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UNRAMIFIED MAPS*

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Abstract

By an unramified map we shall mean a local homeomorphism of locally connected topological spaces that is a cosheaf space in the sense of Funk [Cahiers de Top. et Géom. Diff. Catégoriques 36(1) (1995), 53-93]. Cosheaf spaces have a topological characterization that is almost identical to Fox's notion of a complete spread [R. H. Fox, Covering spaces with singularities, Algebraic Geometry and Topology: A Symposium in Honor of S. Lefschetz, R. H. Fox et al., editors, pp. 243-257, Princeton Univ. Press, Princeton, 1957]. Unramified maps are a generalization of covering spaces. We show that an arbitrary unramified map over a locally connected space has unique path and homotopy lifting. We also show that the pullback of an unramified map along any map of locally connected spaces is again an unramified map. These two results imply that the category of unramified maps is a homotopy invariant of locally path-connected spaces. A covering space of a locally connected space is an unramified map, but over a locally path-connected and semi-locally simply connected space we establish the converse, i.e.,

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we show that an unramified map is a covering space. We provide an example of a connected unramified map over a connected, locally path-connected space that is not a covering space.

0. Introduction

It is well-known that sheaves correspond by an adjointness to sheaf spaces [24]. A sheaf space may be topologically characterized as a local homeomorphism. It is perhaps less well-known that *cosheaves* correspond by an adjointness to *cosheaf locales* [17]. In this paper we shall focus our attention on the category of cosheaf *spaces*, which is equivalent to a full subcategory of cosheaves (called the *spatial cosheaves* in [17]). Cosheaf spaces also have a topological characterization (Proposition 2.1), which is nearly the same as a slightly more general notion due to R. H. Fox [16] called a *complete spread*. Fox had introduced complete spreads as a topological framework by which to explain ramification phenomena such as branching or folding. In Section 2 we review cosheaf spaces and complete spreads for topological spaces, and how they are related.

We shall refer to a map into a locally connected space that is both a sheaf space and cosheaf space as an *unramified map*. For instance, a covering space is an unramified map in this sense (the fact that a covering space is a cosheaf space is explained in [17]). One aim of this paper is to address the question of the converse: when is an unramified map a covering space? This cannot hold in general because a coproduct of covering spaces may not be a covering space, but it is always an unramified map. In Section 3, we provide a positive answer to the converse by showing that over a locally path-connected and semi-locally simply connected space, an unramified map is a covering space. We do this by first showing that over a locally connected space, unramified maps have unique path and homotopy lifting (Theorem 3.1). On the other hand, Example 3.1 describes an unramified map over a locally path-connected space that is not homeomorphic to a small coproduct of covering spaces.

It would be of interest to improve our knowledge of the category of unramified maps (over a locally connected space). For instance, we may immediately deduce from known facts about cosheaf and sheaf spaces that unramified maps are closed under finite limits and small coproducts, which are created in the including category of sheaf spaces. We show that the category of unramified maps is a *homotopy invariant* of locally path-connected spaces (Definition 5.2 and Theorem 5.1). This aspect of unramified maps depends on their stability under pullback, which we establish, and on the fact that they have the homotopy lifting property. Consequently, if two locally path-connected spaces are homotopy equivalent, then their respective categories of unramified maps are equivalent.

Unramified maps, as we have defined them, are connected with topos theory [6, 20, 24]. Though we occasionally comment on this connection (Remarks 3.1 and 4.1 for instance), a familiarity with topos theory is not required to understand our main results. For the interested reader, we include here a brief explanation of unramified maps in the context of toposes. Over a locale, a cosheaf locale and a localic complete spread are equivalent concepts. These localic complete spreads have been generalized to geometric morphisms under the same name complete spread [11, 13]. Complete spread geometric morphisms are related to Lawvere's topos distributions [21, 22], to the symmetric topos [9, 10], and to distribution algebras, introduced in [15]. An unramified map is thus a special case of an object X of a locally connected topos $\mathcal{E}$ with the property that the induced geometric morphism $\mathcal{E}/X \to \mathcal{E}$ is a complete spread. Such *complete spread objects* and their connection with the fundamental group of a topos have been studied in [13]. Those results and the result we establish here for topological spaces (Corollary 3.1) suggest that under reasonably general hypothesis (of the 'locally simply connected' kind), complete spread objects of a locally connected topos are locally constant.

