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INEQUALITIES FOR THE TIMES WE HAVE TO USE THE FOURTH COLOR IN A 4-EDGE COLORING OF A CUBIC GRAPH

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Abstract

For a cubic graph G let the following problem: "Among all proper 4-edge colorings of G find one that minimizes the use of one of the four colors". This paper deals with this problem. We consider a 4-edge coloring of G which minimizes the smallest cardinality m of color classes. Using colors 1, 2, 3 and 4, let this class have the fourth color "4". We introduce a variable D that denotes the distance between pairs of edges having color 4. We also introduce another variable q, which is related to the number of vertices that is necessary to delete, in order to get a 3-edge critical subgraph of G. Our motivation is to give inequalities having parameters m, D and q. Therefore, finding restricted classes of cubic graphs which guarantee that parameters D or q must have specific values, then we can get good upper bounds for the cardinality m. Moreover, we believe that these inequalities are useful tools in the further investigation of 3-edge critical graphs.

1. Introduction

1.1. Preliminaries

We consider simple and finite graphs in this paper. A graph G is called Δ-*regular* if all the vertices have the same degree Δ. A 3-regular

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graph is also called *cubic* graph. An edge-coloring of a simple graph G is an assignment of colors to the edges of G such that no two incident edges have the same color. The *chromatic index* of G, $\chi'(G)$ is the minimum number of colors in an edge-coloring of G. Vizing [6] proved that $\chi'(G) = \Delta(G)$ or $\chi'(G) = \Delta(G) + 1$ for every graph G, where Δ or $\Delta(G)$ is the maximum vertex degree of G. Therefore, each graph belongs to one of the following two categories: either to class 1, which includes all the graphs G, so that $\chi'(G) = \Delta(G)$, or to class 2, which includes all the graphs, so that $\chi'(G) = \Delta(G) + 1$. The problem of classifying a graph is NP-complete ([4]). This problem remains NP-complete for triangle free graphs with maximum degree 3 ([5]). Planar graphs with $\Delta < 7$ have not been completely classified. So, it is reasonable to search for approximating algorithms that decide, in polynomial time, if a graph with $\Delta = k$ is k-edge colorable for some restricted classes of graphs. A classical work in the edge coloring of graphs is in [2].

A different direction to face the k-edge coloring problem is to try to color as many edges as possible using k colors. This approach is known as max edge k-coloring: "Given a graph G and a number k, color as many edges as possible using k colors". In the max edge k-coloring problem we try to find the best possible ratio of the number of edges that are colored with k-colors to the size of the given graph. Some well-known approximating ratios are: $1 - \left(1 - \frac{1}{k}\right)^k$ or $\frac{k}{k+1}$ ([3], [1]).

In this paper we use a variation of the max edge 3-coloring problem: For a cubic graph G let the following problem: "Among all proper 4-edge colorings of G find one that minimizes the use of one of the four colors". Let a 4-edge coloring of G which minimizes the smallest cardinality m of color classes. Using colors 1, 2, 3 and 4 we assign to this class color the *fourth* color 4. Max edge 3-coloring problem allows the use of more than 4 colors so, any ratio that we can achieve in our problem also holds for the original max edge 3-coloring problem. We introduce a variable D to denote the distance between pairs of edges having color 4. We also introduce another variable q, which is related to the number of vertices that is necessary to delete, in order to get a 3-edge critical subgraph of G.

We give inequalities having parameters m, D and q. In other words, our purpose is to find upper bounds for m using parameters D and q. These inequalities are useful, since if we find restricted classes of cubic graphs which guarantee that these parameters must have specific values, then we can get *good* upper bounds for the cardinality m.

1.2. Terminology and definitions

Definition 1. A 4-edge coloring of G which minimizes the smallest cardinality m of color classes is called *optimal*. Distance between edges having the fourth color is intuitively used as a criterion that indicates how large subgraphs of the given graph that we can color with 3 colors are. It is clear that when the *fourth* color is used few times, edges having the *fourth* color are far apart and vice-versa.

Definition 2. The distance d between two edges x and y in a graph G is the length of the longest path that connects them.

Definition 3. The distance between two edges **ab** and **cd** is the minimum distance between the endpoints $\{\mathbf{a}, \mathbf{c}\}$, $\{\mathbf{a}, \mathbf{d}\}$, $\{\mathbf{b}, \mathbf{c}\}$ and $\{\mathbf{b}, \mathbf{d}\}$.

