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# A CONTRACTED PROCEDURE FOR THE UNIQUE SOLVABILITY OF A SEMI-LINEAR WAVE EQUATION ASSOCIATED WITH A LINEAR INTEGRAL EQUATION AT THE BOUNDARY

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**Abstract**

In this paper, the unique solvability of a semi-linear wave equation associated with a linear integral equation at the boundary is proved by a contracted procedure.

## 1. Introduction

We study the solution $u(x, t)$ of the following semi-linear equation:

$$u_{tt} - \mu(t)u_{xx} + F(u, u_t) = f(x, t), \quad 0 < x < 1, \; 0 < t < T \tag{1.1}$$

associated with initial-boundary values given by

$$u(0, t) = 0, \tag{1.2}$$

$$-\mu(t)u_x(1, t) = Q(t), \tag{1.3}$$

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \tag{1.4}$$

where $F(u, u_t) = K|u|^{p-2}u + \lambda|u_t|^{q-2}u_t$, $p, q \geq 2$, $K$ and $\lambda$ are constants;

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$u_0$, $u_1$ and $\mu$ are given real functions satisfying conditions specified later, and $Q(t)$ satisfies the following integral equation:

$$Q(t) = K_1(t)u(1, t) + \lambda_1(t)u_t(1, t) - g(t) - \int_0^t k(t-s)u(1, s)ds, \qquad (1.5)$$

where $g$, $k$, $K_1$ and $\lambda_1$ are given functions.

This problem is the mathematical model describing the shock of a rigid body and the viscoelastic bar (see [1], [5-8], [10], [11], [13]) considered by several authors.

In [1], with $F(u, u_t) = Ku + \lambda u_t$, $f(x, t) = 0$, $\mu(t) \equiv a^2$, An and Trieu studied equation (1.1) in the domain $[0, l] \times [0, T]$ when the initial data are homogeneous, namely $u(x, 0) = u_t(x, 0) = 0$ and the boundary conditions are given by

$$\begin{cases} Eu_x(0, t) = -f(t), \\ u(l, t) = 0, \end{cases} \qquad (1.6)$$

where $E$ is a constant.

In [5], Long and Dinh considered problem (1.1)-(1.4) with $\lambda_1(t) \equiv 0$, $K_1(t) = h \geq 0$, $\mu(t) = 1$, wherein the unknown function $u(x, t)$ and the unknown boundary value $Q(t)$ satisfy the following integral equation:

$$Q(t) = hu(1, t) - g(t) - \int_0^t k(t-s)u(1, s)ds. \qquad (1.7)$$

We note that equation (1.7) is deduced from a Cauchy problem for an ordinary differential equation at the boundary $x = 1$.

In [10], Santos studied the asymptotic behavior of the solution of problem (1.1), (1.2), (1.4) in the case of $F(u, u_t) = 0$, $f(x, t) = 0$ associated with a boundary condition of memory type at $x = 1$ as follows:

$$u(1, t) + \int_0^t g(t-s)\mu(s)u_x(1, s)ds = 0, \quad t > 0. \qquad (1.8)$$

It is noted that the boundary conditions (1.7) and (1.8) are similar since their formal differences can be crossed out after solving the Volterra equation with respect to the variable $u(1, t)$ given by (1.8).

In [6-8], Long et al. gave the unique existence, stability, regularity in time variable and asymptotic expansion for the solution of problem (1.1)-(1.5) when $F(u, u_t) = Ku + \lambda u_t$.

And for $F(u, u_t)$ similar to what in the present problem, Ut [13] proved the unique solvability of the present problem. Furthermore, we also studied the stability of the weak solution with respect to some given parameters.

In [11], Sengul investigated the solvability of equation (1.1) in the case of $F(x, t, u, u_t) = g(u) + \alpha u_t - f(x, t)$ associated homogeneous boundary conditions and the initial conditions are the same to (1.4).

Although there have been many publications related to the present problem, the contracted procedure has not been applied for the solvability, in our knowledge, as in [1], [5-8], [10] and [13], etc.

