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P-ANTIREGULAR GRAPHS

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Abstract

We generalize the definition of an antiregular graph by subjecting antiregularity to the presence of a given set P of graph properties. In this paper, we study the cases where $P = \{\text{vertex-connectivity}\}$ and $P = \{\text{vertex-connectivity, thresholdness}\}$. We discuss the construction of these classes, looking also for structural characteristics and some properties such as hamiltonicity and tree-universality.

1. Introduction

Antiregular graphs were first defined by Behzad and Chartrand [1] (who named them *quasiperfect*) as graphs having exactly two vertices with equal degree values. Merris [13] defines them by considering K_1 (for $n - 1$) and the graphs G whose vertex degrees attain $n - 1$ different values. For each $n > 1$, there are exactly two mutually complementary antiregular graphs, one connected and the other disconnected. These graphs have a number of interesting properties, from which we initially distinguish that they are split graphs ([7, 12, 14]) and also threshold graphs ([14]), all to be defined later in the text.

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Let then P be a set of graph properties associated to a graph class to which we extend the definition of antiregularity, that is, we are interested in graphs having the maximum number of different degrees among these verifying P. For a given P, we will call these graphs *P-antiregular* (*P-AR*) graphs. With this consideration, for a Merris antiregular graph, we have $P = \{i\text{-vertex-connectivity}\}$, for $i \in \{0, 1\}$. In this paper, our concern will be the cases where $P = \{\text{vertex-connectivity}\}$ and $P = \{\text{vertex-connectivity, thresholdness}\}$, where *thresholdness* stands for "the property of being threshold".

The notation is that of [10] with the exception of that given by stated definitions. We will be working with simple, undirected graphs $G = (V, E)$, where we consider the *order* $|V| = n$ and the *size* $|E| = m$. Single vertices will be denoted by x, y or z and single edges by u, v, w or, if convenient, as vertex pairs. The minimum degree of a vertex in a graph G will be denoted as $d_{\min}(G)$. By notational convenience, we speak of a *clique* K_r and also of its *complete graph* $\mathbf{K}_r = (K_r, W_r)$, where $|W_r| = C_{r,2}$. Frequently, we will drop the index of K_r, when there is no doubt about its cardinality. In the definition of a graph, a vertex set referred to as I will be an independent one, unless otherwise specified. Eventually, an edge (x, y), where $x \in A$, $y \in B$, $A, B \subset V$, $A \cap B = \varnothing$, will be described as a (A, B) edge.

Our study will consider the following definitions and the theorem:

Definition 1.1. An integer number sequence with even sum is *graphic* if its elements correspond to the degrees of a graph. It is also called a *degree sequence*.

There are several equivalent theorems giving criteria for a number sequence to be graphic, beginning with that of Erdös and Gallai, as discussed by Sierksma and Hoogeven [15]. We will use a result from Berge, which is expressed as Theorem 1.1 below.

Definition 1.2. A graph $G = (V, E)$ is *P-antiregular* (*P-AR*) if, given a set P of G properties, it has a set $D(G) = \{d_{i1}, \ldots, d_{i\xi}\}$ (where $\xi = \xi_P$)

of different degree values such that its cardinality $\xi_P(G) = |D(G)|$ is maximum among the graphs following P. We also call a *P-AR sequence* a graphic sequence generating a P-AR graph.

As stated earlier, when talking about a graph we will say it is {*κ-connected*}*-AR* or {*κ-connected*, *threshold*}*-AR*, where $\kappa = \kappa(G)$ is the vertex-connectivity (or, from now on, simply connectivity). After the content of P is specified, we will eventually refer to P-AR as AR. For the Merris graphs studied in [13], we will then have $D(G) = \{n-1, n-2, \ldots, 2, 1\}$ for the 1-connected graph and $D(G) = \{n-2, n-3, \ldots, 1, 0\}$ for its complement, thus $\xi_P(G) = n-1$. The degree $\lfloor n/2 \rfloor$ is repeated for the connected graphs.