At times we use category theory concepts to express our explanations. These concepts such as functor adjoint functor, pseudo-functor, natural transformation, equivalent categories, limit, colimit, and 2-category are explained in [6, 7, 23]. A *finite limit* is a limit of finitely many objects and morphisms. A *pullback* is an instance of a finite limit. A *small coproduct* is a coproduct that is indexed by a set.

1. Review of Fibrations and Covering Spaces

The path components of an open subset of a *locally path-connected* space are open (and conversely). Furthermore, a locally path-connected space is locally connected, and the path components of an open subset coincide with its connected components. A locally path-connected space is connected if and only if it is path-connected. If the codomain space of a local homeomorphism is locally connected space, then so is the domain space, and likewise for local path-connectedness.

Throughout, I shall denote the unit interval $[0, 1]$. In [3], a continuous map $\varphi : Y \to X$ is said to have the *homotopy lifting property* (*HLP*) *with respect to a space Z* if in the following diagram of topological spaces, given any two maps $Z \xrightarrow{f} X^I$ and $Z \xrightarrow{g} Y$ such that $d_0 \cdot f = \varphi \cdot g$, there is a unique map $Z \xrightarrow{h} Y^I$ such that $\varphi^I \cdot h = f$ and $d_0 \cdot h = g$.

$$\begin{tikzcd}
Z \arrow[dr, dashed, "h"] \arrow[drr, "g"] \arrow[ddr, "f"'] & & \\
& Y^I \arrow[r, "d_0"'] \arrow[d, "\varphi^I"] & Y \arrow[d, "\varphi"] \\
& X^I \arrow[r, "d_0"'] & X
\end{tikzcd}$$

X^I denotes the exponential space of paths in X (X^I carries the compact-open topology). The maps d_0 send a path p to $p(0)$. For instance, HLP with respect to $\{0\}$ means unique path-lifting. The image of a map that has unique path-lifting must be equal to a union of path-components of the codomain space. The following is easily established:

Proposition 1.1. *A map has the HLP with respect to I if and only if it has the HLP with respect to* $\{0\}$ *and homotopy lifting with respect to I* (without uniqueness).

If φ has the HLP with respect to every topological space, then φ is said to be a *fibration*. Equivalently, φ is a fibration if the above square is a pullback in the category of topological spaces. We shall use the following terminology:

Definition 1.1. A *discrete fibration* is a fibration that is also a local homeomorphism.

We next turn to covering spaces. Following [27], an open set $U \subseteq B$ is *evenly covered by a map* $\pi : Y \to B$ if $\pi^{-1}(U)$ has an open partition such that the restriction of π to each member of the partition is a homeomorphism with U. Then π is a covering space if B has a cover of open sets each of which is evenly covered by π. Implicitly, a covering space is an onto map; however, throughout we shall assume only that the image of a covering space of a space B is equal to a (possibly empty) union of connected components of B.

It is well-known that a covering space of a locally path-connected space is a fibration [8, 27]. Since a covering space is a local homeomorphism, in our terminology a covering space of a locally path-connected space is a discrete fibration. We recall that a space X is said to be *semi-locally simply connected* if X has a cover $\{U_\alpha\}$ of open neighbourhoods such that each U_α has the property that any two paths in U_α with common endpoints are homotopic in X by a homotopy that fixes the endpoints. (This is equivalent to saying that the image groupoid of the induced functor of fundamental groupoids $\pi_1(U_\alpha) \to \pi_1(X)$ is equivalent to a trivial groupoid-a trivial groupoid is just a set.) It follows immediately from [27, Chap. 2, Section 4, Theorem 10] that over a locally path-connected and semi-locally simply connected space, a discrete fibration is a covering space. We have the following:

Proposition 1.2. *Over a locally path-connected and semi-locally simply connected space the following notions are equivalent*:

(1) *covering space* (*whose image is equal to a union of connected components*),

(2) *discrete fibration*, *and*

(3) *a local homeomorphism that has the HLP with respect to* $I = [0, 1]$.