In this paper, we apply a contracted procedure (see [3] and [12]) to obtain the unique solvability of problem (1.1)-(1.5), and it is believed that the essential proofs must be shorter and easier than what has been brought up. What we obtain here is considered as the generalization of those in An and Trieu [1], Long and Dinh [5], Santos [10], and Sengul [11], and of course a more comprehensive part of results in [13].

## 2. Preliminary Results and Notations

First we introduce some preliminary results and notations used in this paper. Put $\Omega = (0, 1)$, $Q_T = \Omega \times (0, T)$, $T > 0$. We omit the definitions of usual function spaces: $C^m(\overline{\Omega})$, $L^p = L^p(\Omega)$, $W^{m,p}(\Omega)$. We denote $W^{m,p} = W^{m,p}(\Omega)$, $L^p = W^{0,p}(\Omega)$, $H^m = W^{m,2}(\Omega)$, $1 \leq p \leq \infty$, $m = 0, 1, \ldots$.

The norm in $L^2$ is denoted by $\|\cdot\|$. We also denote by $\langle\cdot, \cdot\rangle$ the scalar product in $L^2$ or pair of dual scalar product of a continuous linear functional with an element of a function space. We denote by $\|\cdot\|_X$ the norm of a Banach space $X$ and by $X'$ the dual space of $X$. We denote by $L^p(0, T; X)$, $1 \le p \le \infty$ the Banach space of the real functions $u : (0, T) \to X$ measurable, such that

$$\| u \|_{L^p(0,T;X)} = \left( \int_0^T \| u(t) \|_X^p \, dt \right)^{1/p} < \infty \quad \text{for} \quad 1 \le p < \infty,$$

and

$$\| u \|_{L^\infty(0,T;X)} = \operatorname*{esssup}_{0<t<T} \| u(t) \|_X \quad \text{for} \quad p = \infty.$$

In addition, we denote by $C([0, T]; X)$ all of continuous functions

$$u : (0, T) \to X$$

with

$$\| u \|_{C([0,T];X)} := \max_{0 \le t \le T} \| u(t) \|_X < \infty,$$

and $C^1([0, T]; X)$ all of differential functions

$$u : (0, T) \to X$$

with

$$\| u \|_{C^1([0,T];X)} := \max_{0 \le t \le T} (\| u(t) \|_X + \| u'(t) \|_X) < \infty.$$

Let $u(t)$, $u'(t) = u_t(t)$, $u''(t) = u_{tt}(t)$, $u_x(t)$ and $u_{xx}(t)$ denote $u(x, t)$, $\frac{\partial u}{\partial t}(x, t)$, $\frac{\partial^2 u}{\partial t^2}(x, t)$, $\frac{\partial u}{\partial x}(x, t)$ and $\frac{\partial^2 u}{\partial x^2}(x, t)$, respectively.

We put

$$V = \{v \in H^1 : v(0) = 0\}, \tag{2.1}$$

$$a(u, v) = \left\langle \frac{\partial u}{\partial x}, \frac{\partial v}{\partial x} \right\rangle = \int_0^1 \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} \, dx. \tag{2.2}$$

Here $V$ is a closed subspace of $H^1$ and on $V$, $\| v \|_{H^1}$ and $\| v \|_V = \sqrt{a(v, v)}$ are two equivalent norms.

Then we have the following lemma.

**Lemma 1.** *The embedding* $V \hookrightarrow C^0([0, 1])$ *is compact and*

$$\| v \|_{C^0([0,1])} \leq \| v \|_V, \tag{2.3}$$

*for all* $v \in V$.

We omit the detailed proof because of its certainty.

Moreover, there are following results whose proofs are also omitted.

**Lemma 2.** *Suppose* $u \in L^2(0, T; V)$, *with* $u' \in L^2(0, T; H^{-1}(\Omega))$. *Then*

$$u \in C([0, T]; L^2(\Omega))$$

(*after possibly being redefined on a set of measure zero*).

**Lemma 3.** *Suppose* $u \in L^2(0, T; H^2(\Omega))$, *with* $u' \in L^2(0, T; L^2(\Omega))$. *Then*

$$u \in C([0, T]; V)$$

(*after possibly being redefined on a set of measure zero*).

