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MULTIPLE DIRECTIONAL DECISION WITH A CONTROL IN PARALLEL PROFILE MODEL

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Abstract

In profile analysis, three hypotheses as "parallelism", "level hypothesis" and "no condition variation" are considered. This paper gives an approximated multiple three decision procedure with a control in "level hypothesis" under "parallelism".

1. Introduction

Let the p dimensional random vector $\boldsymbol{x}_{ir}$ be independently and normally distributed with mean $\boldsymbol{\mu}_i$ and covariance matrix $\Sigma = (\sigma_{jl})$, that is, $N_p(\boldsymbol{\mu}_i, \Sigma)$ $(i = 0, 1, ..., k,\ r = 1, ..., n_i)$. The mean vectors $\boldsymbol{\mu}_i$'s are the mean profiles of the $k+1$ groups and $\boldsymbol{\mu}_0$ is the control mean. The parallelism hypothesis is

$$H_1 : \boldsymbol{\mu}_1 - \boldsymbol{\mu}_0 = \gamma_1 \mathbf{1}_p, ..., \boldsymbol{\mu}_k - \boldsymbol{\mu}_0 = \gamma_k \mathbf{1}_p,$$

where $\mathbf{1}_p$ is a *p-vector* of ones and $\gamma_1, ..., \gamma_k$ are called the *level differences*. The alternative hypothesis is $A_1 \neq H_1$. When H_1 is true, one may wish to test the level hypothesis

$$H_2 : \boldsymbol{\gamma} = \mathbf{0}$$

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against the alternative $A_2 : \boldsymbol{\gamma} \neq \mathbf{0}$, where $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_k)'$. The likelihood ratio procedures testing "H_1 vs A_1" and "H_2 vs A_2" are given by Srivastava [6]. But Srivastava's procedure cannot decide which γ_i is (are) not 0, when H_2 was rejected.

In this paper, we consider the testing hypothesis

$$H_{2i} : \gamma_i = 0 \quad \text{vs} \quad A_{2i}^{+} : \gamma_i > 0, \quad A_{2i}^{-} : \gamma_i < 0, \tag{1.1}$$

$(i = 1, \ldots, k)$, simultaneously. This testing problem is called *multiple directional decision with a control*. Liu [4] discussed two procedures of this problem for univariate populations. The procedures are the multiple three decision procedure proposed by Bohrer [1] and the extension of Dunnett's [2] procedure for multiple comparisons with a control. The probability of type I error of (1.1) is not controlled at level α in Bohrer's procedure. In the parallel profile model, the significance level is controlled at α. We give an approximated directional decision based on Dunnett's procedure in Section 2. In Section 3, the accuracy of approximation is examined by simulation and the probabilities of type III error and correct decision are also examined by simulation.

2. Directional Decision

Let $\overline{\boldsymbol{x}} = \sum_{i=0}^{k} n_i \overline{\boldsymbol{x}}_i / \nu$ and $V = \sum_{i=0}^{k} \sum_{r=1}^{n_i} (\boldsymbol{x}_{ir} - \overline{\boldsymbol{x}}_i)(\boldsymbol{x}_{ir} - \overline{\boldsymbol{x}}_i)'$, where $\overline{\boldsymbol{x}}_i$ is the usual sample mean based on n_i observations from the ith population and $\nu = n_0 + \cdots + n_k$. The likelihood function is

$$L(\boldsymbol{\gamma}, \overline{\boldsymbol{\mu}}, \Sigma) = (2\pi)^{-p\nu} |\Sigma|^{-\nu/2},$$

$$\exp[(-1/2)\mathrm{tr}\{\Sigma^{-1}(Y - \mathbf{1}_p \boldsymbol{\gamma}')A^{-1}(Y - \mathbf{1}_p \boldsymbol{\gamma}')' + \nu(\overline{\boldsymbol{x}} - \overline{\boldsymbol{\mu}})(\overline{\boldsymbol{x}} - \overline{\boldsymbol{\mu}})' + V\}], \tag{2.1}$$

where $\overline{\boldsymbol{\mu}} = \sum_{i=0}^{k} n_i \overline{\boldsymbol{\mu}}_i / \nu$, $Y = (\overline{\boldsymbol{x}}_1 - \overline{\boldsymbol{x}}_0, \ldots, \overline{\boldsymbol{x}}_k - \overline{\boldsymbol{x}}_0)$ and

$$A = \begin{pmatrix} n_1^{-1} + n_0^{-1} & n_0^{-1} & \cdots & n_0^{-1} \\ n_0^{-1} & n_2^{-1} + n_0^{-1} & \cdots & n_0^{-1} \\ \vdots & \vdots & & \vdots \\ n_0^{-1} & n_0^{-1} & \cdots & n_k^{-1} + n_0^{-1} \end{pmatrix}.$$