In order to compute the minimum distance between more than two edges we have to check the distances for all possible pairs of edges. For example, if we want to compute the minimum distance between edges x, y and z we have to check the distances for the three pairs of edges (x, y), (x, z) and (y, z) and to find the minimum one. In Figure 1, for example, edges **eb** and **gj** have the color 4. Their distance is 7. The graph in Figure 1 is cubic and non 3-edge colorable. In an optimal 4-edge coloring of this graph we need twice the color 4. It is easy to check that the distance between edges that get the fourth color cannot be less than 5 unless you have assigned a non optimal 4-edge coloring to the graph. So, in that case color 4 must be used more than twice.

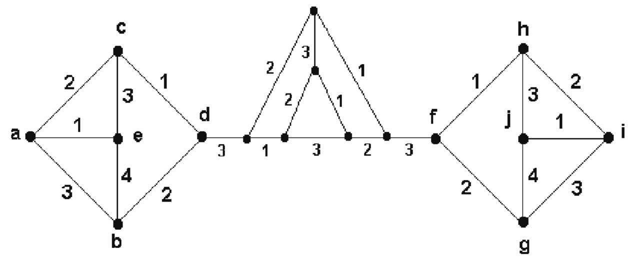

Figure 1. Edges **eb** and **gj** have color "4" and their distance is 7. The distance between edges that get the fourth color can be less than 7 but then we will use more than twice the color "4".

1.3. Basic lemmas

For our purpose we need one more definition and some basic results.

Definition 4. We shall call a graph *edge-critical* if it is connected, it is of class 2 and the removal of any edge transforms it to a graph of class 1. In this paper when we say "critical graph" we mean "edge-critical graph". If a critical graph has maximum degree Δ, then we shall call it Δ-*critical*. Some basic results, see [2], for a 3-edge critical graph G with n vertices are:

(1) It is not 3-regular.

(2) It cannot have exactly two vertices of degree 2.

(3) The number of vertices with degree 2 is at most $\frac{n}{3}$.

(4) The number of edges of G is between $\frac{4n}{3}$ and $\frac{1}{2}(3n-1)$.

(5) It does not contain adjacent vertices of degree 2.

2. The Main Result

2.1. The upper bound

We state the following theorem:

Theorem. *Let G be a cubic graph with n vertices. Suppose that G belongs to class* 2 *and m is the minimum number of times we have used color* 4 (*over all possible* 4-*edge colorings*). *Then the following hold*:

(i) *If in each optimal* 4-*edge coloring the distance between any pair of edges*, *in which the color* "4" *has been assigned*, *is at least D*, *then* $m \leq \max\left\{\frac{n}{2D} + \frac{1}{2} + \frac{3}{2D}, \frac{n}{6} + 1\right\}$.

(ii) *If the number of vertices that are endpoints of edges with color* 4 *must be reduced by q in order to get a* 3-*critical subgraph in G*, *then for* $D \leq 7$ *we get* $m \leq \frac{n}{6} + \frac{(3 - D)q}{6} + 1$ *and for* $D \geq 7$ *we get* $m \leq \frac{n}{6} + 1 - \frac{q(D+1)}{12}$.

Proof. If we assign the fourth color – in a proper way – to m edges, then we get a 4-edge coloring. If the m edges with the fourth color are deleted, then we will get a 3-edge colorable subgraph. Let us delete only $m - 1$ of these edges. We denote by $\mathbf{v}$ the mth edge that remains and by H the resulting subgraph of G. Graph H is not 3-edge colorable, otherwise m is not a minimum.

First, we suppose that H is critical. We notice that the deletion of the $m - 1$ edges generates $2m - 2$ vertices of degree 2. Indeed, it is impossible this deletion to generate less than $2m - 2$ vertices of degree 2, because in that case two of the edges, which we have deleted, must be adjacent. However, this is impossible since the deleted edges have been assigned the same *fourth* color in a legal 4-edge coloring of G.

From now on, we shall call simply the set of the $2m - 2$ vertices of degree 2 with M. Since we suppose that H is critical, we get $2m - 2 \leq \frac{n}{3}$ and finally

$$m \leq \frac{n}{6} + 1. \tag{1}$$

Consider now the case where H is not critical. Graph H can be either connected or disconnected. If H consists of more than one components, then only one of them is not 3-edge colorable, actually the one that has edge $\mathbf{v}$, so all the 3-edge colorable components must be deleted. In both cases the deletion of some other edges except the edge $\mathbf{v}$ will give a subgraph of H that is critical, say H' of order n'.