## 3. Unique Solvability

First and foremost, we make some following essential assumptions:

$(A_\mu)$ $\mu \in H^2(0, T)$, $\mu(t) \geq \mu_0 > 0$;

$(A_F)$ $K, \lambda \in \mathbb{R}$; $p, q \geq 2$;

$(A_f)$ $f, f_t \in L^2(Q_T)$;

$(A_{K_1})$ $K_1 \in H^1(0, T)$, $K_1(t) \geq 0$;

$(A_{\lambda_1})$ $\lambda_1 \in H^1(0, T)$, $\lambda_1(t) \geq \lambda_0 > 0$;

$(A_g)$ $g \in H^1(0, T)$;

$(A_k)$ $k \in H^1(0, T)$;

$(A_{0,1})$ $u_0 \in V \cap H^2$, $u_1 \in H^1$.

In this paper, we say that a function

$$u \in C^1([0, T]; L^2) \cap C([0, T]; V)$$

is a *weak solution* of problem (1.1)-(1.5) iff

$$\begin{cases} \langle u''(t), v\rangle + \mu(t)\langle u_x(t), v_x(t)\rangle + Q(t)v(1) + \langle F(u(t), u'(t)), v\rangle = \langle f(t), v\rangle, \\ u(x, 0) = u_0(x), u'(x, 0) = u_1(x), \\ Q(t) = K_1(t)u(1, t) + \lambda_1(t)u'(1, t) - g(t) - \int_0^t k(t-s)u(1, s)ds, \end{cases}$$

for all $v \in V$. And we can say that problem (1.1)-(1.5) is *solvable* in $C^1([0, T]; L^2) \cap C([0, T]; V)$ with respect to a weak sense.

Then, we have the following theorem:

**Theorem 1.** *Let* $(A_\mu)$, $(A_F)$, $(A_f)$, $(A_{K_1})$, $(A_{\lambda_1})$, $(A_g)$, $(A_k)$ *and* $(A_{0,1})$ *hold. Then, for* $T > 0$, *the problem* (1.1)-(1.5) *has a unique weak solution* $u(x, t)$ *satisfying*

$$u \in C^1([0, T]; L^2) \cap C([0, T]; V). \tag{3.1}$$

**Remark 1.** To be honest, this result still holds when the nonlinearity of damping source $F(u, u_t)$ is more general. However, besides the usage of different comprehensive method or the expected generalizations, we truly want to cover some open questions. For details, let us see the next remark.

**Proof.** The proof consists of several steps as follows.

**Step 1.** The solvability in $C^1([0, T]; L^2)$.

Let the operator $\Xi$ be defined as follows. Given a function $u \in C^1([0, T]; L^2)$, set $£(t) := F(u(t), u_t(t)) - f(t), 0 \le t \le T$. From $(A_F)$ and $(A_f)$, we deduce that

$$£, £' \in L^2(Q_T). \tag{3.2}$$

Then, there is the following lemma whose proof is similar to what in [13].

**Lemma 4.** *With the advent of* (3.2) *and assumptions* $(A_\mu)$, $(A_{K_1})$, $(A_{\lambda_1})$, $(A_g)$, $(A_k)$ *and* $(A_{0,1})$, *the linear initial-boundary value problem given by*

$$\begin{cases} w_{tt} - \mu(t) w_{xx} = -\pounds \text{ in } Q_T, \\ w(0, t) = 0, \\ -\mu(t) w_x(1, t) = P(t), \\ w(x, 0) = u_0(x), \ w_t(x, 0) = u_1(x), \\ P(t) = K_1(t) w(1, t) + \lambda_1(t) w_t(1, t) - g(t) - \int_0^t k(t-s) w(1, s) ds \end{cases} \tag{3.3}$$

*has a unique weak solution* $w(x, t)$ *such that*

$$w \in L^\infty(0, T; V \cap H^2), \quad w' \in L^\infty(0, T; V), \quad w'' \in L^\infty(0, T; L^2). \tag{3.4}$$