The maximum likelihood estimate (MLE) of $\boldsymbol{\gamma}$ is

$$\hat{\gamma} = (Y'V^{-1}\mathbf{1}_p)/(\mathbf{1}_p'V^{-1}\mathbf{1}_p).$$

Let $\hat{\Sigma}$ be the MLE of Σ. Then $\mathbf{1}_p'\hat{\Sigma}^{-1} = \nu\mathbf{1}_p'V^{-1}$. These are given by Srivastava [6]. The Fisher information matrix of $\boldsymbol{\gamma}$ is

$$I(\boldsymbol{\gamma}) = -E\left[\frac{\partial^2}{\partial\boldsymbol{\gamma}\partial\boldsymbol{\gamma}'}\log L(\boldsymbol{\gamma}, \bar{\boldsymbol{\mu}}, \Sigma)\right] = (\mathbf{1}_p'\Sigma^{-1}\mathbf{1}_p)A^{-1}, \tag{2.2}$$

hence the asymptotic distribution of $(\mathbf{1}_p'\Sigma^{-1}\mathbf{1}_p)^{1/2}A^{-1/2}(\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma})$ is $N(\mathbf{0}, I_p)$. If $n_0 = n_1 = \cdots = n_k = n$, then $\sqrt{n\mathbf{1}_p'\Sigma^{-1}\mathbf{1}_p/2}(\hat{\gamma}_i - \gamma_i)$ is distributed as the standard normal $N(0, 1)$ and the correlation coefficient of $\hat{\gamma}_i$ and $\hat{\gamma}_{i'}$ $(i \neq i')$ is $1/2$, that is, the asymptotic covariance matrix of $\hat{\boldsymbol{\gamma}}$ has an intraclass correlation structure. We assume that the sample size from each population is n below, say $n_0 = n_1 = \cdots = n_k = n$. Under the parallelism hypothesis H_1,

$$I(\gamma_i, \sigma_{jl}) = -E\left[\frac{\partial^2}{\partial\gamma_i\partial\sigma_{jl}}\log L(\boldsymbol{\gamma}, \bar{\boldsymbol{\mu}}, \Sigma)\right] = 0. \tag{2.3}$$

Since the asymptotic covariance of $\hat{\boldsymbol{\gamma}}$ and $\hat{\Sigma}$ is O by (2.3) and the MLE has the asymptotic normality, $\hat{\boldsymbol{\gamma}}$ and $\hat{\boldsymbol{\Sigma}}$ are asymptotically independent. But the normal approximation of the distribution of $\hat{\Sigma}$ may not be good. The statistic V has the Wishart distribution with covariance matrix Σ and $\nu - k - 1$ degrees of freedom (df). Hence $(\mathbf{1}_p'\Sigma^{-1}\mathbf{1}_p)/(\mathbf{1}_p'V^{-1}\mathbf{1}_p)$ has the chi-square distribution with $\nu - k - p$ df, $\chi^2_{\nu-k-p}$, by Corollary 2.4.5.1 of Siotani et al. [5]. When the statistics

$$t_i = \sqrt{n(\nu - k - p)/2}\,\frac{\hat{\gamma}_i}{\sqrt{\mathbf{1}_p'V^{-1}\mathbf{1}_p}} \quad (i = 1, \ldots, k) \tag{2.4}$$

would be used for the decision problem (1.1), then

$$P(|\,t_i\,| > |\,d\,|, \text{ for at least one } i \in \{1, \ldots, k\}) \approx \alpha$$

would be held for large n under H_{2i} $(i = 1, \ldots, k)$, where $|d|$ is a constant used in the Dunnett's two-sided multiple comparisons with a control and α is a significance level, see, e.g., Hsu [3] in which the constant $|d|$ is tabulated. Hence, a multiple directional decision procedure would be given by

$$\begin{cases} \gamma_i = 0 & \text{when } |t_i| < |d|, \\ \gamma_i > 0 & \text{when } t_i \geq |d|, \\ \gamma_i < 0 & \text{when } t_i \leq -|d|. \end{cases} \tag{2.5}$$

This procedure is based on Liu [4], so the type I and III errors are approximately α.