A theorem by Whitney [10] states, for a graph G, that $\kappa(G) \leq d_{\min}(G)$, which implies that, for a κ-connected graph G,

$$\xi_P(G) \leq n - \kappa(G). \tag{1}$$

Remark 1.1. It follows immediately from their definition that AR graphs have to verify the equality in (1).

Theorem 1.1 (Berge [2], [15])**.** *Let* $\mathbf{d} = (d_i, 1 \leq i \leq n)$ *be a non-increasing integer sequence with even sum and let* $B(\mathbf{d})$ *be a matrix of order n with a null main diagonal*, *the first other* d_i *elements of each line having unitary values*. *Let* $\delta = \{\delta_i\}$ *be the sequence of i-th column sums in* $B(\mathbf{d})$. *Then* $\mathbf{d}$ *is graphic if and only if*

$$\forall k = 1, \ldots, n : \sum_{i=1}^{k} d_i \leq \sum_{i=1}^{k} \delta_i. \tag{2}$$

Definition 1.3. A *split graph* $G = (I, K, E)$ is a graph whose vertex set can be bipartitioned between an independent set I and a clique K. This partition is also called a *split partition*.

Foldes and Hammer [6] give another characterization of these graphs: *A graph G is split if and only if G does not have an induced subgraph isomorphic to* C_4, C_5 *or* $2K_2$.

We can define a partial order on the n-partition set of a given integer $2m$, which contains the graphic sequences for a graph with n vertices and m edges. This set can be given a lattice structure ([14]), where the first graphic sequences in the associated partial order are called *threshold sequences*, and the graphs associated with them, *threshold graphs*. Every threshold graph is split ([9]) but the converse is not true. Every threshold sequence is *unigraphic*, that is, it has a unique graph associated with it. Merris [14] points out that his {1-connected}-AR graphs are threshold and characterizes these graphs with the aid of Ferrer's diagram, which for a graph G is equivalent to Berge's matrix $B(G)$. As we are already dealing with this matrix, we present such a characterization based on it, as follows:

Let $\mathbf{d}$ be a graphic sequence and $B(\mathbf{d})$ its Berge matrix; then let $U(\mathbf{d})$ $(L(\mathbf{d}))$ be the upper (lower) triangle of $B(\mathbf{d})$. Let $\alpha(\mathbf{d}) = \{U(\mathbf{d})$ rows, non-null positions} and $\beta(\mathbf{d}) = \{L(\mathbf{d})$ columns, non-null positions}.

Lemma 1.2 (Merris [14]). *A sequence* $\mathbf{d}$ *such that* $\alpha(\mathbf{d}) = \beta(\mathbf{d})$ *is a threshold sequence and a graph* G *corresponding to* $\mathbf{d}$ *is a threshold graph.*

Example 1.1. The sequences (7, 6, 5, 4, 4, 3, 2, 1) and (8, 7, 6, 5, 4, 4, 3, 2, 1) are threshold. It is easy to observe on their Berge matrices the symmetry implied by Lemma 1.2.

Some other characterizations are ([9], [11]):

1. A split graph $G = (I, K, E)$ is threshold if and only if the sets of neighbors $N(x)$, $x \in I$ (and therefore $N(y)$, $y \in K$), are totally ordered by inclusion.

2. A graph G is threshold if and only if G does not have an induced subgraph isomorphous to P_4, C_4 or $2K_2$.

Remark 1.2. This last characterization implies that split graphs not containing induced subgraphs isomorphous to P_4 are also threshold graphs.

Remark 1.3. All graphs referred in the text will have their vertices indexed from a higher-degree to a lesser-degree one, following the non-increasing order of their degree sequences.

2. The {1-connected}-AR Graphs

Let us consider the connected Merris AR graphs ([13]), which we call here {1-*connected*}-*AR graphs*. First of all, they satisfy (2) by strict equality. On the other hand, the {1-connected}-AR graphs have minimum size among all connected AR graphs, since a {κ-connected}-AR graph with $\kappa > 1$ would have at least the same number of edges as the corresponding graph with $\kappa = 1$.

