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ON SOME RESULTS IN INTUITIONISTIC FUZZY METRIC SPACE

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Abstract

In this paper, we define a continuous mapping on intuitionistic fuzzy metric space introduced by Park et al. [7] and obtain some similar results as in metric space.

1. Introduction

The theory of fuzzy sets was introduced by Zadeh [10] in 1965. We have introduced the concept of intuitionistic fuzzy metric space ([5-8]). Park et al. [7] have defined the intuitionistic fuzzy metric space which is a little revised from Park [3]. According to this paper, Park et al. [5, 6, 8] have established some results in the intuitionistic fuzzy metric space. Furthermore, Park et al. [7] proved some other results of maps on intuitionistic fuzzy metric spaces. In this paper, we modify the concept of intuitionistic fuzzy metric space introduced by Park et al. [7] and define a continuous mapping on this space. Also, we prove that compactness implies IF-boundedness and completeness, and prove that if a sequence of continuous mappings converges to some mapping, then some mapping is continuous in intuitionistic fuzzy metric space (cf. [2]).

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2. Preliminaries

We give some definitions, properties and notations of the intuitionistic fuzzy metric space as follows (Schweizer and Sklar [9], Grabiec [1] and Park et al. [7]):

Definition 2.1 [9]**.** An operation $* : [0, 1] \times [0, 1] \to [0, 1]$ is *continuous t-norm* if $*$ satisfies the following conditions:

(a) $*$ is commutative and associative,

(b) $*$ is continuous,

(c) $a * 1 = a$ for all $a \in [0, 1]$,

(d) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ $(a, b, c, d \in [0, 1])$.

Definition 2.2 [9]**.** An operation $\diamond : [0, 1] \times [0, 1] \to [0, 1]$ is *continuous t-conorm* if $\diamond$ satisfies the following conditions:

(a) $\diamond$ is commutative and associative,

(b) $\diamond$ is continuous,

(c) $a \diamond 0 = a$ for all $a \in [0, 1]$,

(d) $a \diamond b \geq c \diamond d$ whenever $a \leq c$ and $b \leq d$ $(a, b, c, d \in [0, 1])$.

Remark 2.3 [3]**.** The following conditions are satisfied:

(a) For any $r_1, r_2 \in (0, 1)$ with $r_1 > r_2$, there exist $r_3, r_4 \in (0, 1)$ such that $r_1 * r_3 \geq r_2$ and $r_4 \diamond r_2 \leq r_1$.

(b) For any $r_5 \in (0, 1)$, there exist $r_6, r_7 \in (0, 1)$ such that $r_6 * r_6 \geq r_5$ and $r_7 \diamond r_7 \leq r_5$.

Definition 2.4 [7]**.** The 5-tuple $(X, M, N, *, \diamond)$ is said to be an *intuitionistic fuzzy metric space* if X is an arbitrary set, $*$ is a continuous t-norm, $\diamond$ is a continuous t-conorm and M and N are fuzzy sets on $X^2 \times (0, \infty)$ satisfying the following conditions; for all $x, y, z \in X$, such that

(a) $M(x, y, t) > 0$,

(b) $M(x, y, t) = 1 \Leftrightarrow x = y$,

(c) $M(x, y, t) = M(y, x, t)$,

(d) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,

(e) $M(x, y, \cdot) : (0, \infty) \to (0, 1]$ is continuous,

(f) $N(x, y, t) > 0$,

(g) $N(x, y, t) = 0 \Leftrightarrow x = y$,

(h) $N(x, y, t) = N(y, x, t)$,

(i) $N(x, y, t) \diamond N(y, z, s) \geq N(x, z, t + s)$,

(j) $N(x, y, \cdot) : (0, \infty) \to (0, 1]$ is continuous.

Then (M, N) is called an *intuitionistic fuzzy metric* on X. The functions $M(x, y, t)$ and $N(x, y, t)$ denote the degree of nearness and the degree of non-nearness between x and y with respect to t, respectively.

Remark 2.5 [6]**.** In an intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$, $M(x, y, \cdot)$ is nondecreasing and $N(x, y, \cdot)$ is nonincreasing for all $x, y \in X$.

Example 2.6. Let (X, d) be a metric space. Denote $a * b = \min\{a, b\}$, $a \diamond b = \max\{a, b\}$ for all $a, b \in [0, 1]$ and let M_d and N_d be fuzzy sets on $X^2 \times (0, \infty)$ defined as follows:

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}, \qquad N_d(x, y, t) = \frac{d(x, y)}{t + d(x, y)}.$$

Then (M_d, N_d) is called *intuitionistic fuzzy metric* induced by a metric d the standard intuitionistic fuzzy metric. Also, $(X, M_d, N_d, *, \diamond)$ is an intuitionistic fuzzy metric space.