The deletion of other edges except the $m-1$ ones that we first deleted it is possible to destroy some vertices of degree 2. During the deletion of those edges a sequence of graphs H_0, H_1, ..., H_k is generated.

Obviously in this sequence the first element is H and the last is H'.

Notice that we are not interested in the procedure that leads into the critical subgraph H'.

Let us examine the following cases:

Case 1. In graph H_i there are no adjacent vertices of degree 2.

Suppose that the number $2m-2$ of vertices of degree 2 was reduced by one due to the deletion of an edge adjacent to a vertex of degree 2. However, if we delete one edge adjacent to a vertex of degree 2, then we must also delete the other one as well. That is because a 3-critical graph cannot have vertices of degree 1. Therefore, the deletion of the second edge generates a vertex of degree 2 and at the end of this process two new vertices of degree 2 are generated.

Case 2. In graph H_i there are adjacent vertices of degree 2.

In a 3-critical graph there are no adjacent vertices of degree 2 and these vertices must be deleted.

Let vertices **a** and **b** belong to *M*. For a subgraph H_i vertices **a** and **b** can belong to a path of adjacent vertices of degree 2 for two reasons:

(i) edges **ac** and **bd** in which we have assigned color 4 have distance 1, see Figure 2, and their deletion makes vertices **a** and **b** adjacent;

(ii) edges **ac** and **bd**, in which we have assigned color 4, have distance greater than 1, see Figure 3. So, between them there are vertices of degree 3. In these intermediate vertices we have adjacent edges in which different color than color 4 is assigned. If all these intermediate edges are deleted, then we get a path of adjacent vertices of degree 2 to which vertices **a** and **b** belong.

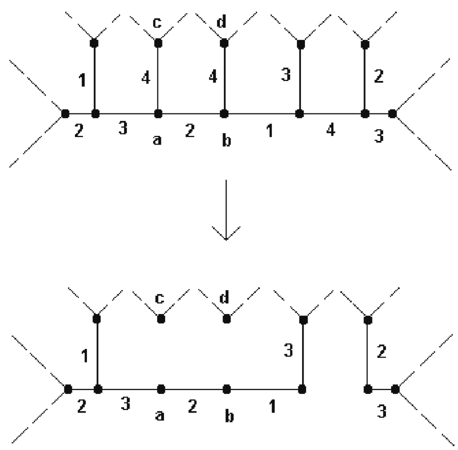

Figure 2. Edges **ac** and **bd** have color 4 and are at distance 1. If these edges are deleted, then vertices **a** and **b** become adjacent.

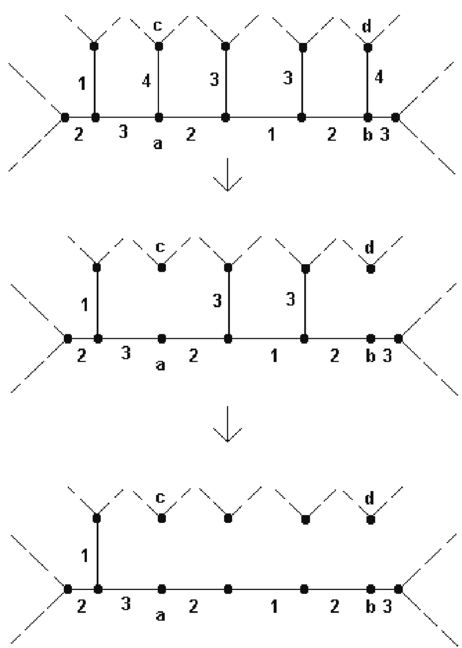

Figure 3. Edges **ac** and **bd** have color 4 and are at distance 3. If these edges are deleted, then vertices **a** and **b** have degree 2 and their distance is 3. Between them there are $3 - 1 = 2$ vertices of degree 3.