In order to simplify the use of the Berge matrix, we partition its index sets I and J (resp., rows and columns) as follows:

$$I = \{I_1 \; I_2\}, \quad J = \{J_1, J_2\}, \tag{3a}$$

where

$$I_1 = J_1 = \{1, \ldots, \lceil n/2 \rceil\} \text{ and } I_2 = J_2 = \{\lceil n/2 \rceil + 1, \ldots, n\} \tag{3b}$$

thus obtaining a 2×2 partition matrix (Figure 2.1):

B_{11}	B_{12}
B_{21}	B_{22}

Figure 2.1. A partitioning of the Berge matrix.

The square Submatrix B_{11} has order $\lceil n/2 \rceil$ and corresponds to a complete subgraph, according to Remark 1.3; both B_{12} and B_{21} contain a sequence of $\lfloor n/2 \rfloor, \ldots, 1$ non-null elements (see Figure 2.2 below). B_{12} contains the (K, I) edges seen from K, while B_{21} contains the same edges, seen from I; B_{22} is null. Let us then call *opposite diagonal* the second main diagonal of a square matrix: we can observe that, for a {1-connected}-AR graph, the last element in each row is on the opposite diagonal of B. Since we are dealing with degree sequences, we can then calculate the size of a {1-connected}-AR graph by summing the number of B_{11} elements to twice the B_{12} (or B_{21}) sequence sum and dividing the result by two,

$$r(n) = \frac{1}{2}\left\{\left\lceil \frac{n}{2} \right\rceil\left(\left\lceil \frac{n}{2} \right\rceil - 1\right) + \left\lfloor \frac{n}{2} \right\rfloor\left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right)\right\}. \tag{4}$$

There will be only one {1-connected}-AR graph for each order, as already observed in [13], where the repeated degree value will be $\lfloor n/2 \rfloor$, occupying the positions $\lceil n/2 \rceil$ and $\lceil n/2 \rceil + 1$ in the sequence.

Example (Figure 2.2)**.** With $n = 8$ and $n = 9$, we will have the sequences (7, 6, 5, 4, 4, 3, 2, 1) and (8, 7, 6, 5, 4, 4, 3, 2, 1), where the value $\lfloor n/2 \rfloor = 4$ is repeated. The corresponding Berge matrices are shown in Figure 2.2, where we indicate the partitioning already expressed by Figure 2.1.

$$\left|\begin{array}{cccc|cccc}
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 1 & 1 & 1 & \\
1 & 1 & 0 & 1 & 1 & 1 & & \\
1 & 1 & 1 & 0 & 1 & & & \\
\hline
1 & 1 & 1 & 1 & 0 & & & \\
1 & 1 & 1 & & & 0 & & \\
1 & 1 & & & & & 0 & \\
1 & & & & & & & 0
\end{array}\right|
\qquad
\left|\begin{array}{ccccc|cccc}
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & \\
1 & 1 & 0 & 1 & 1 & 1 & 1 & & \\
1 & 1 & 1 & 0 & 1 & 1 & & & \\
1 & 1 & 1 & 1 & 0 & & & & \\
\hline
1 & 1 & 1 & 1 & & 0 & & & \\
1 & 1 & 1 & & & & 0 & & \\
1 & 1 & & & & & & 0 & \\
1 & & & & & & & & 0
\end{array}\right|$$

Figure 2.2. Examples of partitioning of Berge matrices.

In what follows, it will be interesting to observe how the size of {κ-connected}-AR graphs behaves with respect to $r(n)$, as we study their construction conditions by starting with the {1-connected}-AR Berge matrix. We will call *size-minimum* the graphs with the least number of edges in their respective classes; this value can be equal to or greater than $r(n)$.