By using the embedding $H^2(0, T) \hookrightarrow C^1([0, T])$ and applying Lemma 2 and Lemma 3, we deduce from (3.4) that

$$w \in C^1([0, T]; L^2) \cap C([0, T]; V). \tag{3.5}$$

And $w$ satisfies

$$\begin{cases} \langle w''(t), v \rangle + \mu(t) \langle w_x(t), v_x(t) \rangle + P(t) v(1) + \langle \pounds(t), v \rangle = 0, \\ w(x, 0) = u_0(x), \ w'(x, 0) = u_1(x), \\ P(t) = K_1(t) w(1, t) + \lambda_1(t) w'(1, t) - g(t) - \int_0^t k(t-s) w(1, s) ds, \end{cases} \tag{3.6}$$

for all $v \in V$.

Define $\Xi : C^1([0, T]; L^2) \to C^1([0, T]; L^2)$ by setting

$$\Xi u = w. \tag{3.7}$$

It is claimed that if $T > 0$ is small enough, then $\Xi$ is a strict contraction. To prove this, choose $u, \tilde{u} \in C^1([0, T]; L^2)$ and define $w = \Xi u$, $\tilde{w} = \Xi \tilde{u}$ as above. As a result, $w$ verifies (3.6) for $\pounds = F(u, u_t) - f$, and $\tilde{w}$ satisfies a similar system to (3.6) for

$$\begin{cases} \tilde{\pounds} := F(\tilde{u}, \tilde{u}_t) - f, \\ \tilde{P}(t) = K_1(t) \tilde{w}(1, t) + \lambda_1(t) \tilde{w}'(1, t) - g(t) - \int_0^t k(t-s) \tilde{w}(1, s) ds. \end{cases} \tag{3.8}$$

In addition, we have that

$$\langle w''(t) - \widetilde{w}''(t),\, v\rangle + \mu(t)\langle w_x(t) - \widetilde{w}_x(t),\, v_x\rangle + (P(t) - \widetilde{P}(t))v(1)$$
$$+ \langle \pounds(t) - \widetilde{\pounds}(t),\, v\rangle = 0, \tag{3.9}$$

for all $v \in V$.

Now, in (3.9), replacing $v$ by $w' - \widetilde{w}'$ and then integrating with respect to $t$, we get

$$S(t) = \int_0^t \mu'(s)\| w_x(s) - \widetilde{w}_x(s)\|^2 ds + \int_0^t K_1'(s)[w(1,\, s) - \widetilde{w}(1,\, s)]^2 ds$$
$$- 2\int_0^t [w'(1,\, s) - \widetilde{w}'(1,\, s)]\left(\int_0^s k(s-\tau)[w(1,\, \tau) - \widetilde{w}(1,\, \tau)]d\tau\right)ds$$
$$+ 2\int_0^t \langle \pounds(s) - \widetilde{\pounds}(s),\, w'(s) - \widetilde{w}'(s)\rangle ds, \tag{3.10}$$

in which

$$S(t) = \| w'(t) - \widetilde{w}'(t)\|^2 + \mu(t)\| w_x(t) - \widetilde{w}_x(t)\|^2 + K_1(t)[w(1,\, t) - \widetilde{w}(1,\, t)]^2$$
$$+ 2\int_0^t \lambda_1(s)[w'(1,\, s) - \widetilde{w}'(1,\, s)]^2 ds. \tag{3.11}$$

From (2.3), (3.2), (3.10), (3.11) and assumptions $(A_\mu)$, $(A_{K_1})$, $(A_\lambda)$, $(A_k)$, we deduce some following estimates:

$$\int_0^t \mu'(s)\| w_x(s) - \widetilde{w}_x(s)\|^2 ds \le \frac{1}{\mu_0}\int_0^t \mu'(s)S(s)ds, \tag{3.12}$$

$$\int_0^t K_1'(s)[w(1,\, s) - \widetilde{w}(1,\, s)]^2 ds \le \int_0^t \frac{K_1'(s)}{\mu_0} S(s)ds, \tag{3.13}$$