Proof. We have only to show that condition (3) is equivalent to the other two. Suppose that X is locally path-connected and semi-locally simply connected. Let $\varphi : Y \to X$ be a local homeomorphism that has the HLP with respect to I. (This amounts to unique path-lifting and homotopy lifting.) We shall show that φ is a covering space. We are assuming that X has a cover of path-connected open neighbourhoods U such that any two paths in U with common endpoints are homotopic in X. Consider one such open set U, and the path components (= connected components) of $\varphi^{-1}(U)$, which are (presumably non-empty) open subsets of $\varphi^{-1}(U)$. Then φ maps every such component homeomorphically onto U. Indeed, let A be a path component of $\varphi^{-1}(U)$. A is open in Y, so the restriction of φ to A is an open map. Since U is path-connected, and by path-lifting, we see immediately that $\varphi : A \to U$ is onto. We have only to show that φ is one-to-one in A. Suppose y and z are two points of A such that $\varphi(y) = \varphi(z) = x$. There is a path f in A joining y and z, so that the image path $p = \varphi \cdot f$ in U begins and ends at x. Since X is semi-locally simply connected, there is a homotopy $H : I \times I \to X$ such that

$$H(0, t) = p(t), \quad H(1, t) = x, \quad H(s, 0) = H(s, 1) = x.$$

By hypothesis, we may lift H to $\overline{H}$ such that $\overline{H}(0, t) = f(t)$. The other three 'edges' $\overline{H}(s, 0)$, $\overline{H}(1, t)$, and $\overline{H}(s, 1)$ of $\overline{H}$ are paths $y \to y' \to z' \to z$ in Y, each lying over the identity path at x. By the uniqueness of path-lifting, we conclude that $y = y' = z' = z$.

2. Review of Cosheaf Spaces and Complete Spreads

We review the notion of cosheaf space [17] and the slightly more general notion of complete spread, due to R. H. Fox. Our terminology is a mixture coming from [5, 16, 17]. We first review complete spreads.

Following [16], a *spread* is a continuous map $\varphi : Y \to X$, where Y is locally connected, such that the connected components of sets $\varphi^{-1}(U)$, for U open in X, are a base for the topology on Y. We shall assume throughout that the domain space of a spread is locally connected, even though the notion can be sensibly generalized to the case of an arbitrary domain space by using quasi-components [16, 25]. In order to formulate completeness we recall that a *cogerm of a map* $\varphi : Y \to X$ *at* $x \in X$ (with Y locally connected) is a consistent choice of connected components $\alpha = \{\alpha_U \subseteq \varphi^{-1}(U)\}$, where U ranges over all neighbourhoods of x. By consistent, we mean that $U \subseteq V \Rightarrow \alpha_U \subseteq \alpha_V$. Then a spread over a space X is *complete* if for every $x \in X$, and every cogerm α at x, the intersection $\bigcap_{x \in U} \alpha_U$ is non-empty.

We next turn to cosheaf spaces. For a given map $\varphi : Y \to X$ (with Y locally connected) consider the collection

$$\widetilde{Y} = \{(x, \alpha) \mid \alpha \text{ is a cogerm of } \varphi \text{ at a point } x \in X\}.$$