If one of the previous reasons occurs, then the number $2m - 2$ of vertices of degree 2 can be reduced. At the same time the order n is also reduced. If the number $2m - 2$ is reduced by q and the order n by Q, then we get the inequality:

$$m \leq \frac{n}{6} + 1 + \frac{3q - Q}{6}. \tag{2}$$

We conclude that we can have a greater upper bound than this in the inequality (1) if $3q - Q$ has a positive value. Trying to examine the possible values of $3q - Q$ we use variables q and Q. Variable Q to count the number of vertices that we have deleted starting from the initial order n. Variable q to count the difference between the number of vertices of degree 2 from the initial number $2m - 2$. Sometimes, we use variable Q to denote the number of vertices that we delete in one step of our procedure and not the number of vertices that we have deleted. In the same way, we can use variable q to denote the number of vertices of degree 2 that are deleted only in one step of our procedure. For example, suppose that in the first step of the procedure one vertex of degree 2 is deleted and its deletion generates two new vertices of degree 2. Then $q = 1 + (-2) = -1$ and $Q = 1$, so $3q - Q = -4$. Suppose that in the second step of the procedure three adjacent vertices of degree 2 are deleted and their deletion generates two new vertices of degree 2. Then $q = 3 + (-2) = 1$ and $Q = 3$, so $3q - Q = 0$. Adding the previous values from step 1 to the variables q and Q, we have their current values, $q = 0$ and $Q = 4$. In other words the current value of $3q - Q$ is -4. It is clear that each step of the procedure contributes an integer value to the expression $3q - Q$ (which we shall call simply $3q - Q$). Now, we are going to examine when $3q - Q$ gets a maximum, since we want to find an upper bound for m. That means that we are not interested in cases where a negative or zero value is contributed to it.

Let a path, in which the endpoints are vertices of degree 3 and all the other vertices are of degree 2.

We will study the following cases:

(i) None of the vertices of degree 2 belong to M. Then their deletion contributes to $3q - Q$ a negative value.

(ii) Between any pair of vertices that belongs to M there are $D - 1$ vertices of degree 2 that do not belong to M. So, in this path vertices that belong to M have been generated due to the deletion of edges with the color 4 at distance D.

Let the number of vertices in M in this path be q_1, see Figure 4. So, the $q + (q - 1)(D - 1)$ vertices of degree 2 must be deleted and this deletion generates two vertices of degree 2. That means that the number $2m - 2$ is reduced by $q_1 - 2$ and the order by $Dq_1 - D + 1$, therefore we contribute to $3q_1 - Q$ the value $3(q_1 - 2) - (Dq_1) + D - 1 = (3 - D)q_1 + D - 7$. If some vertices in M are at greater distance than D, then $3(q_1 - 2) - (Dq_1) + D - 1 \leq (3 - D)q_1 + D - 7$.

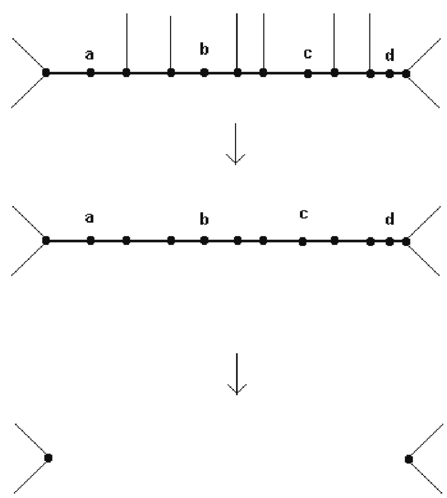

Figure 4. Edges having the color "4" in G are deleted and vertices **a**, **b**, **c** and **d** have degree 2. Between them there are vertices of degree 3 that correspond to edges in G to which we have assigned a color different than color 4. When all these edges are deleted then in the path there are 10 adjacent vertices of degree 2. Vertices **a** and **b**, **b** and **c**, **c** and **d** have distance 3. Deleting them the number $2m - 2$ is reduced by $4 - 2 = 2$, since 2 new vertices of degree 2 are generated. The order of graph G is reduced by 10.

For k paths, see Figure 5, we have: $q_1 + q_2 + \cdots + q_k = q$ and we finally get

$$3q - Q \leq (3 - D)q + k(D - 7). \tag{3}$$

Subcase 1. $D \leq 7$.

For $D \leq 7$ the expression $(3 - D)q + k(D - 7)$ takes a maximum for $k = 1$ and then the expression $3q - Q$ gets the value $(3 - D)q + (D - 7)$. This value be greater if the paths we study form cycles, since then the initial number $2m - 2$ is reduced by q and the order only by Dq.

