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PREDICTION VIA THE CONDITIONAL QUANTILE FOR RIGHT CENSORSHIP MODEL

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Abstract

In this paper we introduce a new smooth estimator of the conditional quantile function in the censorship model. We show that this estimator converges uniformly almost surely and suitably normalized is asymptotically normal. Some simulations have been drawn to lend further support to our theoretical results for the convergence as well as for the normality for the finite samples.

1. Introduction

It is well-known, from the robustness literature, that the mean is sensible to outliers (see Hampel et al. [18]); it may be sensible to use the median, which is a particular case of the quantile, rather than the mean to forecast future since the median is highly resistant against outliers. The nonparametric estimation of conditional quantile has received a great interest since 1969, when Roussas [27] showed the convergence and asymptotic normality of kernel estimates under Markov assumptions. For

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independent and identically distributed (iid) random variables (rv), Stone [32] showed the weak consistency of kernel estimates. The uniform consistency was studied by Schlee [29] for strong mixing case. Samanta [28] proved the asymptotic normality in the iid case. Many other authors considered this problem; without pretending to the exhaustiviteness, we quote Bhattacharya and Gangopadhyay [3], Jones and Hall [20], Mehra et al. [25], Chaudhuri [5], Fan et al. [13], Welsh [38] and Xiang [40]. Honda [19] dealt with the α-mixing case and proved the uniform convergence and asymptotic normality of an estimate of the conditional quantile using polynomial fitting method. Berlinet et al. [2] showed the asymptotic normality of convergent estimates of conditional quantile by considering the particular case of stationary α-mixing process. Gannoun et al. [14] gave a smooth nonparametric conditional median predictor, based on double kernel methods and established its asymptotic normality and proposed an extension to the conditional quantile.

In censoring case, Beran [1] introduced a nonparametric estimate of the conditional survival function and proved some consistency results which were later exposed and extended by Dabrowska [8, 9] in the iid case and Lecoutre and Ould-Saïd [24] studied the consistency in the strong mixing case. Dabrowska [10] established a Bahadur representation of kernel quantile estimator and Xiang [39] obtained the deficiency of sample quantile estimator with respect to a kernel estimator using coverage probability. Leconte et al. [23] built two classes of estimators of the conditional distribution function and the quantile function and showed under some conditions that the two classes are equal. Some simulations have been driven to show that one is better than other in the sense of the mean square error. Further results, including bootstrap approximations, have been gotten by Van Keilegom and Veraverbeke [35, 36]. Recently, Gannoun et al. [15] studied the asymptotic properties of an estimator of the conditional quantile using polynomial. Other large samples properties of the conditional distribution have been studied extensively in the literature (see, e.g., González Manteiga and Cadarso Suárez [16], Stute [33] and Van Keilegom and Veraverbeke [34, 37]). Here we provide consistent and asymptotically normal conditional quantile estimate under the condition C and (T, X) are independent as in the recent paper of Ould-Saïd [26] (see also Carbonnez et al. [4] and Kohler et al. [22]), who established a strong uniform convergence rate of a kernel conditional quantile estimator under censorship model.

Consider a sequence of iid random variables $T_1, T_2, \ldots$ with common unknown absolutely continuous distribution function (df) F. In many situations, we observe only censored lifetimes of items under study. That is, assuming that $\{C_i;\ i \geq 1\}$ is a sequence of iid censoring rv with common unknown df G, we observe only the n pairs $\{(Y_i, \delta_i),$ $i = 1, 2, \ldots, n\}$, with $Y_i = T_i \wedge C_i$ and $\delta_i = \mathbb{I}_{\{T_i \leq C_i\}}$ (where $\mathbb{I}_A$ denotes the indicator function of the set A). We will suppose that T and C are independent to ensure the identifiability of the model.

Let X be a real-valued rv and $F(.|.)$ be the conditional df of T given $X = x$, that is,

$$F(t \mid x) = \mathbb{E}[\mathbb{I}_{\{T \leq t\}} \mid X = x] \tag{1}$$

which can be written as $F(t \mid x) =: \dfrac{F_1(t, x)}{\ell(x)}$, where ℓ is the marginal density of X with respect to Lebesgue measure.

We observe $\{(Y_i, \delta_i, X_i), i = 1, 2, \ldots, n\}$. Now, for any df L, let $\tau_L =$ $\sup\{y, L(y) < 1\}$ be its right endpoint.

Let $p \in (0, 1)$. Then the conditional quantile is defined by

$$\xi_p(x) = \inf\{t : F(t \mid x) \geq p\}. \tag{2}$$

We consider the estimation of the parameter $\xi_p(x)$ which satisfies

$$F(\xi_p(x) \mid x) = p. \tag{3}$$

In this paper we propose a new smooth estimator of the conditional quantile. Simulation study comes to show the well behavior and check the efficiency of our estimator. The remainder of the paper is as follows. In Section 2 we define a new kernel conditional quantile estimator in the censorship model with some notations. In Section 3 we present the assumptions which allow us to get asymptotic results and we give the main results. Some applications and examples are given in Section 4. Simulation results are presented in Section 5. Finally, the proofs of the main results are relegated to Section 6 with some auxiliary results and their proofs.

2. Definition of the Estimator

Throughout this paper we assume that

$$\tau_F < \tau_G, \tag{4}$$

$$C \text{ and } (T, X) \text{ are independent.} \tag{5}$$

Remark 1. In view of (4) and as we need to prove some uniform results which imply a sufficient rate of convergence of G_n (see Lemma 2), we have to consider a set of values of Y_i which do not include τ_G (because a uniform rate for G_n is obtained only for $t < \tau = \min(\tau_F, \tau_G)$, see Deheuvels and Einmahl [11, formula 4.28]).

Condition (5) is plausible whenever the censoring is independent of the modality of the patients. This condition is slightly stronger than the one usually used, that is: T and C are independent conditionally given X (see, e.g., Dabrowska [8, 9, 10]). However, condition (5) is very useful to give an unbiased estimator of $F_1(t, x)$ (when the weights are uniform) which intervenes in our methodology of construction of the estimate of $F(t \mid x)$ (see below).

It is clear that an estimator of (2) will be obtained by estimating the conditional distribution function (1), thus it suffices to estimate $F_1(t, x)$ and $\ell(x)$. The density $\ell(x)$ is not affected by the censoring and therefore can be estimated consistently by the well-known kernel estimator. Furthermore, an unbiased estimator of $F_1(t, x) = \mathbb{E}(\mathbb{I}_{\{T \leq t, X \leq x\}})$ is given by an average mean $\hat{F}_{1,n}(t, x) = \frac{1}{n} \sum_{i=1}^{n} \frac{\delta_i}{\overline{G}(Y_i)} \mathbb{I}_{\{Y_i \leq t, X_i \leq x\}}$.

Indeed, using the properties of the conditional expectation and (5), we get

$$\mathbb{E}[\hat{F}_{1,n}(t, x)] = \mathbb{E}\left[\frac{\delta_1}{\overline{G}(Y_1)} \mathbb{I}_{\{Y_1 \le t, X_1 \le x\}}\right]$$

$$= \mathbb{E}\left[\mathbb{E}\left[\frac{\delta_1}{\overline{G}(Y_1)} \mathbb{I}_{\{Y_1 \le t, X_1 \le x\}} \mid X_1, T_1\right]\right]$$

$$= \mathbb{E}\left[\frac{\mathbb{I}_{\{T_1 \le t, X_1 \le x\}}}{\overline{G}(Y_1)} \mathbb{E}[\mathbb{I}_{\{T_1 \le C_1\}} \mid X_1, T_1]\right]$$

$$= \mathbb{E}[\mathbb{I}_{\{T_1 \le t, X_1 \le x\}}] = F_1(t, x).$$

Now, instead using the uniform weight $\frac{1}{n}$ for all $\{Y_i, \delta_i, X_i\}$ we use the Nadaraya-Watson weight

$$W_{i,n}(x) = \frac{K\left(\frac{x - X_i}{h_n}\right)}{\sum_{i=1}^{n} K\left(\frac{x - X_i}{h_n}\right)}$$

$$= \frac{\frac{1}{nh_n} K\left(\frac{x - X_i}{h_n}\right)}{\ell_n(x)}, \tag{6}$$

where K is a probability density function (so called kernel function), $h_n =: h$ is a sequence of positive real numbers which goes to zero as n goes to infinity (so called bandwidth) and $\ell_n(.)$ is the well-known kernel estimator of $\ell(.)$.

Constructing an appropriate estimator is then obtained by adapting the weight (6), in order to put more emphasis on large values of the interest variable T which are more censored than small values.

Ould-Saïd [26] considered the following weights

$$W_{i,n}(x) = \frac{1}{nh} K\left(\frac{x - X_i}{h}\right) \frac{\delta_i}{\overline{G}(Y_i)\ell_n(x)},$$

and he established a strong uniform convergence of the corresponding kernel conditional quantile estimator; but this estimator is not derivable then we cannot establish its asymptotic normality.