$\widetilde{Y}$ is topologized by the basic sets

$$(U, \beta) = \{(x, \alpha) \mid x \in U, \alpha_U = \beta\},$$

where U is an open set of X, and β is a connected component of $\varphi^{-1}(U)$. We refer to the topological space $\widetilde{Y}$ as the *display space* associated with the *cosheaf* $U \mapsto \pi_0(\varphi^{-1}(U))$, where π_0 denotes connected components. $\widetilde{Y}$ is continuously fibered over X in the obvious way. A fiber over a given $x_0 \in X$ is sometimes called a *costalk*; it consists of the collection of pairs $\{(x_0, \alpha)\}$. A costalk may also be regarded as the limit

$$\lim_{\substack{\leftarrow \\ x_0 \in U}} \pi_0(\varphi^{-1}(U)),$$

taken over the filter of open neighbourhoods of x_0. $\widetilde{Y}$ is locally connected. Every element $y \in Y$ determines a cogerm at $\varphi(y)$: take α_U

to be the unique component of $\varphi^{-1}(U)$ that contains y. This defines a continuous map from Y into $\widetilde{Y}$ (over X), denoted

$$\eta : Y \to \widetilde{Y}; \quad \eta(y) = (\varphi(y), \alpha).$$

It follows easily that the inverse image set $\eta^{-1}(U, \beta)$ is equal to β. A *cosheaf space* is then a map $\varphi : Y \to X$ (with Y locally connected) for which η is a homeomorphism. If Y is locally connected and $Y \xrightarrow{\varphi} X$ is any map, then the canonical projection $\widetilde{Y} \xrightarrow{\widetilde{\varphi}} X$ is a cosheaf space. Furthermore, there are adjoint functors connecting cosheaves and cosheaf spaces, which induces an equivalence between the category of cosheaf spaces over X and a full subcategory of cosheaves on X (called the *spatial cosheaves* in [17, Definition 5.12]). The above map η is the unit of this adjointness. The inclusion of spatial cosheaves in cosheaves has a right adjoint.

Cosheaf spaces and complete spreads are nearly the same. The space $\widetilde{Y}$ is precisely Fox's construction of the completion of a spread. A map with locally connected domain over a T_1 space is a cosheaf space if and only if it is a complete spread with T_1 domain [17, Definition 5.17]. The following proposition may help to clarify the notion of cosheaf space and its connection with complete spreads:

Proposition 2.1 (Topological characterization of cosheaf spaces)**.** *For any map $\varphi : Y \to X$ with locally connected domain, the following are equivalent*:

(1) φ *is a cosheaf space.*

(2) φ *is a spread and* η *is a bijection, in which case the inverse of* η *is given by*:

$$\eta^{-1}(x, \alpha) = \left(\bigcap_{x \in U} \alpha_U\right) \cap \varphi^{-1}(x).$$

(3) φ *is a spread, and for every* $x \in X$ *and every cogerm* α *of* φ *at* x, *the set*

$$\left(\bigcap_{x \in U} \alpha_U\right) \cap \varphi^{-1}(x)$$

is equal to a singleton.

Proof. (1) implies (2) because a cosheaf space is a spread with locally connected domain [17, 5.14, 5.17], and η is a bijection, being a homeomorphism. Conversely, if η is a bijection, then the image set $\eta(\beta)$ is equal to (U, β), where β is a component of $\varphi^{-1}(U)$. If φ is a spread, then the β are a base for Y, so that η is open, hence a homeomorphism. The formula given for η^{-1} is easily seen to hold. The equivalence of (2) and (3) is also easy to establish.

Corollary 2.1. *Over any base space, a cosheaf space is a complete spread.*

Proof. Use Proposition 2.1(3).

Example 2.1. Cosheaf spaces occur in singularity theory. In Arnold [1] we find the "Whitney cusp:" project the surface $x = z^3 + yz$ onto the xy-plane. This projection is a cosheaf space with a fold. The singular set in the xy-plane is a cusp, which is resolved on the surface.

We now define the *pure*, *cosheaf space* factorization of a map with locally connected domain. The display space construction applies to any map $Y \xrightarrow{\varphi} X$ for which Y is locally connected. Thus, we may always factor such a map as

$$Y \xrightarrow{\eta} \widetilde{Y} \xrightarrow{\widetilde{\varphi}} X,$$

where $\widetilde{\varphi}$ is a cosheaf space. The map η is *pure* meaning that for every non-empty, connected open set $V \subseteq \widetilde{Y}$, $\eta^{-1}(V)$ is again non-empty and connected. In particular, η maps Y onto a dense subset of $\widetilde{Y}$. This factorization of a map with locally connected domain into its pure and cosheaf space parts is unique up to unique homeomorphism. Given $Y \xrightarrow{\varphi} X$ and a cosheaf space $Z \xrightarrow{\psi} X$, any map $Y \xrightarrow{\zeta} Z$ such that $\psi \cdot \zeta = \varphi$ factors uniquely through $Y \xrightarrow{\eta} \widetilde{Y}$ over X.