3. Some Possible Constructions of {2-connected}-AR Graphs

Here we discuss the construction of {2-connected}-AR graphs through some examples. This practical approach has the advantage of showing, in simpler structures, some properties which will later be discussed from a theoretical point of view for greater values of connectivity.

From (1), we have $\xi_P(G) \leq n - 2$, as the minimum degree is 2. The degree sequences of 2-connected-AR graphs can have two types of repetitions, either containing three equal values or two different degree pairs. Let us examine some cases of graphic sequences, for $n = 10$ (Table 3.1):

Table 3.1. Graphic sequences generating 2-connected graphs

Sequence	Column sum vector δ	ξ_P	AR	Size-min.
$\mathbf{d}_1 = (9, 8, 7, 6, 5, \underline{4}, \underline{3}, 3, \underline{\mathit{3}}, \underline{\mathit{2}})$	(9, 9, 8, 5, 4, 5, 4, 3, 2, 1)	8	yes	yes
$\mathbf{d}_2 = (9, 8, 7, 6, 5, 5, 4, 2, 2, 2)$	(9, 9, 6, 6, 5, 5, 4, 3, 2, 1)	7	no	yes
$\mathbf{d}_3 = (9, 8, 7, 6, 5, 4, 4, 3, 2, 2)$	(9, 9, 7, 6, 4, 5, 4, 3, 2, 1)	8	yes	yes
$\mathbf{d}_4 = (9, 8, 7, 6, 5, 5, 4, 3, 3, 2)$	(9, 9, 8, 7, 6, 5, 4, 3, 2, 1)	8	yes	no

Throughout this paragraph, we will refer to the Berge matrices of Table 3.1 sequences. Compared with a {1-connected}-AR sequence, Sequence $\mathbf{d}_1$ shows an increase of one unit in the two lesser degrees (underlined italics), which is compensated for by a decrease of one unit in the two first degrees of B_{21} (underlined), thus it is AR. According to Definition 1.2, Sequence $\mathbf{d}_2$ is not AR, as it has $\xi_P = 7$, while $\mathbf{d}_1$ has $\xi_P = 8$ (see Lemma 4.4 below). Sequence $\mathbf{d}_3$ is AR.

These three sequences correspond to graphs having $r(10) = 25$ edges, so (4) will be valid for 2-connected graphs, as long as we have enough space to open a null position on the opposite diagonal to compensate for the second non-null element in the last line, which is necessary to grant the 2-connectivity. In what follows, we will see (Theorem 4.1) that this is possible from $n = 6$ on. For instance, $K_4 - u$ is size-minimum and {2-connected}-AR, but $m(K_4 - u) > r(4)$.

Sequence $\mathbf{d}_4$ generates a graph having 26 edges so, unlike the 1-connected case, we can have non-size-minimum {2-connected}-AR graphs. Here, $\mathbf{d}_4$ corresponds to $\mathbf{d}_3$ with one added edge.

4. Some Structural Issues and Properties

Theorem 4.1. *The minimum order of a* {κ*-connected*}*-AR graph* G *with* $m = r(n)$ *is*

$$n_{\min}(\kappa) = 2\kappa + \kappa(\kappa - 1). \tag{5}$$

Proof. The degree sequence of a {κ-connected}-AR graph has its last element equal to κ. To obtain such a sequence from that of a {1-connected}-AR graph keeping the non-increasing degree order and having $m = r(n)$, we have to exchange (K, I) edges between higher and lower-degree vertices of I. Then we have to add $\kappa - 1$ non-null elements to the last row of the Berge matrix, $\kappa - 2$ ones to the preceding row, ..., 1 element to the $(n - \kappa + 2)$th row. That way, we will obtain κ vertices with degree κ, from the $(n - \kappa + 1)$th to the nth one and we will need to nullify at least $\sum_{i=1}^{k-1} = \kappa(\kappa - 1)2$ positions on B_{21} opposite diagonal, in order to compensate for these inclusions and thus validate (4). Then for the size of a {κ-connected}-AR graph to verify (4), its order should be at least the double of the B_{21} order, which is the second member of (5).