Let us define a smooth estimate of (1) (by substituting the step function $\mathbb{I}_{\{.\}}$ by a smooth df $H(.)$) by

$$\widetilde{F}_n(t \mid x) = \frac{\sum_{i=1}^{n} \frac{\delta_i}{\overline{G}(Y_i)} K\left(\frac{x - X_i}{h}\right) H\left(\frac{t - Y_i}{h}\right)}{\sum_{i=1}^{n} K\left(\frac{x - X_i}{h}\right)}. \tag{7}$$

Recall that (7) can be rewritten as:

$$\widetilde{F}_n(t \mid x) = \frac{\widetilde{F}_{1,n}(t, x)}{\ell_n(x)}, \tag{8}$$

where

$$\widetilde{F}_{1,n}(t, x) = \frac{1}{nh} \sum_{i=1}^{n} \frac{\delta_i}{\overline{G}(Y_i)} K\left(\frac{x - X_i}{h}\right) H\left(\frac{t - Y_i}{h}\right)$$

and

$$\ell_n(x) = \frac{1}{nh} \sum_{i=1}^{n} K\left(\frac{x - X_i}{h}\right).$$

In practice G is usually be unknown, hence it is impossible to use the estimator (7). Then we replace G by the Kaplan-Meier estimate G_n given by

$$1 - G_n(t) = \begin{cases} \prod_{i=1}^{n} \left(1 - \frac{1 - \delta_i}{n - i + 1}\right)^{\mathbb{I}\{Y_i \le t\}}, & \text{if } t < Y_{(n)}, \\ 0, & \text{if } t \ge Y_{(n)}. \end{cases}$$

Therefore the feasible estimator of the conditional df $F(. \mid .)$ is given by

$$F_n(t \mid x) = \frac{\sum_{i=1}^{n} \frac{\delta_i}{\overline{G}_n(Y_i)} K\left(\frac{x - X_i}{h}\right) H\left(\frac{t - Y_i}{h}\right)}{\sum_{i=1}^{n} K\left(\frac{x - X_i}{h_n}\right)} =: \frac{F_{1,n}(t, x)}{\ell_n(x)}, \tag{9}$$

where

$$F_{1,n}(t, x) = \frac{1}{nh} \sum_{i=1}^{n} \frac{\delta_i}{\overline{G}_n(Y_i)} K\left(\frac{x - X_i}{h}\right) H\left(\frac{t - Y_i}{h}\right).$$

Then a natural estimator of $\xi_p(.)$ is given by

$$\xi_{p,n}(x) = \inf\{t;\ F_n(t \mid x) \ge p\} \tag{10}$$

which satisfies

$$F_n(\xi_{p,n}(x) \mid x) = p. \tag{11}$$

We define the first partial derivative with respect to second component of $F_{1,n}(t, x)$ and $\widetilde{F}_{1,n}(t, x)$, respectively, by

$$\frac{\partial F_{1,n}(t, x)}{\partial t} = F'_{1,n}(x, t) = \frac{1}{nh^2} \sum_{i=1}^{n} \frac{\delta_i}{\overline{G}_n(Y_i)} K\left(\frac{x - X_i}{h}\right) H'\left(\frac{t - Y_i}{h}\right)$$

and

$$\frac{\partial \widetilde{F}_{1,n}(t, x)}{\partial t} = \widetilde{F}'_{1,n}(x, t) = \frac{1}{nh^2} \sum_{i=1}^{n} \frac{\delta_i}{\overline{G}(Y_i)} K\left(\frac{x - X_i}{h}\right) H'\left(\frac{t - Y_i}{h}\right),$$

where H' is the derivative of H.

The conditional density estimators are given by

$$f_n(t \mid x) = \frac{F'_{1,n}(t, x)}{\ell_n(x)}$$

and

$$\widetilde{f}_n(t \mid x) = \frac{\widetilde{F}'_{1,n}(t, x)}{\ell_n(x)}.$$

In order to study the behavior of the random variable $(\xi_p(x) - \xi_{p,n}(x))$, we make use of the properties in (3) and (11) and Taylor expansion, we get

$$F(\xi_p(x) \mid x) - F(\xi_{p,n}(x) \mid x) = (\xi_p(x) - \xi_{p,n}(x)) f(\xi^*_{p,n}(x) \mid x), \tag{12}$$

where $\xi^*_{p,n}(x)$ lies between $\xi_p(x)$ and $\xi_{p,n}(x)$. It is clear that equation (12) shows that from the behavior of $(F(\xi_p(x) \mid x) - F(\xi_{p,n}(x) \mid x))$, it is easy to obtain asymptotic results for the sequence $(\xi_p(x) - \xi_{p,n}(x))$. If $f(\xi^*_{p,n}(x) \mid x) \neq 0$ was not satisfied, then we should have increased the order of Taylor expansion.

3. Assumptions and Main Results

Our assumptions are gathered together for easy references.

Let $\Omega_0 = \{x \in \mathbb{R} / \ell(x) > 0\}$, Ω be a compact set such that $\Omega \subset \Omega_0$ and let $\mathcal{C}$ be a compact set included in $]-\infty, \tau_F[$.

A1. The kernel K satisfies:

(i) K is strictly positive valued, bounded, with compact support and there exist constants M^* and m^* such that $0 < M^* < \infty$, $0 < m^* < \infty$, $\sup K(u) = M^*$ and $\inf K(u) = m^*$,

(ii) K is Hölderian of order γ for some $\gamma > 0$,

(iii) $\int_{\mathbb{R}} uK(u)du = 0$, $\int_{\mathbb{R}} |u| K(u)du < +\infty$ and $\int_{\mathbb{R}} u^2 K(u)du < \infty$,

(iv) $\int_{\mathbb{R}} K^2(u)du = \kappa < +\infty$.

A2. The bandwidth h satisfies:

(i) $\sqrt{\dfrac{\log \log n}{n}} = o(h^2)$,

(ii) $nh^5 \to 0$.

A3. The conditional distribution function $F(t \mid x)$ has positive first derivative with respect to t, for all $x \in \Omega$, denoted $f(t \mid x)$ and satisfies:

(i) The Lipschitz condition of order 1 with respect to t and x, such that

$$\forall (t_1, t_2) \in \mathbb{R}^2, \quad \forall (x_1, x_2) \in \Omega^2,$$

$$| F(t_1 \mid x_1) - F(t_2 \mid x_2) | \leq C(| x_1 - x_2 | + | t_1 - t_2 |),$$

(ii) $\int_{\mathbb{R}} |t| f(t \mid x)dt < +\infty$, for all $x \in \Omega$ (that is, the conditional density $f(.|.)$ has a finite first moment),

(iii) There exists a constant $\gamma_1 > 0$ such that $f(t \mid x) > \gamma_1$ for all $x \in \Omega$ and $t \in \mathcal{C}$.

A4. The distribution function H has a first derivative H' which is positive and bounded such that:

(i) There exist two constants $0 < M < \infty$ and $0 < m < \infty$, $\sup_{\mathbb{R}} H'(t) = M$ and $\inf_{\mathbb{R}} H'(t) = m$,

(ii) $\int_{\mathbb{R}} H'(t)dt = 1$ and $\int_{\mathbb{R}} |t| H'(t)dt < +\infty$.

A5. The marginal density $\ell(.)$ satisfies the Lipschitz condition and there exists $\gamma_0 > 0$ such that $\ell(x) > \gamma_0$ for all $x \in \Omega$.

A6. The distribution function of the censored rv, G has bounded first derivative g.

Comments on the assumptions.

Assumptions A1(i), (ii) *and* (iii) *are quite usual in kernel estimation. Condition* A1(iv) *intervenes in the variance terms of* (9) *and* (10). *Assumption* A2(i) *is needed in the study of the behavior of the differences* $(F_{1,n} - \widetilde{F}_{1,n})$ *and* $(f_n(.|.) - \widetilde{f}_n(.|.))$, *and in the proof of the convergence to zero of the bias term of* $\widetilde{F}_n(.,.)$ *in Lemma* 3. *Assumption* A3(i) *is put for technical convenience. Assumption* A4(i) *will be used in the proof of the asymptotic normality. Assumption* A5 *intervenes in the convergence of the kernel density estimator* ℓ_n. A3(ii) *and* A6 *are additional assumptions to get the asymptotic variance term.*

Our first result deals with the uniform almost sure convergence with rate of the conditional df estimator (9) and is stated in Proposition 1. The uniform almost sure convergence of the conditional quantile estimator (10) and its rate will be given in Theorem 1. Next, in Proposition 2, we state the asymptotic normality of the estimator (9) suitably normalized, then it suffices to prove the asymptotic normality of $(F_n(\xi_{n,p}(x)|x) - F_n(\xi_p(x)|x))$ and the convergence in probability of the sequence $f_n(\xi^*_{p,n}(x)|x)$ to obtain the asymptotic normality of the estimator of the conditional quantile ξ_p, by Slutsky's theorem.

Proposition 1. *Under assumptions* A1(i)-(iii), A2(i), A3(i), A4 *and* A5, *and for n large enough, we have*

$$\sup_{x\in\Omega}\sup_{t\in\mathcal{C}} | F_n(t|x) - F(t|x) | = O\left(\max\left\{\sqrt{\frac{\log\log n}{nh}}, h\right\}\right) \quad a.s. \tag{13}$$

Theorem 1. *Under assumptions* A1(i)-(iii), A2(i), A3(i), (iii), A4 *and* A5, *if the conditional density satisfies* $\inf_{x \in \Omega} f(\xi_p(x) | x) > 0,$ *then for n large enough*, *we have*

$$\sup_{x \in \Omega} | \xi_{p,n}(x) - \xi_p(x) | = O\left(\max\left\{ \left(\frac{\log \log n}{nh} \right)^{1/2}, h \right\} \right) \quad a.s. \tag{14}$$

Remark 2. The uniform positiveness assumption on the conditional density (in Theorem 1) implies the uniform unicity of the conditional quantile, that is,

$$\forall \varepsilon > 0, \quad \exists \beta > 0, \quad \forall \eta_p : \Omega \to \mathbb{R},$$

$$\sup_{x \in \Omega} | \xi_p(x) - \eta_p(x) | \geq \varepsilon \Rightarrow \sup_{x \in \Omega} | F(\xi_p(x) | x) - F(\eta_p(x) | x) | \geq \beta. \tag{15}$$

On the other hand, assuming the sole (15) guarantees the consistency of the conditional quantile but permits us to obtain a rate of convergence.