Next, we assume that the covariance matrix has an intraclass correlation structure

$$\Sigma = \sigma^2 \{(1 - \rho) I_p + \rho \mathbf{1}_p \mathbf{1}'_p\}. \tag{2.6}$$

Then the MLE of $\boldsymbol{\gamma}$ is $\tilde{\boldsymbol{\gamma}} = Y' \mathbf{1}_p / p$, whose distribution is

$$N_p(\boldsymbol{\gamma}, (\tau_1 / np)(I_p + \mathbf{1}_p \mathbf{1}'_p)),$$

where $\tau_1 = \sigma^2 \{1 + (p - 1)\rho\}$ is a characteristic root of (2.6). It is well-known that the statistic $\mathbf{1}'_p V \mathbf{1}_p / p\tau_1$ has $\chi^2_{\nu - k - 1}$ and independent of $\bar{\boldsymbol{x}}_i$'s, see, e.g., Chapter 2 of Siotani et al. [5]. Hence, $\hat{\tau}_1 = \mathbf{1}'_p V \mathbf{1}_p / p(\nu - k - 1)$ is an unbiased estimator of τ_1 and is independent of $\tilde{\boldsymbol{\gamma}}$. The statistics

$$\tilde{t}_i = \sqrt{n(\nu - k - 1)/2} \, \frac{\tilde{\gamma}_i}{\sqrt{\mathbf{1}'_p V^{-1} \mathbf{1}_p}} \quad (i = 1, \ldots, k) \tag{2.7}$$

can be used for the decision problem (1.1) instead of (2.4). Then

$$P(|\tilde{t}_i| > |d|, \text{ for at least one } i \in \{1, \ldots, k\}) = \alpha,$$

exactly. Hence, the procedure for (1.1) is same as Liu [4].

3. Simulation

We gave an approximated multiple directional decision procedure

(2.5) in the previous section. In this section, the accuracy of approximation to a significance level is examined by simulation. Further, the probabilities of type III error and correct decision in the procedure (2.5) are also examined. We choose $k = 2, 3,$ $p = 3, 4$ and $\alpha = 0.05$. The means of control and covariance matrices are

$$\boldsymbol{\mu}_0 = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 1 & 0.5 & 0 \\ 0.5 & 1 & 0.5 \\ 0 & 0.5 & 1 \end{pmatrix} \quad \text{for} \quad p = 3$$

and

$$\boldsymbol{\mu}_0 = \begin{pmatrix} 0 \\ 2 \\ 1 \\ 2 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 1 & 0.6 & 0.3 & 0 \\ 0.6 & 1 & 0.6 & 0.3 \\ 0.3 & 0.6 & 1 & 0.6 \\ 0 & 0.3 & 0.6 & 1 \end{pmatrix} \quad \text{for} \quad p = 4,$$

respectively.

3.1. Significance level

We examine the accuracy of approximation to the significance level α. All population means are same as $\boldsymbol{\mu}_0$ above. For the above values, sample size $n = 10, 20, 30, 40$, and $\alpha = 0.05$, 10,000 $(t_1, ..., t_k)$ are computed. The proportion, which at least one of k statistics $t_1, ..., t_k$ is greater than $|d|$, is calculated. The results are given in Table 1.

Table 1. Significance level

		n			
p	k	10	20	30	40
3	2	0.0673	0.0603	0.0556	0.0501
	3	0.0643	0.0599	0.0536	0.0525
4	2	0.0669	0.0592	0.0544	0.0530
	3	0.0653	0.0613	0.0561	0.0523

From Table 1, the approximation may be better under the large sample, which is natural result. The procedure (2.5) would be applicable when the sample size from each population is larger than 30.

3.2. Type III error and correct decision

Next the probabilities of type III error and correct decision are examined. We choose $\alpha = 0.05$ and $n = 20, 30, 40$, because the approximation to the significance level is not good when $n = 10$. The differences (γ_1, γ_2) for $k = 2$ and $(\gamma_1, \gamma_2, \gamma_3)$ for $k = 3$ are chosen as in Tables 2.1 and 2.2. Here, 10,000 $(t_1, ..., t_k)$ are computed. The proportion, which at least one of k statistics $t_1, ..., t_k$ is greater than $|d|$ (or less than $|d|$) when $\gamma_i < 0$ (or $\gamma_i > 0$), is calculated. This is an estimated probability of type III error. The proportion, that all populations are inferred correctly, is also calculated, which is an estimated probability of correct decision. The results are in Tables 2.1 and 2.2, in which the values in the parentheses ( ) are probabilities of correct decision.