The class of cosheaf spaces is closed under composition, and furthermore, if $\varphi \cdot \psi$ and φ are cosheaf spaces (with locally connected domain), then so is ψ. Pure maps are also closed under composition, and if $p \cdot q$ and q are pure maps between locally connected spaces, then so is p. A homeomorphism (of locally connected spaces) is a cosheaf space and a pure map. Conversely, a cosheaf space that is also a pure map is a homeomorphism. All these facts may be established directly (using Proposition 2.1 for instance), or they may deduced from their known counterparts in locale or topos theory.

The following pullback result holds for factorization systems in general [7] , except that here we require a special assumption in order to make the pure, cosheaf space factorization of the pullback available. We shall use this result in Section 5.

Proposition 2.2. *Consider the following pullback of a cosheaf space* ψ (with K locally connected) *along an arbitrary map* f.

$$\begin{tikzcd} P \arrow[r, "g"] \arrow[d, "\tau"'] & K \arrow[d, "\psi"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

If P *is locally connected, then* τ *is a cosheaf space.*

Proof. Consider the pure, cosheaf space factorization η, φ of τ. There is a map $\widetilde{P} \xrightarrow{\delta} K$ such that the following diagram commutes:

$$
\begin{tikzcd}
P \arrow[dr, "\eta"] \arrow[drr, "g"] \arrow[ddr, "\tau"'] & & \\
& \widetilde{P} \arrow[r, "\delta"] \arrow[d, "\varphi"] & K \arrow[d, "\psi"] \\
& X \arrow[r, "f"'] & Y
\end{tikzcd}
$$

Indeed, from the pure, cosheaf space factorizations of $f \cdot \varphi$, respectively g,

$$
\begin{tikzcd}
\tilde{P} \arrow[r, "\alpha"] \arrow[dr, "f \cdot \varphi"'] & Q \arrow[d, "\zeta"] \\
& Y
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
& R \arrow[dr, "\xi"] & \\
P \arrow[ur, "\beta"] \arrow[rr, "g"'] & & K
\end{tikzcd}
$$

we obtain two pure, cosheaf space factorizations of $f \cdot \tau$: one by composing the triangle above (left) with the pure map η, and another one by composing the triangle above (right) with the cosheaf space ψ. Thus, there is a unique homeomorphism $h : Q \cong R$ such that $h \cdot \alpha \cdot \eta = \beta$ and $\psi \cdot \xi \cdot h = \zeta$. This gives the map $\delta = \xi \cdot h \cdot \alpha$ as above. Since P is pullback, there is a map $\widetilde{P} \overset{\varepsilon}{\to} P$ such that $\tau \cdot \varepsilon = \varphi$ and $g \cdot \varepsilon = \delta$. The map ε is inverse to η. In fact, $\varepsilon \cdot \eta = id_P$ follows immediately by the universal property of the pullback P. On the other hand, since we have $\eta \cdot \varepsilon \cdot \eta = \eta$, we obtain $\eta \cdot \varepsilon = id_{\widetilde{P}}$ because φ is a cosheaf space. Hence η is a homeomorphism, so that τ is a cosheaf space.

The above new result has the following consequence, which we shall use later. Corollary 2.2 is already known in topos theory [13].

Corollary 2.2. *Cosheaf spaces are stable under pullback along a local homeomorphism.*

Proof. If in a pullback diagram

$$\begin{tikzcd} P \arrow[r, "g"] \arrow[d, "\tau"'] & K \arrow[d, "\psi"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

the map f is a local homeomorphism, then so is g. Therefore, P is locally connected since K is. The Proposition 2.2 now applies.