Remark 3. If we choose $h = O\left(\frac{\log \log n}{nh} \right)^{1/5}$ which is the optimal bandwidth in density estimation, then for each fixed $p \in (0, 1)$ and for n large enough, we have

$$\sup_{x \in \Omega} | \xi_{p,n}(x) - \xi_p(x) | = O\left(\frac{\log \log n}{nh} \right)^{1/5}.$$

Under the following mild modifications upon assumptions:

A2′. (i) $\sqrt{\frac{\log \log n}{n}} = o(h^3),$

A3′. (i) the df $F(.|.)$ satisfies the Lipschitz condition of order 2 with respect to t and x,

A4′. (ii) $\int_{\mathbb{R}} | t |^2 H'(t) dt < +\infty,$

we get the following rate $O\left(\max\left\{ \sqrt{\frac{\log \log n}{nh}}, h^2 \right\} \right)$ a.s. If we choose $h = O\left(\frac{\log \log n}{nh} \right)^{1/5}$, then we have $\sup_{x \in \Omega} | \xi_{p,n}(x) - \xi_p(x) | = O\left(\frac{\log \log n}{nh} \right)^{2/5}$, which is the rate expected by Mehra et al. [25] and the same rate obtained by Xiang [40] and Ould-Saïd [26]. Here we point out that in Gannoun et al. [15], there is neither uniform result nor rate of convergence. Recall that in this case assumption A2(ii) must be reinforced to get the asymptotic normality.

Remark 4. A generalization of the results to higher dimensions for the covariates, that is, $X \in \mathbb{R}^s$, by adapting the assumptions A1-A2, is straightforward and, for example, (14) becomes:

$$\sup_{x \in \Omega} | \xi_{p,n}(x) - \xi_p(x) | = O\left(\max\left\{ \sqrt{\frac{\log \log n}{nh^s}},\ h \right\} \right) \text{ a.s.}$$

Remark 5. In the proof of Proposition 1 (and therefore Theorem 1), we use Hoeffding's inequality which is a pointwise exponential inequality and we use a standard idea about covering a compact set by a finite number of intervals to get the uniformity. Another interesting idea is to use Vapnik-Cervonenkis theory by using Pollard's inequality (see Devroye et al. [12, Theorem 29.1]) together with bounds on the covering number which gives the uniformity straightforwardly. However, the choice of the kernels in the last case must satisfy K1 condition of Giné and Guillou [17] which is more restrictive than the Hölderian functions. In this case, one finds a rate of convergence of the order $\left(\frac{\log n}{nh^2} \right)^{1/2}$, which is less good than our result.

The following results deal with the asymptotic normality.

Proposition 2. *Under assumptions* A1-A6 *and for any* $x \in \Omega_0$ *such that* $\ell(x) > 0$, *we have*

$$(nh)^{1/2}(F_n(t \mid x) - F(t \mid x)) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma^2(t, x)) \quad as \quad n \to \infty, \tag{16}$$

where $\xrightarrow{\mathcal{D}}$ *denotes the convergence in distribution, and*

$$\sigma^2(t, x) = \frac{\kappa F(t \mid x)(1 - \overline{G}(t) F(t \mid x))}{\ell(x) \overline{G}(t)}.$$

Theorem 2. *Let* $p \in (0, 1)$ *such that* $p = F(\xi_p(x) \mid x)$. *Then under* *assumptions* A1-A6, *for any* $x \in \Omega_0$ *such that* $\Sigma(x, \xi_p(x)) \neq 0$, *we have*

$$\left(\frac{nh}{\Sigma^2(x, \xi_p(x))}\right)^{1/2}(\xi_p(x) - \xi_{p,n}(x)) \xrightarrow{\mathcal{D}} \mathcal{N}(0, 1) \quad as \quad n \to \infty, \tag{17}$$

where

$$\Sigma^2(x, \xi_p(x)) = \frac{\kappa p(1 - p\overline{G}(\xi_p(x)))}{f^2(\xi_p(x) | x)\ell(x)\overline{G}(\xi_p(x))}.$$

4. Applications

4.1. Applications to prediction

It is well-known, from the robustness theory, that the median is more robust than the mean, therefore the conditional median, $\mu(x) = \xi_{1/2}(x)$, is a good alternative to the conditional mean as a predictor for a variable Y given $X = x$. Note that the estimation of $\mu(x)$ is given by $\mu_n(x) = \xi_{\frac{1}{2},n}(x)$. Using this consideration and Section 3, we want to predict the non-observed rv Y_{n+1} (which corresponds to some modality of our problem), from available data $X_1, \ldots, X_n$. Given a new value X_{n+1}, we can predict the corresponding response Y_{n+1} by

$$Y_{n+1} = \mu_n(X_{n+1}) = \xi_{\frac{1}{2},n}(X_{n+1}).$$

Applying the above theorem, we have the following corollary

Corollary 1.

$$\left(\frac{nh}{\Sigma^2(X_{n+1}, \xi_{1/2}(X_{n+1}))}\right)^{1/2}(\xi_{1/2,n}(X_{n+1}) - \xi_{1/2}(X_{n+1})) \xrightarrow{\mathcal{D}} \mathcal{N}(0, 1)$$

$$as \quad n \to \infty.$$

4.2. Confidence intervals

Using a plug-in method by replacing ℓ, $\overline{G}$, f and $\xi_{1/2}$ by their estimates ℓ_n, $\overline{G}_n$, f_n and $\xi_{1/2,n}$, respectively, permits us to obtain a convergent estimate Σ_n of Σ, then we get from Corollary 1

Corollary 2.

$$\left(\frac{nh}{\Sigma_n^2(x,\ \xi_{1/2,n}(x))}\right)^{1/2}(\xi_{1/2,n}(x)-\xi_{1/2}(x)) \xrightarrow{\mathcal{D}} \mathcal{N}(0,1) \quad as \quad n\to\infty. \tag{18}$$

From this corollary, we get for each fixed $\eta \in (0, 1)$, the following approximate $(1-\eta)\%$ confidence interval

$$\xi_{1/2,n}(x) \pm t_{1-\eta/2} \times \frac{\Sigma_n(x,\ \xi_{1/2,n}(x))}{\sqrt{nh}},$$

where $t_{1-\eta/2}$ denotes the $1-\eta/2$ quantile of the standard normal distribution.

5. Simulation Studies

We have conducted a numerical study to examine the performance of our estimator. The first subsection deals with the consistency of the conditional quantile estimator $\xi_{p,n}(x)$ given in (10), whereas the second looks at how good the asymptotic normality is when we deal with a finite sample.

5.1. Consistency

The aim of the following simulations is to examine the performance of our estimator $\xi_{p,n}(x)$ in some particular model. In our simulation, we consider the following model: $T_i = X_i + \sigma\varepsilon_i$, $i = 1, \ldots, n$, where X_i and ε_i are two independent iid sequences distributed as $N(0, 1)$ and σ is a positive constant. The censoring times $\{C_i;\ i = 1, \ldots, n\}$ are generated independently from $N(0, 1)$. Then we compute our estimator with the observed data (X_i, Y_i, δ_i), where $Y_i = T_i \wedge C_i$ and $\delta_i = \mathbb{I}_{\{T_i \le C_i\}}$.

We choose a gaussian kernel and it is well-known that, in nonparametric estimation, optimality (in the MSE sense) is not seriously swayed by the choice of the kernel K but is affected by the choice of the bandwidth h. The bandwidth h is chosen according to the assumption A2, that is, $h = C\left(\frac{\log\log n}{n}\right)^{\frac{1}{4}}$, with an appropriate choice of the constant C.

Furthermore, we choose H as the normal distribution function. We take several values of n and in each case, the conditional quantile estimator, along a grid of 160 equispaced points in $[-2, 2]$ has been calculated. We take several values of n and draw two curves corresponding to the conditional quantile estimator $\xi_{p,n}(x)$ for $x \in [-2, 2]$ and the theoretical conditional quantile $\xi_p(x)$ for $\sigma = 0.45$ and for the two values $p = 1/2$ and $p = 1/4$.

The curves show that our estimator performs well in particular when n increases.

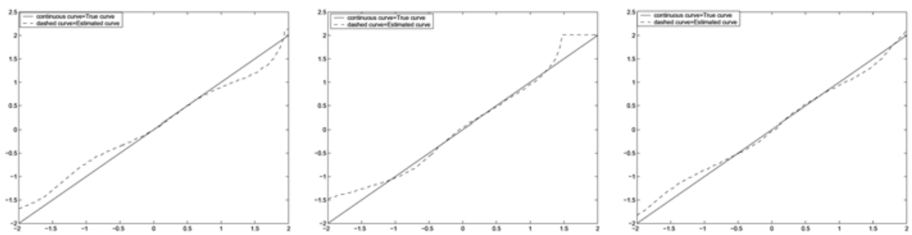

Figure 1. $p = 1/2$; $n = 50$, $n = 100$ and 500, respectively.