Therefore:

$$3q - Q \leq (3 - D)q. \tag{4}$$

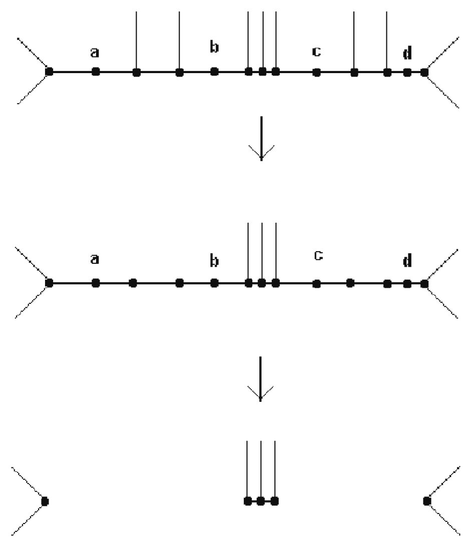

Figure 5. We have the previous example but now different edges are deleted. Actually, 4 edges are deleted: two edges having endpoints between vertices **a** and **b** and two edges having endpoints between vertices **c** and **d**. We have three consecutive paths. The first has 4 adjacent vertices of degree 2. This path is followed by a second path with 2 adjacent vertices of degree 3. This second path is followed by a third path with 4 adjacent vertices of degree 2. In this way the number $2m - 2$ is reduced by $4 - 4 = 0$, since 4 new vertices of degree 2 are generated. At the same time the order is reduced by 8.

One detail in our proof must be clear: We observe that the deletion of a vertex in M, which is not adjacent to other vertices in M, contributes to $3q - Q$ a negative value, since two vertices of degree 2 are generated. But what happens if the two new vertices are intermediate vertices in the paths we study? We do not have the right to count twice these vertices as vertices that reduce the order n. Therefore, the deletion of our paths contributes to $3q - Q$ a greater value than $(3 - D)q$ if $D \geq 3$. We are lucky, since this case cannot occur if $D \geq 3$. Indeed, if this case occurs, then we have a distance between two vertices having color 4 less than D, which is a contradiction.

Subcase 2. $D > 7$.

The value of $(3 - D)q + k(D - 7)$ takes a maximum when k takes a maximum, so for $k = \frac{q}{2}$.

We avoid details that we have already studied in Subcase 1. We only note that inequality (4) does not change if the paths we have described form cycles.

If all adjacent vertices of degree 2 are deleted, then we leave Case 2 and we face again Case 1 for some graph H_j, $j > i$, and so on.

According to the inequalities (3) and (4) and for $D \leq 7$ when we get H' we can assume that the number of $2m - 2$ vertices of degree 2 is totally reduced by q and the size of H' is totally reduced by at least Dq. Let t be the total number of vertices of degree 2 in H' and n' its order.

Then we have $2m - 2 - q = t \leq \frac{n'}{3} \leq \frac{(n - Dq)}{3}$, so $m \leq \frac{n}{6} + \frac{(3 - D)q}{6} + 1$. We consider the following cases:

(1) If $D < 3$, then we get the maximum value for $\frac{n}{6} + \frac{(3 - D)q}{6} + 1$ when $q = 2m - 1$, that is, $\frac{n}{6} + 1$.

(2) If $D = 3$, then the expression $\frac{n}{6} + \frac{(3 - D)q}{6} + 1$ gets the value $\frac{n}{6} + 1$.

(3) If $D > 3$ and $D < 7$, then the expression $\frac{n}{6} + \frac{(3 - D)q}{6} + 1$ achieves the maximum value when q is zero, so $m \leq \frac{n}{2D} + \frac{1}{2} + \frac{3}{2D}$.

That means that in the worst case we have the desired inequality:

$$m \leq \max\left\{\frac{n}{2D} + \frac{1}{2} + \frac{3}{2D}, \frac{n}{6} + 1\right\}.$$

We complete the first and the second parts of the theorem by considering the case where $D > 7$. Using inequality (2) and equation (3) with $k = \frac{q}{2}$ we get $m \leq \frac{n}{6} + 1 - \frac{q(D + 1)}{12}$.

The expression $\frac{n}{6} + 1 - \frac{q(D + 1)}{12}$ takes a maximum for $q = 0$.

Corollary 1. *Let G be a cubic graph with n vertices. Suppose that G belongs to class* 2 *and m is the number of times we have used color* 4 *in a legal* 4*-edge coloring of G. Then the following holds*:

If in the assigned 4*-edge coloring the distance between any pair of edges, in which color* 4 *has been assigned, is at least D and* $m > \max\left\{\frac{n}{2D} + \frac{1}{2} + \frac{3}{2D}, \frac{n}{6} + 1\right\}$, *then this* 4*-edge coloring is not optimal, i.e., there exists a* 4*-edge coloring using less than m times the color* 4.