3. Unramified Maps have Unique Homotopy Lifting

From [16], a surjective covering space of a locally connected space is a complete spread, but in fact a surjective covering space of a locally connected space is a cosheaf space [17, 5.18]. A covering space is also a sheaf space, i.e., a local homeomorphism.

Definition 3.1. We shall refer to a map of locally connected topological spaces that is both a sheaf and cosheaf space as an *unramified map*.

In this section, we shall show that under certain connectedness assumptions on the base space, an unramified map is a covering space.

Proposition 3.1. *Over a locally connected space, an unramified map satisfies the HLP with respect to* $\{0\}$ (i.e., an unramified map has unique path lifting).

Proof. Let $\varphi : Y \to X$ be an unramified map, with X (and hence Y) locally connected. Suppose we have a path $P : I \to X$ and a point $p \in Y$ such that $\varphi(p) = P(0)$. Consider the collection Φ of partial liftings $\overline{P}_D : D \to Y$ of P, where D is a connected interval (open or closed) that contains 0. We are assuming that $\overline{P}_D(0) = p$ and that the following square commutes.

$$\begin{tikzcd} D \arrow[r, "\overline{P}_D"] \arrow[d] & Y \arrow[d, "\varphi"] \\ I \arrow[r, "P"'] & X \end{tikzcd}$$

The members of Φ may be uniquely identified with their domain of definition D because by a closed-open argument any two liftings $\overline{P}_D$, $\overline{P}'_D$ of P must be equal. The collection Φ is non-empty because it has $p = \overline{P}_{\{0\}}$ as a member. We establish the existence of a lifting of P by showing that the subset

$$A = \bigcup_{\overline{P}_D \in \Phi} D,$$

which is non-empty, is both an open and closed subset of I, hence equal to I. Let us first show that A is open. Let $t \in D$, with associated lifting $\overline{P}_D$. First, we find an open neighbourhood $V \subseteq Y$ of $\overline{P}_D(t)$ in which φ is a homeomorphism, and then a connected open interval W containing t such that $P(W) \subseteq \varphi(V)$. We may extend $\overline{P}_D$ to a partial lifting $\overline{P}_{D \cup W}$ of P, where $D \cup W$ is connected and contains 0. Thus, $W \subseteq A$, so A is open. Note that we have a lifting $\overline{P}_A$.

Now, we argue that the open subset A is closed. Since A is connected, the closure $\overline{A}$ is also connected. If we can produce a lifting $\overline{P}_{\overline{A}}$, then $\overline{A} \subseteq A$, so that A is closed. Let $t \in \overline{A}$. We produce a cogerm at $P(t)$ as follows. Let U be an arbitrary open neighbourhood of $P(t)$. There is a connected open neighbourhood B of t such that $P(B) \subseteq U$. Furthermore, $B \cap A$ is non-empty, hence connected. Then the connected set $\overline{P}_A(B \cap A)$ uniquely determines a component of $\varphi^{-1}(U)$, and furthermore the component of $\varphi^{-1}(U)$ obtained in this way does not depend on the choice of B. It follows that this defines a cogerm at $P(t)$. This cogerm corresponds uniquely to a point of the fiber $\varphi^{-1}(P(t))$, which we define to be $\overline{P}_{\overline{A}}(t)$. This defines a function $\overline{P}_{\overline{A}} : \overline{A} \to Y$ over P, which agrees with $\overline{P}_A$ on A. Our assumption that φ is a spread enters into the verification that $\overline{P}_{\overline{A}}$ is continuous, because we have only to check that $\overline{P}_{\overline{A}}^{-1}(\beta)$ is an open set, where β is a component of a $\varphi^{-1}(U)$.

The next lemma is used to argue in Theorem 3.1 that homotopies can be lifted.