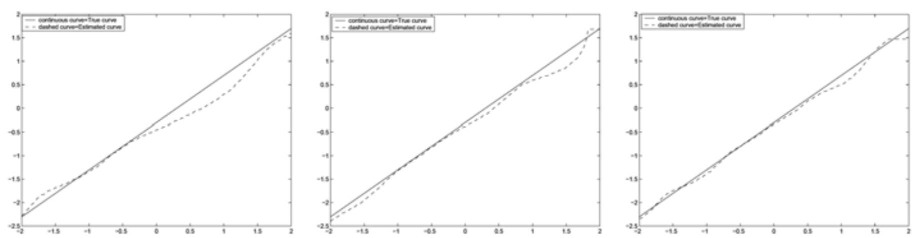

Figure 2. $p = 1/4$; $n = 50$, $n = 100$ and $n = 500$, respectively.

5.2. Asymptotic normality

We now consider the problem of asymptotic normality. We show how good the normality is when dealing with samples of finite size which is the case in practice. The data arise from the same distribution as previously for a given size n, we estimate the conditional quantile function as before and calculate the normalized deviation between this estimate and the theoretical conditional quantile for $x = 0$ and

$$p = 1/2; \;\; \overline{\xi}_{p,n}(x) = \sqrt{nh}\left(\frac{\kappa p(1 - p\overline{G}_n(\xi_p(x))}{f_n^2(\xi_p(x)\,|\,x)\ell_n(x)\overline{G}_n(\xi_p(x))}\right)^{-1/2}(\xi_{p,n}(x) - \xi_p(x))$$

which becomes $\overline{\xi}_{p,n}(0) = \sqrt{nh}\left(\frac{\kappa\frac{1}{2}\left(1 - \frac{1}{2}\overline{G}_n(0)\right)}{f_n^2(0\,|\,0)\ell_n(0)\overline{G}_n(0)}\right)^{-1/2}\left(\xi_{\frac{1}{2},n}(x) - 0\right).$

We draw, using this scheme, B independent n-samples. The bandwidth h is chosen according to assumption A2. In order to estimate the density function of $\overline{\xi}_{1/2,n}(x)$ (by the kernel method), we make the classical bandwidth choice (see, e.g., Silverman [31, p. 40]) $h' = Cn^{-1/5}$, where the constant C is appropriately chosen.

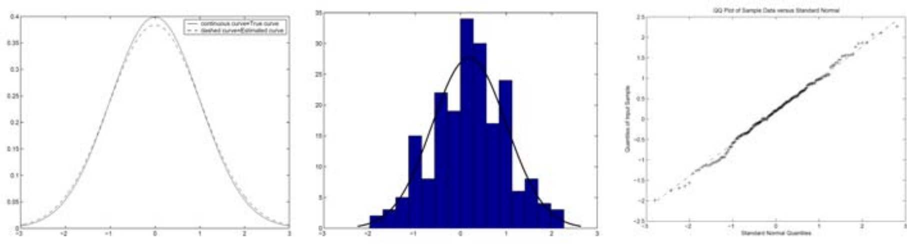

Figure 3. $n = 100$, $B = 200$.

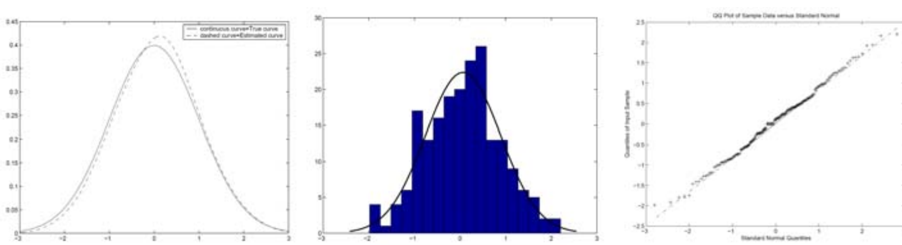

Figure 4. $n = 300$, $B = 200$.

We see that the normality of the rv $\overline{\xi}_{1/2,n}(.)$ is better as n increases, which clearly appears in the Q.Q. plot.

5.3. Confidence curves

We construct approximate $(1 - \eta)\%$ confidence curves obtained from

Corollary 2, that is,

$$\overline{\xi}_{1/2,n}(x) \pm t_{1-\eta/2} \times \frac{\Sigma_n(x_n \xi_{1/2,n}(x))}{\sqrt{nh}}.$$

We draw in Figure 5, for several values of n, two curves corresponding to the approximate $(1-\eta/2)\%$ lower and upper confidence curves for $p = 1/2$, $x \in [-2, 2]$, $\sigma = 0.45$ and $\eta = 5\%$. The bandwidth h is chosen as in Subsection 5.1.

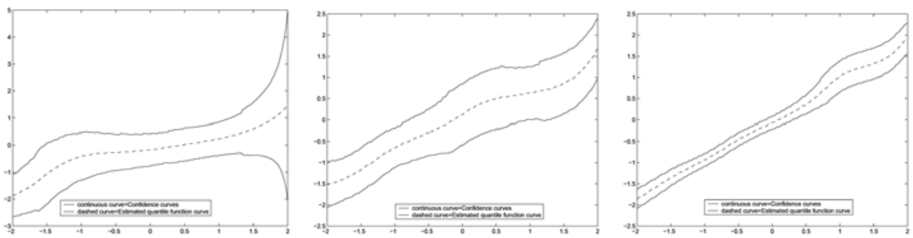

Figure 5. $p = 1/2$; $N = 50$, $N = 100$ and $n = 500$, respectively.

It is clear, from Figure 5, that the confidence interval becomes more precise as the sample size increases.

6. Auxiliary Results and Proofs

The proof of our main results is split up into several lemmas. The first lemma deals with the behavior of the difference between $\mathbb{E}[\widetilde{F}_{1,n}(t, x)]$ and $F_1(t, x)$.

Lemma 1. *Under assumptions* A1(iii), A3(i), A4 *and* A5, *then for n large enough*, *we have*

$$\sup_{x\in\Omega}\sup_{x\in\mathcal{C}} | \mathbb{E}[\widetilde{F}_{1,n}(t, x)] - F_1(t, x) | = O(h) \quad a.s. \tag{19}$$

Proof.

$$\mathbb{E}(\widetilde{F}_{1,n}(t, x)) = \frac{1}{h}\mathbb{E}\left[\frac{\delta_1}{\overline{G}(Y_1)} K\left(\frac{x - X_1}{h}\right) H\left(\frac{t - Y_1}{h}\right)\right]$$

$$= \frac{1}{h}\mathbb{E}\left[K\left(\frac{x - X_1}{h}\right)\mathbb{E}\left[\frac{\delta_1}{\overline{G}(Y_1)} H\left(\frac{t - Y_1}{h}\right) | X_1\right]\right].$$

Moreover we have by integration by parts and changing variables,

$$\mathbb{E}\left[\frac{\delta_1}{\overline{G}(Y_1)} H\left(\frac{t-Y_1}{h}\right) \mid X_1\right] = \mathbb{E}\left[\mathbb{E}\left(\frac{\delta_1}{\overline{G}(Y_1)} H\left(\frac{t-Y_1}{h}\right) \mid T_1\right) \mid X_1\right]$$

$$= \mathbb{E}\left[H\left(\frac{t-T_1}{h}\right) \mid X_1\right]$$

$$= \int H\left(\frac{t-y}{h}\right) f(y \mid X_1) dy$$

$$= \int H'(z) F(t-zh \mid X_1) dz$$

$$= \int H'(z)[F(t-zh \mid X_1) - F(t \mid x)] dz + F(t \mid x),$$

by the first part of assumption A4.

Thus, we have

$$\mathbb{E}[\widetilde{F}_{1,n}(t, x)] = \frac{1}{h} \mathbb{E}\left[K\left(\frac{x-X_1}{h}\right) \int H'(z)[F(t-zh \mid X_1) - F(t \mid x)] dz\right]$$

$$+ \frac{F(t \mid x)}{h} \mathbb{E}\left[K\left(\frac{x-X_1}{h}\right)\right]$$

$$=: \mathcal{I}_1 + \mathcal{I}_2.$$

Making use of the first part of assumption A5, the second term $\mathcal{I}_2$ tends to $F_1(t, x)$ as n goes to infinity.

By assumptions A3(i) and A4(i), we have

$$\mathcal{I}_1 \leq \int_{\mathbb{R}} H'(z) | F(t-zh \mid X_1) - F(t \mid x) | dz$$

$$\leq C \int_{\mathbb{R}} H'(z) | h + zh | dz$$

$$\leq Ch + Ch \int_{\mathbb{R}} |z| H'(z) dz. \tag{20}$$

Making use of A4(ii) and A1(iii), it is clear that $\mathcal{I}_1 = O(h)$, this completes the proof of Lemma 1.

Remark 6. The last argument shows that

$$\mathbb{E}\left[H\left(\frac{y - Y_1}{h}\right) | X_1\right] - F(y | x) \to 0 \quad \text{as} \quad n \to \infty. \tag{21}$$

The second lemma deals with the behavior of the difference between $F_{1,n}$ and $\widetilde{F}_{1,n}$.

