{n-1}}(y_1, \ldots, y_{n-1}) = \int_{-\infty}^{\infty} \prod_{j=1}^{n-1} f_0(y_j + x) f_0(x) dx,$$

$$-\infty < y_1, \ldots, y_{n-1} < \infty. \tag{2.1}$$

The result of Corollary 2.1 below is directly obtained from the paraphrased form of Lemma 2.1.

Corollary 2.1. *Let $X_1, \ldots, X_n$, $n \geq 3$, be a random sample from a density $f(x)$ with a non-vanishing characteristic function, and let f_0 be a given density with a non-vanishing characteristic function. Further, let $Y_1 = X_1 - X_n, \ldots, Y_{n-1} = X_{n-1} - X_n$. Then $X_1, \ldots, X_n$ are i.i.d. from the density $f_0(x - \mu)$ for some $-\infty < \mu < \infty$, if and only if the joint density of $Y_1, \ldots, Y_{n-1}$ is given by* (2.1).

Lemma 2.2. *If $X_1, \ldots, X_n$ are continuous random variables with a joint density $f(x_1, \ldots, x_n)$ interchangeable in $x_1, \ldots, x_n$ and if $Y_1 = X_{(1)}, \ldots, Y_n = X_{(n)}$ are the corresponding order statistics of $X_1, \ldots, X_n$, then the joint distribution of $X_1, \ldots, X_n$ is determined if and only if the joint distribution of $Y_1, \ldots, Y_n$ is given.*

Proof. From the exchangeability of the joint density of $X_1, \ldots, X_n$, they are identically distributed, then if the joint density of $X_1, \ldots, X_n$ is $f(x_1, \ldots, x_n)$, $-\infty < x_1, \ldots, x_n < \infty$, by a transformation technique, the joint density of $Y_1, \ldots, Y_n$ is $n!\, f(y_1, \ldots, y_n)$, $-\infty < y_1 < \cdots < y_n < \infty$. Therefore the joint density of $X_1, \ldots, X_n$ is completely determined if and only if the joint density of $Y_1, \ldots, Y_n$ is given.

Combining the results of Corollary 2.1 and Lemma 2.2, the following result is obtained.

Theorem 2.1. *Let* $X_1, \ldots, X_n$, $n \geq 3$ *be a random sample from a continuous distribution with a density* $f(x)$ *and with a non-vanishing characteristic function, and let* f_0 *be the density of a given continuous distribution with a non-vanishing characteristic function. Further, let* $Y_1 = X_1 - X_n, \ldots, Y_{n-1} = X_{n-1} - X_n$, $W_1 = Y_{(1)}, \ldots, W_{n-1} = Y_{(n-1)}$. *Then* $X_1, \ldots, X_n$ *are i.i.d. according to the density* $f_0(x - \mu)$ *for some* $-\infty < \mu < \infty$ *if and only if the joint density of* $W_1, \ldots, W_{n-1}$ *is*

$$f_{W_1, \ldots, W_{n-1}}(w_1, \ldots, w_{n-1}) = (n-1)! \int_{-\infty}^{\infty} \prod_{i=1}^{n-1} f_0(w_i + x) f_0(x)\, dx,$$

$$-\infty < w_1 < \cdots < w_{n-1} < \infty. \tag{2.2}$$

Proof. If $X_1, \ldots, X_n$ are i.i.d. according to $f_0(x)$, then the joint density of $Y_1, \ldots, Y_{n-1}$ is given by (2.1) and is exchangeable in $y_1, \ldots, y_{n-1}$. The conclusion of this corollary then follows by Lemma 2.2 and Corollary 2.1.

3. A Scale Family Defined by a Symmetric Distribution

Let $X_1, \ldots, X_n$, $n \geq 3$, be a random sample from the density $(1/\sigma) f_0(x/\sigma)$. Assume that $P(X_j = 0) = 0$, and $\ln|X_j|$ has a non-vanishing characteristic function, $j = 1, \ldots, n$. Their joint density is

$$f_{X_1, \ldots, X_n}(x_1, \ldots, x_n) = (1/\sigma^n) \prod_{j=1}^{n} f_0(x_j/\sigma), \quad -\infty < x_1 < \cdots < x_n < \infty. \tag{3.1}$$