Lemma 3.1. *Let X be locally connected, and let $\varphi : Y \to X$ be a local homeomorphism that is also a spread. Then Y has a base of open neighbourhoods $\{V\}$ such that each member V of the base is a component of $\varphi^{-1}(U)$ for some open U, and such that φ restricts to a homeomorphism $\varphi : V \to U$.*

Proof. Y has a base of open sets A such that φ restricts to a homeomorphism in each A. Each of these A is equal to a union of open subsets V_W, such that V_W is a connected component of $\varphi^{-1}(W)$, for W open in X. For each such V_W, take $U = \varphi(V_W)$. Then V_W is a component of $\varphi^{-1}(U)$ since $U \subseteq W$. And φ restricts to a homeomorphism in V_W since $V_W \subseteq A$.

The compactness of I enters into the argument of the following result.

Theorem 3.1. *Over a locally connected space, an unramified map has the HLP with respect to I* (i.e., unique homotopy lifting).

Proof. Suppose we have a commutative diagram

$$\begin{tikzcd} I \arrow[r, "h"] \arrow[d, tail] & Y \arrow[d, "\varphi"] \\ I \times I \arrow[r, "H"'] & X \end{tikzcd}$$

where φ is an unramified map, with X locally connected. The left vertical map is the map $s \mapsto (0, s)$. We may define a function $\overline{H} : I \times I \to Y$ by lifting, for each s, the path $H(t, s), 0 \le t \le 1$, such that $\overline{H}(0, s) = h(s)$ (Proposition 3.1). Clearly, $\overline{H}$ is a lifting of H. We argue that $\overline{H}$ is continuous as follows. Let

$$A = \bigcup_{D \subseteq I} D$$

denote the union of (closed or open) connected intervals D that contain 0, such that $\overline{H}$ is continuous in $D \times I$. Now A is non-empty because it has 0 as a member. We show that A is open and closed, hence equal to I. First, we argue that A is open. Let $t_0 \in D$, where $\overline{H}$ is continuous in $D \times I$, and consider a point (t_0, s). There is a neighbourhood $V \subseteq Y$ of $\overline{H}(t_0, s)$ with the property guaranteed by Lemma 3.1. Then we may find an open disk B of (t_0, s) such that $H(B) \subseteq \varphi(V)$. Thus, we have $\overline{H}(B) \subseteq \varphi^{-1}(\varphi(V))$. We claim that, in fact, we have $\overline{H}(B) \subseteq V$. Indeed, let $(t', s') \in B$. $\overline{H}$ is continuous in $B \cap (D \times I)$, and also along the line $L = \{(t, s') \mid 0 \le t \le 1\}$. Note that (t', s') and (t_0, s) both lie in the connected set $B \cap (L \cup (D \times I))$, which is contained in B. Furthermore, $\overline{H}$ is continuous in this connected set. Hence, the image of this set under $\overline{H}$ is connected, and is therefore contained in the component V of $\varphi^{-1}(\varphi(V))$. In particular, $\overline{H}(t', s') \in V$. We conclude our proof that A is open by using the compactness of I to find an r such that the tubular neighbourhood $(t_0 - r, t_0 + r) \times I$ is contained in the union of finitely many open disks B_i centered at (t_0, s_i), such that for each i we have $H(B_i) \subseteq V_i$, where $V_i \subseteq Y$ is a neighbourhood of $\overline{H}(t_0, s_i)$ with the property guaranteed by Lemma 3.1. It follows easily that $\overline{H}$ is continuous in this tubular neighbourhood. Hence $[0, t_0 + r) \subseteq A$, so that A is open. It can be argued that A is closed in exactly the same manner except that the second step involving the compactness of I is not required.

We have the following:

Corollary 3.1. *Over a locally path-connected* (hence locally connected) *and semi-locally simply connected space*, *the notions of covering space and unramified map are equivalent*. (The image of such a map is equal to a union of connected components of the base space.)

Proof. We have already mentioned the known fact that in general a surjective covering space of a locally connected space is an unramified map. As always, we are assuming that the image of a covering space of a locally connected space is equal to a union of connected components of the base space, so we may apply the surjective case to each connected component separately, and then use the easily established fact that the disjoint union of a collection of unramified maps is again an unramified map. For the converse, by Propositions 1.2 and 3.1, and Theorem 3.1, under the given assumptions an unramified map is a covering space.