Lemma 2. *Under assumptions* A1(i) *and* A2(i), *then for n large enough, we have*

$$\sup_{x \in \Omega} \sup_{t \in \mathcal{C}} | F_{1,n}(x, t) - \widetilde{F}_{1,n}(t, x) | = O(h) \quad a.s. \tag{22}$$

Proof.

$$| F_{1,n}(t, x) - \widetilde{F}_{1,n}(t, x) |$$

$$\leq \frac{1}{nh} \sum_{i=1}^{n} \left| \delta_i K\left(\frac{x - X_i}{h}\right) H\left(\frac{t - Y_i}{h}\right) \right| \left| \frac{1}{\overline{G}_n(Y_i)} - \frac{1}{\overline{G}(Y_i)} \right|$$

$$\leq \frac{1}{h} \frac{M^*}{\overline{G}_n(\tau_F)\overline{G}(\tau_F)} \sup_{t \in \mathcal{C}} | G_n(t) - G(t) | \frac{1}{n} \sum_{i=1}^{n} \delta_i.$$

Since $\overline{G}(\tau_F) > 0$, in conjunction with the SLLN and the LIL on the censoring law (see formula 4.28 in Deheuvels and Einmahl [11]), we have

$$\sup_{x \in \Omega} \sup_{t \in \mathcal{C}} | F_{1,n}(x, t) - \widetilde{F}_{1,n}(t, x) | \leq \frac{cM^*}{\overline{G}^2(\tau_F)} \frac{1}{h} \left(\frac{\log \log n}{n}\right)^{1/2} \quad \text{a.s.},$$

where c is a positive constant. Assumption A2(i) concludes the proof.

Lemma 3. *Under assumptions* A1(i) *and* (ii), A2(i), *then we have*

$$\sup_{x \in \Omega} \sup_{t \in \mathcal{C}} | \widetilde{F}_{1,n}(t, x) - \mathbb{E}[\widetilde{F}_{1,n}(t, x)] | \to 0 \quad a.s. \quad as \quad n \to \infty.$$

Furthermore, for all n large enough,

$$\sup_{x \in \Omega} \sup_{t \in \mathcal{C}} | \widetilde{F}_{1,n}(t, x) - \mathbb{E}[\widetilde{F}_{1,n}(t, x)] | = O\left(\sqrt{\frac{\log \log n}{nh}}\right) \quad a.s. \tag{23}$$

Proof. As Ω and $\mathcal{C}$ are compact sets, then they can be covered by finite numbers s_n and d_n of intervals centered at x_k and t_j of length h^{λ} and h^{μ}, respectively, such that $\lambda \geq \frac{\gamma+3}{\gamma}$ and $\mu \geq 4$. Since Ω and $\mathcal{C}$ are bounded, there exist two constants A_1 and A_2 such that $s_n \leq A_1 h^{-\lambda}$ and $d_n \leq A_2 h^{-\mu}$.

Now put

$$k(x) = \arg\min_{k=1,2,\ldots,s_n} | x - x_k |$$

and

$$j(t) = \arg\min_{j=1,2,\ldots,d_n} | t - t_j |.$$

Thus we have the following decomposition

$$
\begin{aligned}
& | \widetilde{F}_{1,n}(t, x) - \mathbb{E}[\widetilde{F}_{1,n}(t, x)] | \\
& \leq \underbrace{\sup_{x\in\Omega} \sup_{t\in\mathcal{C}} | \widetilde{F}_{1,n}(t, x) - \widetilde{F}_{1,n}(t, x_{k(x)}) |}_{\mathcal{I}_1} \\
& \quad + \underbrace{\sup_{x\in\Omega} \sup_{t\in\mathcal{C}} | \widetilde{F}_{1,n}(t, x) - \widetilde{F}_{1,n}(t_{j(t)}, x_{k(x)}) |}_{\mathcal{I}_2} \\
& \quad + \underbrace{\sup_{x\in\Omega} \sup_{t\in\mathcal{C}} | \widetilde{F}_{1,n}(t_{j(t)}, x_{k(x)}) - \mathbb{E}[\widetilde{F}_{1,n}(t_{j(t)}, x_{k(x)})] |}_{\mathcal{I}_3} \\
& \quad + \underbrace{\sup_{x\in\Omega} \sup_{t\in\mathcal{C}} | \mathbb{E}[\widetilde{F}_{1,n}(t_{j(t)}, x_{k(x)})] - \mathbb{E}[\widetilde{F}_{1,n}(t, x_{k(x)})] |}_{\mathcal{I}_4} \\
& \quad + \underbrace{\sup_{x\in\Omega} \sup_{t\in\mathcal{C}} | \mathbb{E}[\widetilde{F}_{1,n}(t, x_{k(x)})] - \mathbb{E}[\widetilde{F}_{1,n}(t, x)] |}_{\mathcal{I}_5}.
\end{aligned}
$$

Concerning $\mathcal{I}_1$ and $\mathcal{I}_5$. By A1(ii) and the fact that is bounded, we have

$$\sup_{x\in\Omega} \sup_{t\in\mathcal{C}} | \widetilde{F}_{1,n}(t, x) - \widetilde{F}_{1,n}(t, x_{k(x)}) | \leq \frac{c}{\overline{G}(\tau_F)} h^{\gamma\lambda-(\gamma+1)}.$$

By assumption A2(ii) on h and the condition upon λ, we get

$$\sqrt{\frac{nh}{\log\log n}}\sup_{x\in\Omega}\sup_{t\in\mathcal{C}}|\widetilde{F}_{1,n}(t, x)-\widetilde{F}_{1,n}(t, x_{k(x)})| = o(1). \tag{24}$$

Concerning $\mathcal{I}_2$ and $\mathcal{I}_4$. In the same way and by the fact that K is bounded and A4 (which implies that H is Lipschitzian), we have

$$\sup_{x\in\Omega}\sup_{t\in\mathcal{C}}|\widetilde{F}_{1,n}(t, x_{k(x)})-\widetilde{F}_{1,n}(t_{j(t)}, x_{k(x)})| \leq \frac{c}{\overline{G}(\tau_F)}h^{\mu-2}.$$

The condition upon μ implies that

$$\sqrt{\frac{nh}{\log\log n}}\sup_{x\in\Omega}\sup_{t\in\mathcal{C}}|\widetilde{F}_{1,n}(t, x_{k(x)})-\widetilde{F}_{1,n}(t_{j(t)}, x_{k(x)})| = o(1). \tag{25}$$

Concerning $\mathcal{I}_3$, for all $\varepsilon > 0$, we have

$$\mathbb{P}\left\{\sup_{x\in\Omega}\sup_{t\in\mathcal{C}}|\widetilde{F}_{1,n}(t_{j(t)}, x_{k(x)})-\mathbb{E}[\widetilde{F}_{1,n}(t_{j(t)}, x_{k(x)})]| > \varepsilon\right\}$$

$$= \mathbb{P}\left\{\max_{k=1,2,\ldots,s_n}\max_{j=1,2,\ldots,d_n}|\widetilde{F}_{1,n}(t_{j(t)}, x_{k(x)})-\mathbb{E}[\widetilde{F}_{1,n}(t_{j(t)}, x_{k(x)})]| > \varepsilon\right\}$$

$$\leq s_n d_n \mathbb{P}\{|\widetilde{F}_{1,n}(t_{j(t)}, x_{k(x)})-\mathbb{E}[\widetilde{F}_{1,n}(t_{j(t)}, x_{k(x)})]| > \varepsilon\}. \tag{26}$$

Now, we have

$$\mathbb{E}[\widetilde{F}_{1,n}(t_{j(t)}, x_{k(x)})-\mathbb{E}[\widetilde{F}_{1,n}(t_{j(t)}, x_{k(x)})]] = 0$$

and

$$|\widetilde{F}_{1,n}(t_{j(t)}, x_{k(x)})-\mathbb{E}[\widetilde{F}_{1,n}(t_{j(t)}, x_{k(x)})]| \leq \frac{2M^*}{nh\overline{G}(\tau_F)}.$$

Hoeffding's inequality (cf. Shorack and Wellner [30, p. 855]) yields

$$\mathbb{P}\{|\widetilde{F}_{1,n}(t_{j(t)}, x_{k(x)})-\mathbb{E}[\widetilde{F}_{1,n}(t_{j(t)}, x_{k(x)})]| > \varepsilon\} \leq \exp\{-\varepsilon^2 cn^2h^2\},$$

where c is a universal constant. Then (26) becomes

$$\mathbb{P}\left\{\sup_{x\in\Omega}\sup_{t\in\mathcal{C}}|\widetilde{F}_{1,n}(t_{j(t)}, x_{k(x)}) - \mathbb{E}[\widetilde{F}_{1,n}(t_{j(t)}, x_{k(x)})]| > \varepsilon\right\}$$

$$\leq A_1A_2h^{-(\lambda+\mu)}\exp\{-c\varepsilon^2n^2h^2\}$$

$$= A_1A_2(nh)^{-(\lambda+\mu)}n^{\lambda+\mu-c\varepsilon^2n^2h^2/\log n}, \tag{27}$$

where c is a positive constant depending only on M^* and $\overline{G}(\tau_F)$.