Theorem 4.2. *The* {κ*-connected*}*-AR graphs* G *with* $m = r(n)$ *and* $n \geq n_{\min}(\kappa)$ *are split graphs.*

Proof. Let G be a {1-connected}-AR graph. It is therefore a split graph, with size $m = r(n)$. Starting with G, to keep size and to obtain a given connectivity κ_0, we have to exchange (K, I) edges adjacent to a set of higher-degree I vertices with the set of the $\kappa_0 - 1$ lower degree I vertices, as in the proof of Theorem 4.1. Since this process deals only with (K, I) edges and we started with a split graph, the {κ-connected}-AR graph obtained is also split, provided there are enough edges to be exchanged as stated by Theorem 4.1.

Remark 4.1. The edge set of a split graph is the sum of two disjoint edge sets, one from a complete and other from a bipartite graph. For such a graph, this implies that the difference between the degree sum and the double of the number of $\mathbf{K}_{\lceil n/2 \rceil}$ edges (which we call here the *external clique degree*) is equal to the degree sum on I.

Example 4.1. Sequences $\mathbf{d}_3$ and $\mathbf{d}_4$ from Table 3.1 above. According to (4), $\mathbf{d}_3$ generates size-minimum graphs. The degree sum on I for a corresponding graph is equal to the external clique degree, and thus we have split partitions, in accordance with Theorem 4.2. But $\mathbf{d}_4$ is not size-minimum, as we can find an excess of 2 degree units in I degree sum: this corresponds to one edge between two vertices in I, which in this case is not an independent set. Thus $\mathbf{d}_4$ does not generate split graphs.

Figure 4.1 shows examples of graphs generated by these sequences. The contour around the upper vertices means they induce a complete subgraph.

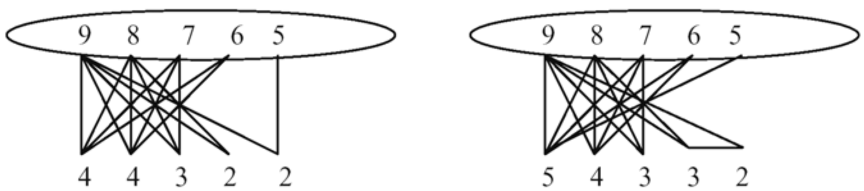

Figure 4.1. Graphs generated by $\mathbf{d}_3$ and $\mathbf{d}_4$.

Remark 4.2. The {κ-connected}-AR graphs with $\kappa > 1$ are not threshold. The last row in the Berge matrix should have at least κ non-null elements (compensated for by zeroes on the opposite diagonal, as already discussed), while its last column will have but one non-null element, which breaks the symmetry implied by Lemma 1.2. For instance, in the case of $\kappa = 2$, we would have to increase by one unit the second element in the degree sequence in order to restore the symmetry. But this creates a new degree repetition, and the new graph will no longer be {κ-connected}-AR.

Theorem 4.3. *For a* {κ-*connected*}-*AR graph G with* $d_{\min}(G) < \lceil n/2 \rceil$ *and* $m = r(n)$, *we have*

$$\kappa(G) = d_{\min}(G), \tag{5}$$

where $\kappa(G)$ *is the connectivity and* $d_{\min}(G)$ *is the minimum degree of G.*

Proof. From Theorem 4.2, G is split. The subgraph $\mathbf{K}_{\lceil n/2 \rceil}$ has connectivity $\lceil n/2 \rceil - 1$. A vertex of I is connected to the remaining graph through the edges it shares with one or more K vertices. Let x be a minimum degree vertex in G. The thesis condition guarantees that $x \in I$. If we remove from G the neighborhood $N(x) \subseteq K$, whose cardinality is $d_{\min}(G)$, then x will become trivial. Since x has minimum degree, $N(x)$ is a minimum cardinality vertex cutset.