Throughout this paper, **N** denotes the set of natural numbers and X denotes an intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$ and Y stands for $(Y, \overline{M}, \overline{N}, *, \diamond)$ with the following properties:

$$\lim_{t\to\infty} M(x_1, x_2, t) = 1, \quad \lim_{t\to\infty} N(x_1, x_2, t) = 0 \quad \text{for all } x_1, x_2 \in X,$$

$$\lim_{t\to\infty} \overline{M}(y_1, y_2, t) = 1, \quad \lim_{t\to\infty} \overline{N}(y_1, y_2, t) = 0 \quad \text{for all } y_1, y_2 \in Y.$$

Definition 2.7 [3]**.** Let X be an intuitionistic fuzzy metric space and let $r \in (0, 1)$, $t > 0$ and $x \in X$. Then the set $B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r, N(x, y, t) < r\}$ is called the *open ball* with center x and radius r with respect to t.

Remark 2.8 [3]**.** Every open ball $B(x, r, t)$ is an open set.

Definition 2.9 [3]**.** Let X be an intuitionistic fuzzy metric space. Then a subset C of X is said to be *IF-bounded* if there exist $t > 0$ and $r \in (0, 1)$ such that $M(x, y, t) > 1 - r$ and $N(x, y, t) < r$ for all $x, y \in C$.

Remark 2.10 [3]**.** Let X be an intuitionistic fuzzy metric space induced by a metric d on X. Then $A \subset X$ is *IF-bounded* iff it is bounded.

Definition 2.11 [5]**.** Let X be an intuitionistic fuzzy metric space. Then

(a) a sequence $\{x_n\}$ in X is called *Cauchy sequence* iff for each $\varepsilon \in (0, 1)$ and each $t > 1$, there exists $n_0 \in \mathbf{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$, $N(x_n, x_m, t) < \varepsilon$ for all $m, n \geq n_0$;

(b) a sequence $\{x_n\}$ in X is convergent to x in X iff $\lim_{n\to\infty} M(x_n, x, t) = 1$, $\lim_{n\to\infty} N(x_n, x, t) = 0$ for each $t > 0$;

(c) X is said to be *complete* if every Cauchy sequence is convergent in X.

(d) X is called *compact* if every sequence contains a convergent subsequence in X.

3. Some Results

In this section, we prove that compactness implies IF-boundedness and completeness. Also, we define a continuous mapping on intuitionistic fuzzy metric space, and prove that if a sequence of continuous mappings converges to some mapping, then some mapping is continuous in intuitionistic fuzzy metric space. Our research is an extension of Kreyszig's result [2].

Theorem 3.1. *A compact subset C of intuitionistic fuzzy metric space X is IF-bounded and complete.*

Proof. Fixed $t > 0$ and $r \in (0, 1)$. Consider an open cover $\{B(x, r, t) : x \in C\}$ of C. Since C is compact, there exist $x_1, x_2, ..., x_n \in C$ such that $C \subseteq \bigcup_{i=1}^{n} B(x_i, r, t)$. Let $x, y \in C$. Then $x \in B(x_i, r, t)$ and $y \in B(x_j, r, t)$ for some i, j. Thus we have $M(x, x_i, t) > 1 - r$, $N(x, x_i, t) < r$, $M(y, x_j, t) > 1 - r$ and $N(y, x_j, t) < r$. Let $\alpha_1 = \min\{M(x_i, x_j, t) : 1 \leq i, j \leq n\}$ and $\alpha_2 = \max\{N(x_i, x_j, t) : 1 \leq i, j \leq n\}$. Then $\alpha_1, \alpha_2 > 0$.

Now, we have

$$
\begin{aligned}
M(x, y, 3t) &\geq M(x, x_i, t) * M(x_i, x_j, t) * M(x_j, y, t) \\
&\geq (1 - r) * \alpha_1 * (1 - r) \\
&> 1 - s_1, \\
N(x, y, 3t) &\leq N(x, x_i, t) \diamond N(x_i, x_j, t) \diamond N(x_j, y, t) \\
&\leq r \diamond \alpha_2 \diamond r \\
&< s_2,
\end{aligned}
$$

for some $0 < s_1, s_2 < 1, s_1, s_2, r, 1 - s_1 < \alpha_1$ and $\alpha_2 < s_2$. Taking $s = \max\{s_1, s_2\}$ and $3t = t'$, $M(x, y, t') > 1 - s$ and $N(x, y, t') < s$ for all $x, y \in C$. Hence, C is IF-bounded.