A2(i) implies that $\frac{n^2h^2}{\log n} \to +\infty$, which yields that the last term of (27) is the general term of a convergent series; then by Borel-Cantelli's Lemma, the first term of (26) goes to zero almost surely. Otherwise, if we replace ε by $\varepsilon_0\sqrt{\frac{\log\log n}{nh}}$, for some $\varepsilon_0 > 0$ in all steps of lemma, then we have

$$\sqrt{\frac{nh}{\log\log n}}\sup_{x\in\Omega}\sup_{t\in\mathcal{C}}|\widetilde{F}_{1,n}(t_{j(t)}, x_{k(x)}) - \mathbb{E}[\widetilde{F}_{1,n}(t_{j(t)}, x_{k(x)})]| = o(1) \quad \text{a.s.}$$

in conjunction with (24) and (25) we complete the proof of the lemma.

Lemma 4. *Under assumptions* A1(iii), A2(i) *and* A4(i), *we have*

$$\operatorname{var}(\widetilde{F}'_{1,n}(t, x)) \to 0 \quad as \quad n \to \infty.$$

Proof. Recall that

$$\widetilde{F}'_{1,n}(t, x) = \frac{1}{nh^2}\sum_{i=1}^{n}\frac{\delta_i}{\overline{G}(Y_i)}K\left(\frac{x - X_i}{h}\right)H'\left(\frac{t - Y_i}{h}\right),$$

then

$$\operatorname{var}(\widetilde{F}'_{1,n}(t, x)) = \frac{1}{nh^4}\operatorname{var}\left[\frac{\delta_1}{\overline{G}(Y_1)}K\left(\frac{x - X_1}{h}\right)H'\left(\frac{t - Y_1}{h}\right)\right]$$

$$= \frac{1}{nh^4}\mathbb{E}\left[\frac{\delta_1}{\overline{G}^2(Y_1)}K^2\left(\frac{x - X_1}{h}\right)H'^2\left(\frac{t - Y_1}{h}\right)\right]$$

$$- \frac{1}{nh^4}\left[\mathbb{E}\left[\frac{\delta_1}{\overline{G}(Y_1)}K\left(\frac{x - X_1}{h}\right)H'\left(\frac{t - Y_1}{h}\right)\right]\right]^2$$

$$= \frac{1}{nh^4}\,\mathbb{E}\left[K^2\left(\frac{x - X_1}{h}\right)\mathbb{E}\left(H'^2\left(\frac{t - Y_1}{h}\right)\frac{\delta_1}{\overline{G}(Y_1)} \mid X_1\right)\right]$$

$$- \frac{1}{nh^4}\left[\mathbb{E}\left[K\left(\frac{x - X_1}{h}\right)\mathbb{E}\left(H'\left(\frac{t - Y_1}{h}\right)\frac{\delta_1}{\overline{G}(Y_1)} \mid X_1\right)\right]\right]^2$$

$$=: J_1 - J_2.$$

By assumptions A1(iii) and A4(i), we have

$$J_1 \le \frac{1}{nh^4}\,\frac{M^2}{\overline{G}^2(\tau_F)}\,\mathbb{E}\left[K^2\left(\frac{x - X_1}{h}\right)\right]$$

$$\le \frac{1}{nh^3}\,\frac{M^2}{\overline{G}^2(\tau_F)}\,\kappa\ell(x) + O\left(\frac{1}{nh^3}\right).$$

Then by A2(i), $J_1 \to 0$ as $n \to +\infty$.

For the same arguments as before, we have

$$J_2 \le \frac{1}{nh^4}\,\frac{M^2}{\overline{G}^2(\tau_F)}\left[\mathbb{E}\left[K\left(\frac{x - X_1}{h}\right)\right]\right]^2$$

$$= \frac{1}{nh^2}\,\frac{M^2}{\overline{G}^2(\tau_F)}\,\ell^2(x) + O\left(\frac{1}{nh^2}\right)$$

and J_2 tends again to zero as n goes to infinity. This completes the proof of Lemma 3.

Remark 7. Under the assumptions of Lemma 3, and making use of Tchebychev's inequality, we have

$$\widetilde{F}'_{1,n}(t, x) \to F'_1(t, x), \text{ in probability, as } n \to \infty$$

and thus if we add assumption A5, then

$$\widetilde{f}_n(t \mid x) = \frac{\widetilde{F}'_{1,n}(t, x)}{\ell_n(x)} \to f(t \mid x), \text{ in probability, as } n \to \infty. \tag{28}$$

The following lemma deals with the convergence of the conditional probability density estimator $f_n(t \mid x)$ to $f(t \mid x)$.

Lemma 5. *Under assumptions* A1(iii), A2(i), A4 *and* A5, *we have*

$$f_n(t \mid x) \to f(t \mid x), \quad \textit{in probability}, \quad \textit{as} \quad n \to \infty. \tag{29}$$

Proof. We have

$$\begin{aligned}
& f_n(t \mid x) - f(t \mid x) \\
&= (f_n(t \mid x) - \widetilde{f}_n(t \mid x)) + (\widetilde{f}_n(t \mid x) - f(t \mid x)), \\
& \mid f_n(t \mid x) - \widetilde{f}_n(t \mid x) \mid \\
&= \frac{1}{\ell_n(x)} \mid (F'_{1,n}(t, x) - \widetilde{F}'_{1,n}(t, x)) \mid \\
&= \frac{1}{\ell_n(x)} \frac{1}{nh^2} \left| \sum_{i=1}^{n} \delta_i K\left(\frac{x - X_i}{h}\right) H'\left(\frac{t - Y_i}{h}\right)\left(\frac{1}{\overline{G}_n(Y_i)} - \frac{1}{\overline{G}(Y_i)}\right) \right| \\
&\leq \frac{1}{\ell_n(x)} \frac{1}{h^2} \frac{M^* M}{\overline{G}_n(\tau_F) \overline{G}(\tau_F)} \sup_{t \in \mathcal{C}} \mid G_n(t) - G(t) \mid \frac{1}{n} \sum_{i=1}^{n} \delta_i .
\end{aligned}$$

Since $\ell_n(x) \geq \inf_x \ell_n(x) \geq \gamma > 0$ and $\overline{G}(\tau_F) > 0$, in conjunction with the SLLN and the LIL on the censoring law, we have

$$\mid f_n(t \mid x) - \widetilde{f}_n(t \mid x) \mid \leq \frac{cM^* M}{\overline{G}^2(\tau_F)} \frac{1}{h^2} \left(\frac{\log \log n}{n}\right)^{1/2} \quad \text{a.s.},$$

where c is a positive constant. Assumption A2(i) gives us that

$$\mid f_n(t \mid x) - \widetilde{f}_n(t \mid x) \mid \to 0 \quad \text{a.s.} \quad \text{as} \quad n \to \infty.$$

The proof can be concluded by Remark 6.

Proof of Proposition 1. In view of (9), we have

$$\sup_{x \in \Omega} \sup_{t \in \mathcal{C}} \mid F_n(t \mid x) - F(t \mid x) \mid$$

$$\leq \frac{1}{\gamma - \sup_{x\in\Omega}|\ell_n(x) - \ell(x)|}\left\{\sup_{x\in\Omega}\sup_{t\in\mathcal{C}}|F_{1,n}(x,t) - F_1(x,t)|\right.$$

$$\left. + \sup_{x\in\Omega}\sup_{t\in\mathcal{C}}|F(t|x)|\sup_{x\in\Omega}|\ell_n(x) - \ell(x)|\right\}. \quad (30)$$

The kernel estimator $\ell_n(x)$ is almost surely bounded away from 0 on Ω because of the second part of assumption A5.

Now, under assumptions A5 and A1(i)-(iii), we have

$$\sup_{x\in\Omega}|\mathbb{E}(\ell_n(x)) - \ell_n(x)| \to 0 \quad \text{as} \quad n \to \infty.$$

Using Taylor's expansion up to the first order to $\ell(.)$, for n large enough, we have

$$\sup_{x\in\Omega}|\mathbb{E}(\ell_n(x)) - \ell(x)| = O(h). \quad (31)$$

Furthermore, under A1(i), (ii), (iii) and A2(i), an analogous proof as in Lemma 3 (by taking $\Delta_i(x) = \frac{1}{nh}K\left(\frac{x - X_i}{h}\right) - \mathbb{E}\left[\frac{1}{nh}K\left(\frac{x - X_i}{h}\right)\right]$ and by replacing ε by $\varepsilon_0\left(\frac{\log\log n}{nh}\right)^{1/2}$, for some $\varepsilon_0 > 0$) yields to

$$\sup_{x\in\Omega}|\mathbb{E}(\ell_n(x)) - \ell_n(x)| = O\left(\left(\frac{\log\log n}{nh}\right)^{1/2}\right) \text{ a.s.} \quad (32)$$

Then the proposition can be obtained straightforwardly from (30)-(32), Lemmas 1, 2 and 3.

Proof of Theorem 1. Let $x \in \Omega$. As $F(.|x)$ and $F_n(.|x)$ are continuous, we have $F(\xi_p(x)|x) = F_n(\xi_{p,n}(x)|x) = p$. Then

$$|F(\xi_{p,n}(x)|x) - F(\xi_p(x)|x)| = |F(\xi_{p,n}(x)|x) - F_n(\xi_{p,n}(x)|x)|$$

$$\leq \sup_{t\in\mathcal{C}}|F_n(t|x) - F(t|x)|. \quad (33)$$

Then the consistency of $\xi_{n,p}(x)$ follows immediately from Proposition 1 and continuity of $F(.|x)$. Now, a Taylor expansion of $F(.|x)$ in a neighborhood of $\xi_p(x)$, implies that

$$F(\xi_{p,n}(x)|x) - F(\xi_p(x)|x) = (\xi_{p,n}(x) - \xi_p(x))f(\xi_p^*(x)|x),$$

where $\xi_p^*(x)$ is between $\xi_p(x)$ and $\xi_{n,p}(x)$. Then, by (33), we have

$$\sup_{t\in\Omega}|\xi_{p,n}(x) - \xi_p(x)||f(\xi_p^*(x)|x)| \leq \sup_{x\in\Omega}\sup_{t\in\mathcal{C}}|F_n(t|x) - F(t|x)|.$$

Then the result is a consequence of Proposition 1 and the assumption of $f(\xi_p(.)|.)$ being uniformly bounded away from zero.