Lemma 4.4. *The* {κ-*connected*}-*AR graphs* $G = (I, K, E)$ *with* $m = r(n)$ *do not have repetition of the* $\lfloor n/2 \rfloor$ *degree.*

Proof. The Berge matrix of a {1-connected}-AR graph contains a repetition of $\lfloor n/2 \rfloor$. For G to be κ-connected, the last row of the Berge matrix must have κ non-null entries, which can be obtained through the construction utilized in the proof of Theorem 4.1. But the zeroes referred to in that proof have to be created at the end of the first $\kappa(\kappa - 1)B_{21}$ rows, in order to eliminate the original $\lfloor n/2 \rfloor$ repetition and to avoid building a new one, which would lower the value of ξ_P.

Example 4.2 (Figure 4.2)**.** For $\kappa = 2$ and $n = 11$, we show the Submatrix B_{21} with the addition of a neighbor row and one or two columns. The creation of a zero in the first B_{12} row (matrix farthest to the right) gives us a sequence generating {2-connected}-AR graphs with $m = r(n)$, while the strategies shown by the second and the third matrices do not arrive at this result. The repetitions are in italics.

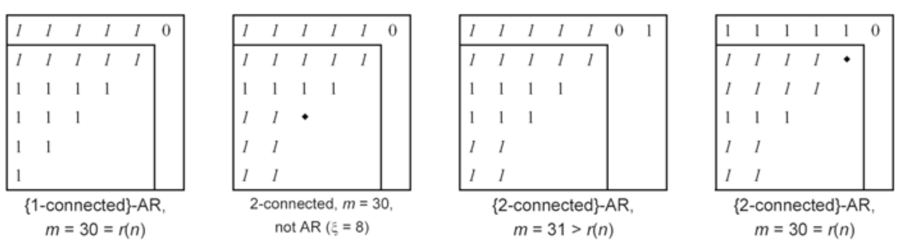

Figure 4.2. Repetition of $\lfloor n/2 \rfloor$.

Theorem 4.5. *The* {κ-*connected*}-*AR graphs* $G = (I, K, E)$ *with* $m = r(n)$ *are not Hamiltonian.*

Proof. If n is even, a vertex y of minimum degree in K has degree $n/2$, and thus has exactly one neighbor in I. The bipartite spanning subgraph $G' = (I, K, E')$, with $E' = \{(x, y) | x \in I, y \in K\}$ is balanced ([8, p. 262]), so every Hamiltonian cycle in G will also be in G'. But G' will have a vertex of degree one, and thus cannot be Hamiltonian.

If n is odd, $\lceil n/2 \rceil - \lfloor n/2 \rfloor = 1$, then K has one vertex more than I. The vertex z of lesser degree in K has degree $\lfloor n/2 \rfloor$, thus its neighborhood will be $N(y) = K$, as it belongs to a complete subgraph with this degree. Then $G - z$ is {κ-connected}-AR of even order with $m_1 = r(n-1)$ and, as we have shown earlier, it is not Hamiltonian. G thus cannot be Hamiltonian, because by putting back z and its adjacent edges we will not create any new connection with I.

As the connectivity grows, there are progressively more size-minimum AR graphs with $m > r(n)$ which can possibly be Hamiltonian, although the most commonly known sufficient conditions (such as the Bondy-Chvátal theorem [3], [4]) cannot guarantee it.

Example 4.3. With 3-connected graphs $(\xi = n - 3)$ and $n < n_{\min}(\kappa) = 12$, we can have the following situations:

- the graph is not split;
- the graph is not AR;
- the graph has a size $m > r(n)$.

Table 4.1, below, shows some examples of sequences generating {3-connected}-AR graphs, from $n = 10$ to $n = 13$ for comparison. There, ECD stands for External Clique Degree and DSI for Degree Sum in I. The column ECD shows the number of edges between K and I. As already discussed, the equality of entries between ECD and DSI columns, for a given sequence, implies the corresponding graph(s) is (are) split. We can also observe that for $n < n_{\min}$ (in the case at hand, for $n < 12$), we always have $m > r(n)$.