$$- 2\int_0^t [w'(1,\, s) - \widetilde{w}'(1,\, s)]\left(\int_0^s k(s-\tau)[w(1,\, \tau) - \widetilde{w}(1,\, \tau)]d\tau\right)ds$$
$$\le \varepsilon\frac{S(t)}{2\lambda_0} + \frac{T}{\varepsilon\mu_0}\| k\|^2_{L^2(0,T)}\int_0^t S(s)ds, \tag{3.14}$$

$$2\int_0^t \langle \mathcal{L}(s) - \widetilde{\mathcal{L}}(s),\ w'(s) - \widetilde{w}'(s)\rangle ds \leq \int_0^t \| \mathcal{L}(s) - \widetilde{\mathcal{L}}(s) \|^2 ds + \int_0^t S(s) ds, \quad (3.15)$$

for some $\varepsilon > 0$.

With the relevant choice of $\varepsilon$, namely $\varepsilon = \lambda_0$, also using Gronwall's inequality, we conclude from (3.10)-(3.15), that

$$S(t) \leq \left(2\int_0^t \| \mathcal{L}(s) - \widetilde{\mathcal{L}}(s) \|^2 ds\right) \exp[E(t)], \quad (3.16)$$

where

$$E(t) = 2\int_0^t \left[1 + \frac{1}{\mu_0}\left(\mu'(s) + K_1'(s) + \frac{T}{\lambda_0} \| k \|^2_{L^2(0,\, T)}\right)\right] ds. \quad (3.17)$$

Using assumptions $(A_\mu)$, $(A_{K_1})$ and $(A_k)$, we deduce from (3.17) that there exists a constant $E_0 > 0$ (independent of $t$) such that

$$2\exp[E(t)] \leq E_0, \quad \forall t \in [0, T]. \quad (3.18)$$

From (3.18), the assumption $(A_F)$ and the following inequality

$$|\, | x |^\alpha x - | y |^\alpha y \,| \leq (\alpha + 1)[\max\{| x |, | y |\}]^\alpha | x - y |, \quad \forall x, y \in \mathbb{R}, \forall \alpha > 0,$$

we discover that (3.12) is equivalent to

$$S(t) \leq 2TE_0 \| u - \widetilde{u} \|^2_{C^1([0, T]; L^2)}, \quad \forall t \in [0, T]. \quad (3.19)$$

Combining (3.11) and (3.19), it follows that

$$\| w'(t) - \widetilde{w}'(t) \|^2 \leq 2TE_0 \| u - \widetilde{u} \|^2_{C^1([0, T]; L^2)}, \quad \forall t \in [0, T]. \quad (3.20)$$

Moreover, it is not difficult to affirm that

$$\| w(t) - \widetilde{w}(t) \|^2 \leq 2T^3 E_0 \| u - \widetilde{u} \|^2_{C^1([0, T]; L^2)}, \quad \forall t \in [0, T]. \quad (3.21)$$

Hence, after maximizing the left hand sides of (3.20) and (3.21) with respect to $t$, we discover

$$\| w - \widetilde{w} \|^2_{C^1([0, T]; L^2)} \leq 2E_0 T(T^2 + 1) \| u - \widetilde{u} \|^2_{C^1([0, T]; L^2)}. \quad (3.22)$$

Thus,

$$\| \Xi u - \Xi \tilde{u} \|_{C^1([0,T]; L^2)} \leq \sqrt{2E_0 T(T^2 + 1)} \| u - \tilde{u} \|_{C^1([0,T]; L^2)} \tag{3.23}$$

and then $\Xi$ is a strict contraction, provided $T > 0$ is so small that

$$\sqrt{2E_0 T(T^2 + 1)} = \alpha < 1.$$

As a result, with the application of Banach's fixed point theorem, we conclude that problem (1.1)-(1.5) is solvable in $C^1([0, T]; L^2)$ with respect to a weak sense.

In the case of $T > 0$ given, we select $T_1 > 0$ so small that

$$\sqrt{2E_0 T_1(T_1^2 + 1)} < 1.$$

Then we are able to apply Banach's fixed point theorem to find a weak solution $u$ of problem (1.1)-(1.5) existing on the time interval $[0, T_1]$. Due to $u(t)$, $u'(t) \in L^2(\Omega)$ for a.e. $0 \leq t \leq T_1$, we can continue redefining $T_1$ if necessary by assuming $u(T_1)$, $u'(T_1) \in L^2(\Omega)$. We can then repeat the argument above to extend our solution to the time interval $[T_1, 2T_1]$. Continuing, after finitely many steps, we construct a weak solution existing on the full interval $[0, T]$.