In order to prove Proposition 2, we use the following decomposition and the lemmas below:

$$\widetilde{F}_n(t|x) - F(t|x)$$

$$= \frac{\ell(x)}{\ell_n(x)}\left[\frac{\widetilde{F}_{1,n}(t, x) - \mathbb{E}(\widetilde{F}_{1,n}(t, x)) - F(t|x)[\ell_n(x) - \mathbb{E}(\ell_n(x))]}{\ell_n(x)}\right]$$

$$- \frac{1}{\ell_n(x)}[F_1(t, x) - \mathbb{E}(\widetilde{F}_n(t|x)) - F(t|x)[\ell(x) - \mathbb{E}(\ell_n(x))]].$$

Note that

$$(nh)^{1/2}(F_n(t|x) - F(t|x)) = \frac{\ell(x)}{\ell_n(x)}((nh)^{1/2}A_n(t, x)) + (nh)^{1/2}C_n(t, x), \quad (34)$$

where

$$A_n(t, x) = \frac{\widetilde{F}_{1,n}(t, x) - \mathbb{E}(\widetilde{F}_{1,n}(t, x)) - F(t|x)[\ell_n(x) - \mathbb{E}(\ell_n(x))]}{\ell(x)},$$

$$B_n(t, x) = \frac{F_1(t, x) - \mathbb{E}(\widetilde{F}_n(t|x)) - F(t|x)[\ell(x) - \mathbb{E}(\ell_n(x))]}{\ell_n(x)}$$

and

$$C_n(t, x) = (F_n(t|x) - \widetilde{F}_n(t|x)) - B_n(t, x).$$

The next lemmas show the asymptotic normality of $(nh)^{1/2}A_n(x, y)$ and the convergence in probability of $(nh)^{1/2} C_n(x, y)$ to zero. We begin to show the second part:

Lemma 6. *Under assumptions* A1(i)-(iii), A2, A3(i), A4 *and* A5, *we have*

$$(nh)^{1/2} C_n(x, y) \to 0, \quad \textit{in probability}, \quad \textit{as} \quad n \to \infty.$$

Proof. Let us use the following decomposition:

$$(nh)^{1/2} C_n(x, y) = \frac{(nh)^{1/2}}{\ell_n(x)} [(F_{1,n}(t, x) - \widetilde{F}_{1,n}(t, x))$$

$$- (F_1(t, x) - \mathbb{E}(\widetilde{F}_{1,n}(t, x))) + F(t \mid x)(\ell(x) - \ell_n(x))].$$

On the one hand, making use of Lemmas 1 and 2 we have that each term in brackets goes to zero in probability as n goes to infinity.

On the other hand,

$$\frac{(nh)^{1/2}}{\ell_n(x)} = \frac{(nh)^{1/2} \widetilde{f}_n(t \mid x)}{\ell_n(x) \widetilde{f}_n(t \mid x)} = \frac{(nh)^{1/2} \widetilde{f}_n(t \mid x)}{\widetilde{F}'_{1,n}(t \mid x)},$$

where

$$\widetilde{F}'_{1,n}(t, x) = \frac{1}{nh^2} \sum_{i=1}^{n} K\left(\frac{x - X_i}{h}\right) H'\left(\frac{t - Y_i}{h}\right) \frac{\delta_i}{\overline{G}(Y_i)}.$$

By assumptions A1(i) and A4(i), we obtain that

$$\widetilde{F}'_{1,n}(t, x) \geq \frac{m^* m}{h^2} \frac{1}{n} \sum_{i=1}^{n} \delta_i.$$

Therefore

$$\frac{(nh)^{1/2}}{\ell_n(x)} \leq \frac{(nh^5)^{1/2}}{m^* m} \frac{\widetilde{f}_n(t \mid x)}{\frac{1}{n} \sum_{i=1}^{n} \delta_i}.$$

By A2(ii), Lemma 3 and in conjunction with the SLLN the right hand side of the last inequality goes to zero in probability as n goes to infinity, which completes the proof of Lemma 6.

Lemma 7. *Under assumptions* A1(iii), A3, A4, A5 *and* A6, *we have*

$$nh\mathrm{Var}[A_n(t, x)] \to (\sigma(t, x))^2 \quad as \quad n \to \infty.$$

Proof. Clearly, we have

$$A_n(t, x)$$

$$= \frac{1}{nh\ell(x)} \sum_{i=1}^{n} \left[K\left(\frac{x - X_i}{h}\right) H\left(\frac{t - Y_i}{h}\right) \frac{\delta_i}{\overline{G}(Y_i)} - \mathbb{E}\left[K\left(\frac{x - X_i}{h}\right) H\left(\frac{t - Y_i}{h}\right) \frac{\delta_i}{\overline{G}(Y_i)} \right] \right]$$

$$- F(t \mid x) \sum_{i=1}^{n} \left[K\left(\frac{x - X_i}{h}\right) - \mathbb{E}\left(K\left(\frac{x - X_i}{h}\right) \right) \right]$$

$$=: \frac{1}{nh\ell(x)} \sum_{i=1}^{n} N_i(t, x),$$

where

$$N_i(t, x) = K\left(\frac{x - X_i}{h}\right) \left[K\left(\frac{t - Y_i}{h}\right) \frac{\delta_i}{\overline{G}(Y_i)} - F(t \mid x) \right]$$

$$- \mathbb{E}\left[K\left(\frac{x - X_i}{h}\right) \left(H\left(\frac{t - Y_i}{h}\right) \frac{\delta_i}{\overline{G}(Y_i)} - F(t \mid x) \right) \right].$$

Then

$$nh\mathrm{Var}(A_n(t, x))$$

$$= \frac{1}{h\ell^2(x)} \mathrm{Var}(N_1(t, x))$$

$$= \frac{1}{h\ell^2(x)} \mathbb{E}(N_1^2(t, x))$$

$$= \frac{1}{h\ell^2(x)} \mathbb{E}\left[K^2\left(\frac{x - X_1}{h}\right) \left(\frac{\delta_1}{\overline{G}(Y_1)} H\left(\frac{t - Y_i}{h}\right) - F(t \mid x) \right)^2 \right]$$

$$- \frac{1}{h\ell^2(x)} \left[\mathbb{E}\left[K\left(\frac{x - X_1}{h}\right) \left(\frac{\delta_1}{\overline{G}(Y_1)} H\left(\frac{t - Y_1}{h}\right) - F(t \mid x) \right) \right] \right]^2. \qquad (35)$$

We can write the second term of the right hand side of (35) as:

$$\frac{1}{h\ell^2(x)}\left[\mathbb{E}\left[K\left(\frac{x-X_1}{h}\right)\left(\frac{\delta_1}{\overline{G}(Y_1)}H\left(\frac{t-Y_1}{h}\right)-F(t\,|\,x)\right)\right]\right]^2$$

$$=\frac{h}{\ell^2(x)}[\mathbb{E}(\widetilde{F}_{1,n}(t,\,x))-F(t\,|\,x)\mathbb{E}(\ell_n(x))]^2.$$

Using Lemma 1, we can conclude that the second term of (35) tends to zero as n goes to infinity. Now let us turn to the first term of (35), we have

$$\frac{1}{h\ell^2(x)}\mathbb{E}\left[K^2\left(\frac{x-X_1}{h}\right)\left(\frac{\delta_1}{\overline{G}(Y_1)}H\left(\frac{t-Y_1}{h}\right)-F(t\,|\,x)\right)^2\right]$$

$$=\frac{1}{h\ell^2(x)}\mathbb{E}\left[K^2\left(\frac{x-X_i}{h}\right)\mathbb{E}\left[\left(\frac{\delta_1}{\overline{G}(Y_1)}H\left(\frac{t-Y_i}{h}\right)-F(t\,|\,x)\right)^2\,|\,X_1\right]\right].$$