Table 4.1. Some examples of {3-connected}-AR sequences

Example	Order	ξ	Sequence	ECD	DSI	Split	m	$r(n)$
1	10	7	(9, 8, 7, 6, 5, 4, 4, 3, 3, 3)	15	17	no	26	25
2		7	(9, 9, 8, 7, 6, 5, 4, 4, 3, 3)	19	19	yes	29	25
3	11	8	(10, 9, 8, 7, 6, 5, 4, 4, 3, 3, 3)	15	17	no	31	30
4		8	(10, 10, 9, 8, 7, 6, 5, 5, 4, 3, 3)	20	20	yes	35	30
5	12	9	(11, 10, 9, 8, 7, 6, 5, 4, 3, 3, 3, 3)	21	21	yes	36	36
6	13	10	(12, 11, 10, 9, 8, 7, 6, 5, 4, 3, 3, 3, 3)	21	21	yes	42	42

Remark 4.3. The examples of 10- and 11-vertex AR sequences corresponding to split graphs of size greater than minimum (Examples 2 and 4 in Table 4.1), generate Hamiltonian graphs. Using the already defined indexing, we can build graphs presenting respectively the Hamiltonian cycles (5, 10, 2, 9, 3, 8, 1, 7, 4, 6, 5) and (6, 11, 2, 10, 3, 9, 4, 8, 5, 7, 1, 6). As we can observe, these sequences use a repetition in Submatrix B_{12} to balance the non-null values in Submatrix B_{21}.

5. About the {κ-connected, threshold}-AR Graphs

Remark 4.2 suggests the definition of a new class of P-AR graphs: the {κ-*connected*, *threshold*}-*AR graphs*. The symmetry constraint on the Berge matrix is a very practical instrument for the construction of threshold graphs. We will use it to build some sequences and graphs from this class and to study some of its characteristics.

To begin with, we cannot expect these graphs to verify (4), since the first movement made to recover the symmetry we lose when building {κ-connected}-AR graphs with $\kappa > 1$ will be the introduction of new non-null elements in the last $\kappa - 1$ columns of the Berge matrix. This way, we have new edges in the graph between vertices with maximum and minimum degrees; and in addition, the equilibrium we obtained meets the Berge criterion by strict equality. Then, for instance, for {2-connected, threshold}-AR graphs, we have $\xi_P = n - 3$, because the degree $n - 2$ disappears from the sequence.

We could consider compensating for this loss of adherence to (4) by creating holes in the opposite diagonal as we have already done, but here it will not be possible to avoid a decay in ξ_P value: at least one degree value would disappear in the process, as we would have to create the holes symmetrically in order to keep the thresholdness. For {2-connected, threshold}-AR graphs, we have thus a minimum size of $r(n)+1$ and, for greater values of κ, we can apply reasonings similar to those already presented.

We can build {2-connected, threshold}-AR graphs with size greater than minimum through the addition of edges, as long as we do not make any degree disappear. See the lines 2 (compared with 1) and 4 (compared with 3) in Table 5.1 below.

Example 5.1 (Table 5.1)**.**

Table 5.1. Some {2-connected, threshold}-AR sequences

Example	n	Sequence	ξ	m	AR	Size-min.
1	8	(7, 7, 5, 4, 4, 3, 2, 2)	5	17	yes	yes
2	.	(7, 7, 6, 4, 4, 3, 3, 2)	5	18	yes	no
3	9	(8, 8, 6, 5, 4, 4, 3, 2, 2)	6	21	yes	yes
4	.	(8, 8, 7, 5, 4, 4, 3, 3, 2)	6	22	yes	no

We can immediately generalize this view for a given κ. Starting once more with a connected Merris graph, which is {1-connected, threshold}-AR, we have to add 1, 2, ..., $\kappa-1$ non-null elements, *both to the first and to the last* $\kappa-1$ rows of its Berge matrix. There is no minimum order to consider, because we cannot obtain AR graphs by creating holes in the opposite diagonal as was discussed above. For instance, the sequence (11, 11, 11, 8, 7, 6, 6, 5, 4, 3, 3, 3) is {3-connected, threshold}-AR. It generates the graph depicted in Figure 5.1 below.