**Step 2.** The solvability in $C([0, T]; V)$.

Repeating the technical arguments in Step 1 in which $C^1([0,T]; L^2)$ is replaced by $C([0, T]; V)$, we also define the operator

$$\Xi : C([0, T]; V) \to C([0, T]; V).$$

And we are going to show that $\Xi$ is also strict contracted in $C([0, T]; V)$.

Indeed, from (3.10)-(3.15), we compute that

$$S(t) \leq 2\widetilde{E_0}\left( \int_0^t \| u(s) - \tilde{u}(s) \|^2 ds \right), \tag{3.24}$$

where $\widetilde{E_0}$ is a positive constant independent of $t$ such that

$$\widetilde{E_0} \geq 2\exp\left(\int_0^t \left[3 + \frac{1}{\mu_0}\left(\mu'(s) + K_1'(s) + \frac{T}{\lambda_0}\| k \|^2_{L^2(0,T)}\right)\right] ds\right), \tag{3.25}$$

for all $t \in [0, T]$.

Combining (3.11) and (3.25) and using the embedding $V \hookrightarrow L^2$, we discover

$$\| w(t) - \tilde{w}(t) \|_V^2 \leq \frac{2\widetilde{E_0}T}{\mu_0} \max_{0\leq s\leq T} \| u(s) - \tilde{u}(s) \|_V^2, \quad \forall t \in [0, T]$$

$$\leq \frac{2\widetilde{E_0}T}{\mu_0} \| u - \tilde{u} \|^2_{C([0,T];V)}, \quad \forall t \in [0, T]. \tag{3.26}$$

After maximizing the left hand side of (3.26) with respect to $t$, we obtain

$$\| w - \tilde{w} \|^2_{C([0,T];V)} \leq \frac{2\widetilde{E_0}T}{\mu_0} \| u - \tilde{u} \|^2_{C([0,T];V)}. \tag{3.27}$$

Consequently, we receive

$$\| \Xi u - \Xi \tilde{u} \|_{C([0,T];V)} \leq \sqrt{\frac{2\widetilde{E_0}T}{\mu_0}} \| u - \tilde{u} \|^2_{C([0,T];V)}, \tag{3.28}$$

and $\Xi$ is also contracted, provided $T > 0$ is so small that

$$\sqrt{\frac{2\widetilde{E_0}T}{\mu_0}} < 1.$$

So, applying Banach's fixed point theorem, we deduce that problem (1.1)-(1.5) is solvable in $C([0, T]; V)$ in a weak sense. Certainly, this still holds when $T > 0$ is arbitrarily given, a weak solution existing on the full interval $[0, T]$.

**Step 3.** The uniqueness of the weak solution.

To demonstrate uniqueness, suppose both $u$ and $\widetilde{u}$ are two weak solutions of problem (1.1)-(1.5). Then we have $w = u$, $\widetilde{w} = \widetilde{u}$ in (3.24), hence we discover

$$\| u(t) - \tilde{u}(t) \|_V^2 \leq \frac{2\widetilde{E_0}}{\mu_0} \int_0^t \| u(s) - \tilde{u}(s) \|_V^2 \, ds. \tag{3.29}$$

Because of Gronwall's inequality, we deduce from (3.29), that $u \equiv \tilde{u}$.

The above three steps show that the proof of the theorem is complete.

**Remark 2.** In fact, for Galerkin approximation, if the damping source of related problems is $F(u, u_t) = K|u|^{p-2}u + \lambda|u_t|^{q-2}u_t$, $p, q \geq 2$, then both parameters $K$ and $\lambda$ are mostly non-negative, in our knowledge. In the above theorem, the unique solvability of problem (1.1)-(1.5) really holds in the case of $K, \lambda \in \mathbb{R}$. It is strongly enjoyable to affirm that the open problems in [13, Remark 2], [9], etc., have been completely solved.

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