Using the definition of the conditional variance, we have

$$\mathbb{E}\left[\left(\frac{\delta_1}{\overline{G}(Y_1)}H\left(\frac{t-Y_1}{h}\right)-F(t\,|\,x)\right)^2\,|\,X_1\right]$$

$$=\mathrm{Var}\left[\frac{\delta_1}{\overline{G}(Y_1)}H\left(\frac{t-Y_1}{h}\right)|\,X_1\right]+\left[\mathbb{E}\left(\frac{\delta_1}{\overline{G}(Y_1)}H\left(\frac{t-Y_1}{h}\right)|\,X_1\right)-F(t\,|\,x)\right]^2$$

$$=: \mathcal{V}_1+\mathcal{V}_2.$$

Using Lemma 1, we have

$$\mathcal{V}_2=O(h^2)\quad\text{as}\quad n\to+\infty.$$

Let us now examine the term $\mathcal{V}_1$,

$$\mathcal{V}_1=\mathbb{E}\left[\frac{\delta_1}{\overline{G}^2(Y_1)}H^2\left(\frac{t-Y_1}{h}\right)|\,X_1\right]-\left[\mathbb{E}\left(\frac{\delta_1}{\overline{G}(Y_1)}H\left(\frac{t-Y_1}{h}\right)|\,X_1\right)\right]^2$$

$$=: \mathcal{J}_1+\mathcal{J}_2.$$

The first term of the last equality can be developed as follows:

$$\begin{aligned}
\mathcal{J}_1 &= \mathbb{E}\left[\mathbb{E}\left(H^2\left(\frac{t-Y_1}{h}\right)\frac{\delta_1}{\overline{G}^2(Y_1)}\,|\,T_1\right)|\,X_1\right] \\
&= \mathbb{E}\left[H^2\left(\frac{t-T_1}{h}\right)\frac{1}{\overline{G}(T_1)}\,|\,X_1\right] \\
&= \int_{\mathbb{R}} H^2\left(\frac{t-y}{h}\right)\frac{1}{\overline{G}(y)}\,f(y\,|\,X_1)dy \\
&= \int_{\mathbb{R}} H^2(z)\frac{1}{\overline{G}(t-hz)}\,dF(t-hz\,|\,X_1).
\end{aligned}$$

By Taylor's expansion, we have

$$\begin{aligned}
\mathcal{J}_1 &= \int_{\mathbb{R}} H^2(z)\frac{1}{\overline{G}(t)}\,dF(t-hz\,|\,X_1) \\
&\quad + \frac{h}{\overline{G}^2(t)}\int_{\mathbb{R}} zH(z)\overline{G}'(t^*)dF(t-hz\,|\,X_1) + o(1) \\
&=: \mathcal{J}'_1 + \mathcal{J}'_2, \qquad (36)
\end{aligned}$$

where t^* is between t and $t - hz$.

Under assumption A6, the second term of (36) can be bounded by

$$\mathcal{J}'_2 \le \frac{h^2 \sup_{x\in\mathbb{R}}|\,g(x)\,|}{\overline{G}^2(\tau_F)}\int_{\mathbb{R}} zf(t-hz\,|\,X_1)dz,$$

then under assumption A3(ii), we get that $\mathcal{J}'_2 = O(h^2)$.

On the other hand, by integrating by parts, we have

$$\begin{aligned}
\mathcal{J}'_1 &= \frac{1}{\overline{G}(t)}\int_{\mathbb{R}} 2H'(z)H(z)F(t-hz\,|\,X_1)dz \\
&= \frac{1}{\overline{G}(t)}\int_{\mathbb{R}} 2H'(z)H(z)(F(t-hz\,|\,X_1) - F(t\,|\,x))dz \\
&\quad + \frac{1}{\overline{G}(t)}\int_{\mathbb{R}} 2H'(z)H(z)F(t\,|\,x)dz.
\end{aligned}$$

Remark that $\int_{\mathbb{R}} 2H(z)H'(z)F(t\,|\,x)dz = F(t\,|\,x)$ and using the same idea as in (20), we get

$$\int_{\mathbb{R}} 2H'(z)H(z)(F(t - hz\,|\,X_1) - F(t\,|\,x))dz = O(h) \quad \text{as} \quad n \to +\infty$$

and

$$\int_{\mathbb{R}} H^2(z)\frac{1}{\overline{G}(t)}dF(t - hz\,|\,X_1) \to \frac{F(t\,|\,x)}{\overline{G}(t)} \quad \text{as} \quad n \to +\infty.$$

Therefore

$$\begin{aligned}\mathcal{V}_1 &= \frac{1}{\overline{G}(t)}\int_{\mathbb{R}} 2H'(z)H(z)F(t\,|\,x)dz \\ &\qquad + O(h^2) + O(h) - \left[\mathbb{E}\left(\frac{\delta_1}{\overline{G}(Y_1)}H\left(\frac{t - Y_1}{h}\right)|\,X_1\right)\right]^2 \\ &= \frac{1}{\overline{G}(t)}\int_{\mathbb{R}} 2H'(z)H(z)F(t\,|\,x)dz \\ &\qquad + O(h) - \left[\mathbb{E}\left(\frac{\delta_1}{\overline{G}(Y_1)}H\left(\frac{t - Y_1}{h}\right)|\,X_1\right)\right]^2.\end{aligned}$$

Then we get

$$\begin{aligned}&\lim_{n\to\infty} nh\mathrm{Var}(A_n(t,\, x)) \\ &= \lim_{n\to\infty}\frac{1}{h\ell^2(x)}\mathbb{E}\left[K^2\left(\frac{x - X_1}{h}\right)\left(\frac{1}{\overline{G}(t)}\int_{\mathbb{R}} 2H'(z)H(z)F(t\,|\,x)dz + O(h)\right)\right] \\ &\quad - \lim_{n\to\infty}\frac{1}{h\ell^2(x)}\mathbb{E}\left[K^2\left(\frac{x - X_1}{h}\right)\left[\mathbb{E}\left(\frac{\delta_1}{\overline{G}(Y_1)}H\left(\frac{t - Y_1}{h}\right)|\,X_1\right)\right]^2\right] \\ &= \frac{\kappa}{\ell(x)}\left[\frac{F(t\,|\,x)}{\overline{G}(t)} - (F(t\,|\,x))^2\right] \\ &= (\sigma(t,\, x))^2.\end{aligned}$$

Let $Z_{ni}(t, x) = \dfrac{N_i(t, x)}{(nh\ell^2(x))^{1/2}}$. Remark that

$$(nh)^{1/2} A_n(x, y) = \sum_{i=1}^{n} Z_{ni}(x, y),$$

so to prove the asymptotic normality of $(nh)^{1/2} A_n(t, x)$, it suffices to show that $\sum_{i=1}^{n} Z_{ni}(t, x)$ satisfies the Lindberg-Feller condition, which is given in the following lemma.

Lemma 8. *Under assumptions* A1(i) *and* (iii), A2(i), A3, A4, A5 *and* A6 *for all* $\varepsilon > 0$, *we have*

$$\sum_{i=1}^{n} \int_{\left\{Z_{ni}^2(t, x) > \varepsilon^2 \operatorname{Var}\left(\sum_{i=1}^{n} Z_{ni}(t, x)\right)\right\}} Z_{ni}^2(t, x) d\mathbb{P}_{(X_i, Y_i)} \to 0 \quad as \quad n \to \infty.$$

Proof. By Lemma 7, we have clearly

$$\operatorname{Var}\left(\sum_{i=1}^{n} Z_{ni}(t, x)\right) \to (\sigma(t, x))^2 \quad \text{as} \quad n \to \infty,$$

therefore for n large enough,

$$\left\{Z_{ni}^2(t, x) > \varepsilon^2 \operatorname{Var}\left(\sum_{i=1}^{n} Z_{ni}(t, x)\right)\right\} \subset \left\{Z_{ni}^2(t, x) > \frac{\varepsilon^2}{2} (\sigma(t, x))^2\right\}.$$

On the one hand, using the fact that $\left| H\left(\dfrac{t - Y_i}{h}\right) \dfrac{\delta_i}{\overline{G}(t)} - F(t \mid x) \right| \leq 2$ and condition A1(i), we have

$$\begin{aligned} Z_{ni}^2(x, y) &= \frac{N_i^2(t, x)}{nh} \\ &< \frac{2}{nh\ell^2(x)} \left[K\left(\frac{x - X_n}{h}\right) \left(H\left(\frac{t - Y_i}{h}\right) \frac{\delta_i}{\overline{G}(t)} - F(t \mid x) \right) \right]^2 \\ &\quad + \frac{2}{nh\ell^2(x)} \mathbb{E}^2 \left[K\left(\frac{x - X_n}{h}\right) \left(H\left(\frac{t - Y_i}{h}\right) \frac{\delta_i}{\overline{G}(t)} - F(t \mid x) \right) \right] \end{aligned}$$

$$\leq \frac{8}{nh}\left[K^2\left(\frac{x - X_n}{h}\right) + \mathbb{E}^2\left(K\left(\frac{x - X_n}{h}\right)\right)\right]$$

$$\leq \frac{16M^*}{nh\ell^2(x)}.$$

On the other hand, the second part of assumption A2(i) implies that $nh \to +\infty$ as $n \to +\infty$, therefore

$$Z_{ni}^2(t, x) \to 0 \quad \text{as} \quad n \to \infty,$$

then for n large enough, we have

$$Z_{ni}^2(t, x) \leq \frac{\varepsilon^2}{2}(\sigma(t, x))^2.$$

Finally, this shows clearly, that for n large enough, the set

$$\left\{Z_{ni}^2(t, x) > \varepsilon^2 \mathrm{Var}\left(\sum_{i=1}^{n} Z_{ni}(t, x)\right)\right\}$$

is empty, then this completes the proof of Lemma 8 and therefore Proposition 2.

Proof of Theorem 2. Using a Taylor expansion, we have

$$\xi_{p,n}(x) - \xi_p(x) = \frac{F_n(\xi_{p,n}(x) | x) - F_n(\xi_p(x) | x)}{f_n(\xi_{p,n}^{\star}(x) | x)},$$

where $\xi_{p,n}^{\star}(x)$ lies between $\xi_p(x)$ and $\xi_{p,n}(x)$.

The continuity of $f(.|x)$, Theorem 1 and Lemma 5 implies the convergence in probability of the above denominator to $f(\xi_p(x)|x)$. Proposition 2 is used to finish the proof.

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