In the figure, for the sake of clarity, the 18 (K, I) edges adjacent to the vertices of degree 11 were not represented. To keep track of that, we nullified the degree of those vertices and we subtracted 5 from the lower-degree K vertices and 3 from the degrees of the I vertices. The null degrees were not indicated in the figure.

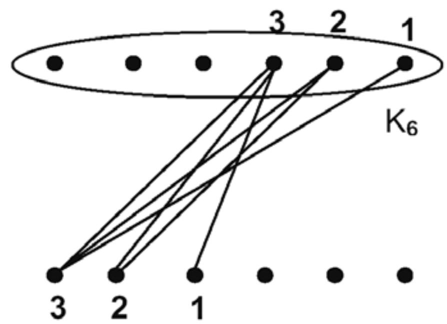

Figure 5.1. A {3-connected, threshold}-AR graph.

The unigraphic character of this sequence thus becomes evident, as is the total ordering of the neighborhoods of both K and I vertices by inclusion.

Here we have $\xi_P = n - 5$ and a (minimum) number of 39 edges. We can observe that the structural characteristics related to the vertices of minimum degree in $\mathbf{K}_{\lceil n/2 \rceil}$ and in the whole graph, which allowed us to propose Theorems 4.3 and 4.5, are also found here. As a consequence, the {κ-connected, threshold}-AR size-minimum graphs follow these theorems, thus they have their connectivity strictly equal to their minimum degree and are not Hamiltonian. Finally, it is already known that every threshold graph is split.

6. Tree-universality

A problem that has been studied for some time is how to find graphs containing isomorphs for every tree of the same order as (spanning) subgraphs. Chung and Graham [5] look for bounds on the minimum size for tree-universal graphs, while Merris [13] proves the tree-universality of {1-connected}-AR graphs.

Definition 6.1. A graph G with n vertices is *universal for trees* (or *tree-universal*) if every tree on n vertices is isomorphic to a subgraph of G.

We present the Merris theorem ([13]), translated into our notation.

Theorem 6.1 (Merris). *The* {1-*connected*}*-AR graphs are universal for trees*.

Proof. The proof is by induction, using sum, join and complement operations to build the reasoning on two graph families, the {1-connected}-AR graphs and their disconnected complements.

In order to investigate the occurrence of this property in the broader classes defined here, we make use of a straightforward conclusion:

Remark 6.1. If a graph $G = (V, E)$ is tree-universal, then every graph $H = (V, E')$ such that $E' \supset E$ is also tree-universal.

This property allows us to look for graphs whose edge set contains that of the {1-connected}-AR graph with the same order.

We can find {κ-connected}-AR graphs, not minimum-sized, whose edge sets have this property. For instance, Sequences 2 and 4 from Table 3.1 generate tree-universal graphs, as their elements dominate the corresponding ones in the {1-connected}-AR sequence of same order. For the same reason, the family of {κ-connected, threshold}-AR graphs is also tree-universal.

7. Conclusions

In the discussion we have defined the class of P-antiregular graphs, subjecting the antiregularity to a set P of properties verified by the graphs, thus extending the original class to admit new structures not conforming to the original definition. When connectivity is specified as the sole property to be considered, we are able to build {κ-connected}-AR graphs by using the matrix defined by Berge, in the context of his theorem for characterizing graphic sequences. We have considered order and size limits for these graphs and presented some discussion on their hamiltonicity. The same was done when considering thresholdness as well, which allowed us to define the family of {κ-connected, threshold}-AR graphs. A brief discussion concerning the tree-universality of these families is also presented. We believe that these extended classes of graphs have interesting research possibilities, especially if P is considered to contain other properties.

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