Table 2.1. Probabilities of type III error and correct decision $(k = 2)$

p	γ_1	γ_2	n 20	30	40
3	0.1	0	0.0065 (0.0395)	0.0042 (0.0461)	0.0024 (0.0553)
	0.2	0	0.0023 (0.0903)	0.0016 (0.1269)	0.0007 (0.1606)
	0.4	0	0.0001 (0.3144)	0.0001 (0.4628)	0.0000 (0.5988)
	0.7	0	0.0000 (0.7789)	0.0000 (0.9138)	0.0000 (0.9532)
3	0.1	−0.1	0.0098 (0.0003)	0.0080 (0.0001)	0.0072 (0.0006)
	0.2	−0.2	0.0040 (0.0014)	0.0013 (0.0031)	0.0010 (0.0064)
	0.4	−0.4	0.0006 (0.0505)	0.0000 (0.1554)	0.0000 (0.3105)
	0.7	−0.7	0.0000 (0.6140)	0.0000 (0.8813)	0.0000 (0.9670)
4	0.1	0	0.0067 (0.0367)	0.0050 (0.0460)	0.0026 (0.0535)
	0.2	0	0.0018 (0.0892)	0.0009 (0.1272)	0.0005 (0.1603)
	0.4	0	0.0001 (0.3217)	0.0000 (0.4662)	0.0000 (0.6003)
	0.7	0	0.0000 (0.7794)	0.0000 (0.9168)	0.0000 (0.9601)
4	0.1	−0.1	0.0093 (0.0001)	0.0078 (0.0000)	0.0044 (0.0001)
	0.2	−0.2	0.0023 (0.0010)	0.0019 (0.0015)	0.0002 (0.0029)
	0.4	−0.4	0.0001 (0.0377)	0.0000 (0.1153)	0.0000 (0.2351)
	0.7	−0.7	0.0000 (0.5621)	0.0000 (0.8488)	0.0000 (0.9551)

Table 2.2. Probabilities of type III error and correct decision $(k = 3)$

p	γ_1	γ_2	γ_3	n 20	30	40
3	0.1	−0.1	0	0.0067 (0.0000)	0.0054 (0.0002)	0.0035 (0.0002)
	0.2	−0.2	0	0.0016 (0.0004)	0.0016 (0.0016)	0.0007 (0.0030)
	0.4	−0.4	0	0.0001 (0.0322)	0.0000 (0.1069)	0.0000 (0.2335)
	0.7	−0.7	0	0.0000 (0.5436)	0.0000 (0.8395)	0.0000 (0.9378)
3	0.1	0.1	−0.1	0.0116 (0.0000)	0.0095 (0.0000)	0.0062 (0.0000)
	0.2	0.2	−0.2	0.0022 (0.0001)	0.0013 (0.0002)	0.0010 (0.0004)
	0.4	0.4	−0.4	0.0002 (0.0090)	0.0001 (0.0424)	0.0000 (0.1498)
	0.7	0.7	−0.7	0.0000 (0.4473)	0.0000 (0.8009)	0.0000 (0.9396)
4	0.1	−0.1	0	0.0081 (0.0000)	0.0058 (0.0000)	0.0038 (0.0000)
	0.2	−0.2	0	0.0016 (0.0008)	0.0013 (0.0016)	0.0003 (0.0028)
	0.4	−0.4	0	0.0002 (0.0385)	0.0000 (0.1163)	0.0000 (0.2463)
	0.7	−0.7	0	0.0000 (0.5473)	0.0000 (0.8425)	0.0000 (0.9451)
4	0.1	0.1	−0.1	0.0094 (0.0000)	0.0063 (0.0000)	0.0045 (0.0000)
	0.2	0.2	−0.2	0.0040 (0.0000)	0.0022 (0.0000)	0.0005 (0.0007)
	0.4	0.4	−0.4	0.0001 (0.0089)	0.0000 (0.0498)	0.0000 (0.1502)
	0.7	0.7	−0.7	0.0000 (0.4568)	0.0000 (0.8007)	0.0000 (0.9450)

From Tables 2.1 and 2.2, all probabilities of type III error are less than 0.01 except for the case $k = 3$, $p = 3$, $n = 20$ and $\gamma_i = \pm 0.1$. These are much smaller than $\alpha = 0.05$. Each probability of type III error for $p = 4$ is less than that for $p = 3$, when γ_i's are ± 0.1. When $\gamma_i \neq 0$ for all i, probability of correct decision for $k = 2$ is uniformly greater than that for $k = 3$, which is natural result. If one of γ_i is 0, then there is not much difference for p.

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