Let $Z_1 = \frac{X_1}{X_n}, \ldots, Z_{n-1} = \frac{X_{n-1}}{X_n}$. Then the joint density of $Z_1, \ldots, Z_{n-1}$ is

$$f_{Z_1, \ldots, Z_{n-1}}(z_1, \ldots, z_{n-1}) = \int_{-\infty}^{\infty} \prod_{j=1}^{n-1} f_0(z_j u) f_0(u) |u|^{n-1} du,$$

$$-\infty < z_1, \ldots, z_{n-1} < \infty. \tag{3.2}$$

Hence, the distribution given by (3.2) has a non-vanishing joint characteristic function. From (3.2), the joint density of $|Z_1| = \frac{|X_1|}{|X_n|}, \ldots,$ $|Z_{n-1}| = \frac{|X_{n-1}|}{|X_n|}$ is

$$f_{|Z_1|, \ldots, |Z_{n-1}|}(z_1, \ldots, z_{n-1}) = \sum_{a_1, \ldots, a_{n-1}} f_{Z_1, \ldots, Z_{n-1}}(a_1 z_1, \ldots, a_{n-1} z_{n-1}),$$

$$z_1, \ldots, z_{n-1} > 0. \tag{3.3}$$

The summation in (3.3) is taken over all values of $a_j = \pm 1$, $j = 1, \ldots, n-1$. Hence, the joint density function of $Z_1, \ldots, Z_{n-1}$ uniquely determines the joint density of $|Z_1|, \ldots, |Z_{n-1}|$. The following result gives a way to determine the common distribution of $X_1, \ldots, X_n$ if the joint distribution of $Z_1, \ldots, Z_{n-1}$ is given. Since $Z_j = \frac{X_j}{X_n} = \frac{-X_j}{-X_n}$, the two densities $f_0(x)$ and $f_0(-x)$ give the same joint distribution to $Z_1, \ldots, Z_{n-1}$. In general it is not expected that the common distribution of $X_1, \ldots, X_n$ is determined uniquely up to a change of scale.

Theorem 3.1. *Let $X_1, \ldots, X_n$, $n \geq 3$, be i.i.d. according to a distribution F with density $f(x)$ and with no mass point at zero, and let F_0 be a distribution with density f_0 symmetric about zero, with no mass point at zero, and if X is distributed according to F_0, then* $\ln|X|$ *has a non-vanishing characteristic function. Let $Z_i = \frac{X_i}{X_n}$, $i = 1, \ldots, n-1$. Then $f(x) = (1/\sigma) f_0(x/\sigma)$ if and only if the joint density of $Z_1, \ldots, Z_{n-1}$ is given by* (3.2).

Proof. If $X_1, \ldots, X_n$ are i.i.d. according to the density $(1/\sigma)f_0(x/\sigma)$, then the joint density of $Z_j = X_j/X_n$, $j = 1, \ldots, n-1$ is given by (3.2). Conversely, if the joint density of $Z_1, \ldots, Z_{n-1}$ is given by (3.2), then the joint distribution of $|Z_1|, \ldots, |Z_{n-1}|$ is given by (3.3). Hence, the common density of $|X_1|, \ldots, |X_{n-1}|$ is uniquely identified by the result of Kotlarski (Lemma 2.1). This common density is $(1/\sigma)(f_0(x/\sigma) + f_0(-x/\sigma))$ $= (2/\sigma)f_0(x/\sigma)$ for $x > 0$. Hence, the common density of $X_1, \ldots, X_n$ must be of the form $(2/\sigma)f_0(x/\sigma)h(x/\sigma)$, where $h(x) + h(-x) = 1$, for $-\infty < x$ $< \infty$. Since the joint density of $Z_1, \ldots, Z_{n-1}$ is given by (3.2), the density of Z_1 is given by

$$\int_{-\infty}^{\infty} f_0(x_1 u) f_0(u) |u| du = 4\int_{-\infty}^{\infty} f_0(x_1 u) f_0(u) h(x_1 u) h(u) |u| du. \tag{3.4}$$

For $x_1 = -1$, since $h(u) + h(-u) = 1$, (3.4) can be written as

$$\int_{-\infty}^{\infty} f_0^2(u)(h(u) + h(-u))^2 |u| du = 4\int_{-\infty}^{\infty} f_0^2(u) h(-u) h(u) |u| du. \tag{3.5}$$

Expand and simplify (3.5),

$$\int_{-\infty}^{\infty} f_0(u)^2 |u| (h(u) - h(-u))^2 du = 0. \tag{3.6}$$

Therefore $h(x) = h(-x) = 1/2$, and $X_1, \ldots, X_n$ are i.i.d. according to the density $1/\sigma f_0(x/\sigma)$.