Proof. The proof follows from our theorem.

Corollary 2. *Let G be a cubic graph with n vertices. Suppose that G belongs to class* 2 *and m is the minimum number of times we have used color* 4 (*over all possible* 4*-edge colorings*). *Then the following hold*:

(i) *for $D < 3$, we get* $q > \frac{6(m - 1) - n}{3 - D}$, *for $D > 3$ and $D < 7$, we get* $q < \frac{6(m - 1) - n}{3 - D}$, *and for $D > 7$, we get* $q \leq 3 - \frac{12 + 2n - 12m}{D + 1}$;

(ii) *for $q > 0$ and $D \leq 7$, we get* $D \leq 3 - \frac{6(m - 1) - n}{q}$ *and for $D > 7$, we get* $D \leq \frac{12 + 2n - 12m - q}{q}$;

(iii) $m \geq \frac{(7+D)q}{20} - \frac{(n+6)}{10}.$

Proof. The proof for the first and second parts follows from the inequalities in our theorem.

We shall prove the third part using the configuration shown in Figure 5. This behaves as a single edge in a 3-edge coloring, in the sense that edges ab and cd must get the same color. In any edge having color 4 are adjacent four edges, each one having one of the colors 1, 2 or 3. We pick all these four edges and replace each one by the configuration in Figure 6. It is clear that our 4-edge coloring remains optimal in the resulting new graph. Also the new graph remains connected, bridgeless and cubic. The order n and the minimum distance D have changed. Indeed, the configuration which we use has four vertices and the sorter path that connects vertices a and d has length 4. Therefore, the order of the new graph is $n + 16m$ and its minimum distance is $D + 6.$ Using the inequality, for $D \geq 7,$ we have proved in the second part of our theorem, we are finished.

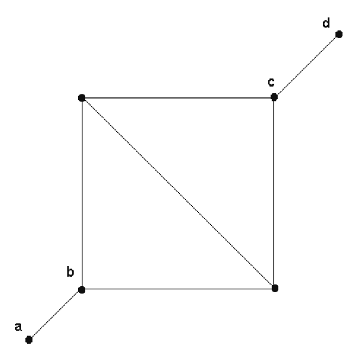

Figure 6. This configuration behaves as a single edge in a 3-edge coloring, since edges ab and cd must get the same color.

2.2. Criteria that guarantee bounds for the distance D

One can ask “Is it necessary to assign 4-edge colorings to G in order to find the minimum distance between edges with the fourth color?” In other words “Do we have any criteria that predetermine the distance between edges that are colored with the fourth color in an optimal 4-edge coloring?" The answer in the second question is "Yes". We will give two simple conditions below:

(1) Let graph G contain a non 3-edge colorable subgraph H, so that in every 4-edge coloring of H we have to use at least 2 times color 4. Let an optimal 4-edge coloring of G. Then some edges, to which we have assigned color 4, have distance less than the diameter of H.

(2) Suppose that a connected and cubic graph G is constructed by two (or more) connected and 3-edge colorable cubic graphs in the following manner: One edge ab from a cubic graph G_1 is deleted and one edge cd from a cubic graph G_2 is deleted. Then graphs G_1 and G_2 are connected by edges ac and bd. We can repeat this procedure by picking edges with endpoints vertices of degree 3. Suppose that the resulting graph G belongs to class 2. Then in an optimal 4-edge coloring of G we know that only in edges connecting these subgraphs we will need the color 4. So, the distance between pairs of edges to which we have assigned the *fourth* color is the distance between pairs of edges that connects these subgraphs. Note that the cardinality of the color class of color 4 cannot be greater than the one of the cardinalities of color classes for colors 1, 2 or 3. Indeed, for each connected edge that gets the color 4 there are four adjacent edges. Three of these edges get colors 1, 2 and 3, respectively. The fourth edge gets one of the colors 1, 2 or 3.

(3) Let D be the minimum distance. If we know a priori that in any pair of edges having color 4, we can find a path of length d between them such as $d \leq kD$, where $k \leq 1$. Then for each such pair there is a cycle having in its edges the two edges that get color 4 and two paths connecting them: one with D edges and another with kD edges. So the length of the cycle is $2 + D + kD$. If g is the girth of G, i.e., the length of the sorter cycle in G, then $2 + D + kD \geq g$ or $D \geq \frac{g-2}{1+k}$.

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