Since $C \subset X$ is compact, let $\{x_n\}$ be a Cauchy sequence in C and $\{x_{n_i}\} \subset \{x_n\}$ that converges to x. Also, let $t > 0$ and $\varepsilon \in (0, 1)$. Choose $r \in (0, 1)$ such that $(1 - r) * (1 - r) \geq 1 - \varepsilon$ and $r \diamond r \leq \varepsilon$. Since $\{x_n\}$ is Cauchy sequence, there exists $n_0 \in \mathbf{N}$ such that

$$M\left(x_m, x_n, \frac{t}{2}\right) > 1 - r, \quad N\left(x_m, x_n, \frac{t}{2}\right) < r$$

for all $m, n \geq n_0$. Also, since $x_{n_i} \to x$, there is a positive integer n_p such that $n_p > n_0$,

$$M\left(x_{n_p}, x, \frac{t}{2}\right) > 1 - r, \quad N\left(x_{n_p}, x, \frac{t}{2}\right) < r.$$

Thus, if $n \geq n_0$, then

$$\begin{aligned} M(x_n, x, t) &\geq M\left(x_n, x_{n_p}, \frac{t}{2}\right) * M\left(x_{n_p}, x, \frac{t}{2}\right) \\ &> (1 - r) * (1 - r) \\ &\geq 1 - \varepsilon, \\ N(x_n, x, t) &\leq N\left(x_n, x_{n_p}, \frac{t}{2}\right) \diamond N\left(x_{n_p}, x, \frac{t}{2}\right) \\ &< r \diamond r \\ &\leq \varepsilon. \end{aligned}$$

Hence for arbitrary $\varepsilon \in (0, 1)$,

$$\lim_{n \to \infty} M(x_n, x, t) = 1$$

and

$$\lim_{n \to \infty} N(x_n, x, t) = 0.$$

Therefore, $x_n \to x$. That is, C is complete.

Definition 3.2. Let $(X, M, N, *, \diamond) = X$ and $(Y, \overline{M}, \overline{N}, *, \diamond) = Y$ be intuitionistic fuzzy metric spaces. Then a mapping $T : X \to Y$ is *continuous* at a point $x_0 \in X$ if for every $r > 0$, there is $s > 0$ with $s < r$ such that

$$\overline{M}(Tx, Tx_0, t) > 1 - r, \quad \overline{N}(Tx, Tx_0, t) < r$$

for all x satisfying

$$M(x, x_0, t) > 1 - s, \quad N(x, x_0, t) < s,$$

where $(1 - r) * (1 - r) > 1 - s$ and $r \diamond r < s$.

T is said to be *continuous* if it is continuous at every point of X.

Theorem 3.3. *Let X and Y be intuitionistic fuzzy metric spaces. Then a mapping T of X into Y is continuous at a point $x_0 \in X$ iff $x_n \to x_0$ implies $Tx_n \to Tx_0$.*

Proof. Assume T to be continuous at x_0. For a given $r \in (0, 1)$, we choose $s \in (0, 1)$ with $s < r$ such that $(1 - r) * (1 - r) > 1 - s$ and $r \diamond r < s$. Then by Definition 3.2 $M(x, x_0, t) > 1 - s$ and $N(x, x_0, t) < s$ imply $\overline{M}(Tx, Tx_0, t) > 1 - r$ and $\overline{N}(Tx, Tx_0, t) < r$. Let $x_n \to x_0$. Then we have $M(x_n, x_0, t) > 1 - s$ and $N(x_n, x_0, t) < s$.

Hence for all $n > n_0$,

$$\begin{aligned}\overline{M}(Tx_n, Tx_0, t) &\geq \overline{M}\left(Tx_n, Tx, \frac{t}{2}\right) * \overline{M}\left(Tx, Tx_0, \frac{t}{2}\right)\\ &> (1 - r) * (1 - r)\\ &> 1 - s\\ &> 1 - r,\\ \overline{N}(Tx_n, Tx_0, t) &\leq \overline{N}\left(Tx_n, Tx, \frac{t}{2}\right) \diamond \overline{N}\left(Tx, Tx_0, \frac{t}{2}\right)\\ &< r \diamond r\\ &< s\\ &< r.\end{aligned}$$

Thus $Tx_n \to Tx_0$.

Conversely, we assume that $x_n \to x_0$ implies $Tx_n \to Tx_0$ and prove that T is continuous at x_0. Suppose that this is false. Then there is $r > 0$ such that for every $s > 0$ with $s < r$, there is $x \neq x_0$ satisfying $M(x, x_0, t) > 1 - s$ and $N(x, x_0, t) < s$, but $\overline{M}(Tx, Tx_0, t) \leq 1 - r$ and $\overline{N}(Tx, Tx_0, t) \geq r$. In particular, for $s = \frac{1}{n}$, there is x_n satisfying $M(x_n, x_0, t) > 1 - \frac{1}{n}$ and $N(x_n, x_0, t) < \frac{1}{n}$, but $\overline{M}(Tx_n, Tx_0, t) \leq 1 - r$ and $\overline{N}(Tx_n, Tx_0, t) \geq r$. Therefore $x_n \to x_0$, but Tx_n does not converge to Tx_0. This contradicts $Tx_n \to Tx_0$ and proves this theorem.