Combining the results of Lemma 2.2 and Theorem 3.1 the following result is obtained.

Corollary 3.1. *Let* $X_1, \ldots, X_n$, $n \geq 3$, *be i.i.d. according to a distribution F with density* $f(x)$ *and with no mass point at zero, and let* F_0 *be a distribution with density* f_0 *symmetric about zero with no mass point at zero, and if X is distributed according to* F_0, *then* $\ln|X|$ *has a non-vanishing characteristic function. Let* $Z_i = \dfrac{X_i}{X_n}$, $i = 1, \ldots, n-1$, *and* *let* $W_i = Z_{(i)}, \; i = 1, \ldots, n-1,$ *be the order statistics of* $Z_i, \; i = 1, \ldots, n-1.$ *Then* $F(x) = F_0(x/\sigma),$ *that is, with density* $\frac{1}{\sigma} f_0(x/\sigma)$ *if and only if the joint density of* $W_1, \ldots, W_{n-1}$ *is*

$$f_{W_1, \ldots, W_{n-1}}(w_1, \ldots, w_{n-1}) = (n-1)! \int_{-\infty}^{\infty} \prod_{i=1}^{n-1} f_0(w_i u) f_0(u) | u |^{n-1} du,$$

$$-\infty < w_1 < \cdots < w_{n-1} < \infty. \tag{3.7}$$

4. A Location-scale Family Defined by a Symmetric Distribution

Let $X_1, \ldots, X_n, \; n \geq 6,$ be i.i.d. according to a continuous distribution with density $(1/\sigma) f_0((x - \mu)/\sigma),$ where $-\infty < \mu < \infty, \; \sigma > 0,$ and $Z_1 = \frac{X_1 - X_n}{X_{n-1} - X_n}, \ldots, \; Z_{n-2} = \frac{X_{n-2} - X_n}{X_{n-1} - X_n}.$ Then the joint density of $Z_1, \ldots,$ Z_{n-2} is

$$f_{Z_1, \ldots, Z_{n-2}}(z_1, \ldots, z_{n-2})$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \prod_{j=1}^{n-2} f_0(z_j u + x) f_0(u + x) f_0(x) | u |^{n-2} dx du,$$

$$-\infty < z_1 < \cdots < z_{n-1} < \infty. \tag{4.1}$$

Assume that (4.1) has a non-vanishing characteristic function. The question to be asked here is: If the joint density of $Z_1, \ldots, Z_{n-2}$ is given by (4.1), then does it imply that $X_1, \ldots, X_n$ are i.i.d. according to a distribution with density $(1/\sigma) f_0((x - \mu)/\sigma)$? If $f_0(x)$ is not a symmetric density about zero, then the two densities $f_0(-x)$ and $f_0(x)$ are different, but they determine the same joint density of $Z_1, \ldots, Z_{n-2}$ given by (4.1). Hence, assume that f_0 is a symmetric density.

Theorem 4.1. *Let* $X_1, \ldots, X_n, \; n \geq 6,$ *be i.i.d. according to a continuous symmetric distribution* F *with density* f, *and let* F_0 *be a given distribution symmetric about zero with density* f_0. *For* $Y_1, \; Y_2$ *i.i.d. according to* F_0, *assume that* $\ln|Y_1 - Y_2|$ *has a non-vanishing characteristic function. Then* $X_1, \ldots, X_n$ *are i.i.d. with density* $(1/\sigma)f_0((x-\mu)/\sigma)$ *if and only if the joint distribution of* $Z_j = \dfrac{X_j - X_n}{X_{n-1} - X_n}$, $j = 1, \ldots, n-2$ *is given by* (4.1).

Proof. Assume that $X_1, \ldots, X_n$, $n \geq 6$, are i.i.d. according to the distribution $F_0((x-\mu)/\sigma)$ about the point μ with the density $(1/\sigma)f_0((x-\mu)/\sigma)$. As noted above, the joint density of $Z_1, \ldots, Z_{n-2}$ is given by (2.10), and the joint distribution of $Z_1, \ldots, Z_{n-2}$ does not depend on the location parameter μ and the scale parameter σ of X_j, $j = 1, \ldots, n$. Then without loss of generality, assume that $\mu = 0$, and $\sigma = 1$.