Theorem 3.4 (Continuous mapping)**.** *Let X and Y be intuitionistic fuzzy metric spaces and let $\{T_n\} : X \to Y$ be a sequence of continuous mappings. If sequence $\{T_n\}$ converges to $T : X \to Y$, then T is a continuous mapping.*

Proof. Let $\{T_n\}$ be a sequence of continuous mappings. Then for all $\{x_n\} \subset X$ with $x_n \to x_0$,

$$\overline{M}(T_n x_n, T_n x_0, t) > 1 - s, \quad \overline{N}(T_n x_n, T_n x_0, t) < s \quad \text{for } s \in (0, 1).$$

We choose $r \in (0, 1)$ such that $(1 - s) * (1 - s) * (1 - s) > 1 - r$ and $s \diamond s \diamond s < r$. Since $\{T_n\}$ converges to T, for given $t > 0$ and $r \in (0, 1)$, there exists $n_0 \in \mathbf{N}$ such that

$$\overline{M}\left(T_n x_n, Tx_n, \frac{t}{3}\right) > 1 - s, \quad \overline{N}\left(T_n x_n, Tx_n, \frac{t}{3}\right) < s,$$

$$\overline{M}\left(T_n x_0, Tx_0, \frac{t}{3}\right) > 1 - s, \quad \overline{N}\left(T_n x_0, Tx_0, \frac{t}{3}\right) < s$$

for all $n \geq n_0$ and for all $\{x_n\} \subset X$ with $x_n \to x_0$. Also, since T_n is a continuous mapping for all $n \in \mathbf{N}$,

$$\overline{M}\left(T_n x_n, T_n x_0, \frac{t}{3}\right) > 1 - s, \quad \overline{N}\left(T_n x_n, T_n x_0, \frac{t}{3}\right) < s$$

for all $\{x_n\} \subset X$ with $x_n \to x_0$.

Now,

$$\overline{M}(Tx_n, Tx_0, t)$$

$$\geq \overline{M}\left(Tx_n, T_n x_n, \frac{t}{3}\right) * \overline{M}\left(T_n x_n, T_n x_0, \frac{t}{3}\right) * \overline{M}\left(T_n x_0, Tx_0, \frac{t}{3}\right)$$

$$> (1 - s) * (1 - s) * (1 - s)$$

$$> 1 - r,$$

$$\overline{N}(Tx_n, Tx_0, t)$$

$$\leq \overline{N}\left(Tx_n, T_n x_n, \frac{t}{3}\right) \diamond \overline{N}\left(T_n x_n, T_n x_0, \frac{t}{3}\right) \diamond \overline{N}\left(T_n x_0, Tx_0, \frac{t}{3}\right)$$

$$< s \diamond s \diamond s$$

$$< r.$$

Hence T is continuous for all $\{x_n\} \subset X$ such that $x_n \to x_0$.

References

[1] M. Grabiec, Fixed point in fuzzy metric spaces, Fuzzy Sets and Systems 27 (1988), 385-389.

[2] E. Kreyszig, Introductory Functional Analysis with Applications, John Wiley and Sons, 1978.

[3] J. H. Park, Intuitionistic fuzzy metric spaces, Chaos Solitons Fractals 22(5) (2004), 1039-1046.

[4] J. S. Park and S. Y. Kim, A fixed point theorem in a fuzzy metric space, Far East J. Math. Sci. (FJMS) 1(6) (1999), 927-934.

[5] J. S. Park and Y. C. Kwun, Some fixed point theorems in the intuitionistic fuzzy metric spaces, Far East J. Math. Sci. (FJMS) 24(2) (2007), 227-239.

[6] J. S. Park, Y. C. Kwun and J. H. Park, A fixed point theorem in the intuitionistic fuzzy metric spaces, Far East J. Math. Sci. (FJMS) 16(2) (2005), 137-149.

[7] J. H. Park, J. S. Park and Y. C. Kwun, A common fixed point theorem in the intuitionistic fuzzy metric space, Advances in Natural Comput. Data Mining, Proc. 2nd ICNC and 3rd FSKD, 2006, 293-300.

[8] J. S. Park, J. H. Park and Y. C. Kwun, Fixed point theorems in intuitionistic fuzzy metric space (I), JP Jour. Fixed Point Theory & Appl. 2(1) (2007), 79-89.

[9] B. Schweizer and A. Sklar, Statistical metric spaces, Pacific J. Math. 10 (1960), 314-334.

[10] L. A. Zadeh, Fuzzy sets, Inform. and Control 8 (1965), 338-353.

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