Conversely, given that $X_1, \ldots, X_n$ are i.i.d. according to the distribution F symmetric about zero with the density f, and if the joint density of $Z_1, \ldots, Z_{n-2}$ is given by (4.1), we determine the common distribution of $X_1, \ldots, X_n$. Assume that $\phi_{Z_1,\ldots,Z_{n-2}}(s_1, \ldots, s_{n-2})$ is the joint characteristic function of $Z_1, \ldots, Z_{n-2}$, then

$$\phi_{Z_1,\ldots,Z_{n-2}}(s_1, \ldots, s_{n-2})$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\prod_{j=1}^{n-2}\phi(s_j/u)e^{-i\sum_{j=1}^{n-2}(x/u)s_j} f_0(u+x)f_0(x)\,du\,dx, \tag{4.2}$$

where $\phi(s)$ is the characteristic function of F. Since the joint density of $Z_1, \ldots, Z_{n-2}$ is known and is given by (4.1), assume that the characteristic function of this distribution is $H(s_1, \ldots, s_{n-2})$. The joint characteristic function of $W_1 = \dfrac{X_1 - X_2}{X_{n-1} - X_n} = Z_1 - Z_2$, $W_2 = \dfrac{X_3 - X_4}{X_{n-1} - X_n} = Z_3 - Z_4$ is $H(s_1, -s_1, s_3, -s_3, 0, \ldots, 0)$ and is obtained by letting $s_2 = -s_1$, $s_4 = -s_3$, and the other $s_j = 0$ in the given joint characteristic function $H(s_1, \ldots, s_{n-2})$ of $Z_1, \ldots, Z_{n-2}$. Hence, the joint distribution of $\dfrac{X_1 - X_2}{X_{n-1} - X_n}$ and $\dfrac{X_3 - X_4}{X_{n-1} - X_n}$ is known and since $X_1 - X_2$, $X_3 - X_4$,

$X_{n-1} - X_n$ are i.i.d. with a symmetric distribution and $\ln| X_1 - X_2 |$ has a non-vanishing characteristic function, by Theorem 3.1, the distribution of $X_1 - X_2$, is uniquely determined and with a real valued characteristic function $M(s)$. Since the common distribution of $X_1, \ldots, X_n$ is symmetric about zero, the characteristic function of this distribution is a real valued quantity $A(s)$ and then the characteristic function of $X_1 - X_2$ is $A^2(s) = M(s)$. Hence, $A(s) = \sqrt{M(s)}$ is uniquely determined. Therefore the common density of $X_1, \ldots, X_n$ is $(1/\sigma) f_0((x - \mu)/\sigma)$.

Corollary 4.1. *Let* $X_1, \ldots, X_n$, $n \geq 6$, *be i.i.d. according to a continuous symmetric distribution F with density f, and let* F_0 *be a given distribution symmetric about zero with density* f_0. *For* Y_1, Y_2 *i.i.d. according to* F_0, *assume that* $\ln| Y_1 - Y_2 |$ *has a non-vanishing characteristic function. Then* $X_1, \ldots, X_n$ *are i.i.d. with density* $\frac{1}{\sigma} f_0((x - \mu)/\sigma)$ *if and only if the joint density of* $W_1, \ldots, W_{n-2}$ *is*

$$f_{W_1, \ldots, W_{n-2}}(w_1, \ldots, w_{n-2})$$

$$= (n-2)! \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \prod_{j=1}^{n-2} f_0(w_j u + x) f_0(u + x) f_0(x) | u |^{n-2} du dx,$$

$$-\infty < w_1 < \cdots < w_{n-2} < \infty. \tag{4.3}$$

References

[1] R. B. D'Agostino and M. A. Stephens, Goodness-of-fit Techniques, Marcel Dekker Inc., New York, 1986.

[2] N. L. Johnson, S. Kotz and N. Balakrishnan, Continuous Univariate Distributions 2, 2nd ed., Wiley & Sons Inc., New York, 1995.

[3] I. Kotlarski, On characterizing the normal distribution by Student's t law, Biometrika 53 (1966), 603-606.

[4] S. Kotz, T. J. Kozubowski and K. Podgorski, The Laplace Distribution and Generalizations, Birkhäuser Inc., Boston, 2001.

[5] E. L. Lehmann and G. Casella, Theory of Point Estimation, 2nd ed., Springer Inc., New York, 1998.

[6] T. T. Nguyen, A. K. Gupta, L. Pardo and K. T. Dinh, Some characterizations of Laplace distribution, Technical Report, Department of Mathematics and Statistics, Bowling Green State University, Bowling Green, OH 43403, 2007.

[7] M. Rosenblatt, Remark on multivariate transformation, Ann. Math. Statist. 23 (1952), 470-472.

■