We conclude this section with the following example:

Example 3.1. We describe a connected unramified map over a connected, locally path-connected space (which is not semi-locally simply connected) that is not a covering space. (By a connected map we mean a map whose domain space is connected. The significance of having a connected example is explained in Remark 3.1.) We take for the base space X a 'pencil' of tangent circles C_n of radius $\frac{1}{n}$, $n = 1, 2, 3, \ldots$, topologized as a subspace of the Euclidean plane. In other words, let

$$X = \bigcup_{n=1}^{\infty} C_n$$

with a single tangent point a such that $\forall m \neq n, C_m \cap C_n = \{a\}$. The domain space Y consists of countably many copies of the real line and of X, topologized as a subset of Euclidean 3-space. To be precise, let

$$Y = \left(\bigcup_{n=1}^{\infty} U_n\right) \cup \left(\bigcup_{|z|=1}^{\infty} X_z\right),$$

where n is a natural number and z is an integer, and let $\psi : Y \to X$ be the map such that:

(1) each U_n is a homeomorphic copy of the real line, and ψ restricted to U_n is a universal covering map $U_n \to C_n$,

(2) $\psi^{-1}(a) = \{..., -2, -1, 1, 2, ...\}$ ordered consecutively on U_1, and $\psi^{-1}(a) \cap U_n = \{..., -n-1, -n, n, n+1, ...\}$,

(3) ψ carries X_z homeomorphically onto $\bigcup_{j=|z|+1}^{\infty} C_j$, $|z| = 1, 2, ...$,

(4) each $y \in Y - \psi^{-1}(a)$ has an open neighbourhood that is homeomorphic to the real line,

(5) $X_z \cap (\bigcup_{n=1}^{\infty} U_n) = \{z\}, |z| = 1, 2, \dots$.

The space Y is connected and locally path-connected. We readily see that the map ψ is a local homeomorphism, even at the points of the fiber $\psi^{-1}(a)$. Furthermore, ψ is a spread, and it also holds that the fiber of any point of X is in bijection with its cogerms, so that ψ is a cosheaf space. On the other hand, ψ is not a covering space because the point $a \in X$ does not have an evenly covered neighbourhood. Indeed, any neighbourhood B of a contains a circle C_n, for some n. For this n, the *point* n of $\psi^{-1}(a)$ (according to our naming convention) is a member of U_n. The connected component of $\psi^{-1}(B)$ that contains this point must contain all of U_n, so that ψ cannot restrict to a homeomorphism of this component onto B.

Remark 3.1. Let $Sh(B)$ denote the category of sheaf spaces over a (locally connected) space B. Let $\prod_1(B)$ denote the full subcategory of $Sh(B)$ whose objects are all small coproducts of covering spaces. (For a topos-theory perspective of covering spaces and the fundamental group of a space we refer the reader to Grothendieck [19]. More recent developments of this theory and further information concerning the category $\prod_1(B)$ may be found in [2, 12] and the references cited therein.) The space Y described in Example 3.1 is connected, so ψ cannot be homeomorphic to a coproduct of covering spaces. The unramified map ψ is thus not a member of $\prod_1(X)$, where X is the space of Example 3.1. However, ψ has unique path-lifting (Proposition 3.1).

4. The Category $U(X)$

It would be of interest to know what kind of category the unramified maps form, and what are its properties. In this section, we collect some facts about this category. Let $\mathcal{U}$ denote the category of locally connected spaces and unramified maps (with the usual composition). This is a category because both sheaf spaces and cosheaf spaces are closed under composition.

If A is an object of a category $\mathcal{C}$, then $\mathcal{C}/\mathcal{A}$ denotes the category whose objects are all morphisms $B \to A$. The morphisms of $\mathcal{C}/\mathcal{A}$ are commutative triangles over A. $\mathcal{C}/\mathcal{A}$ is sometimes called a *slice category*.

Definition 4.1. For a locally connected space X, let $U(X)$ denote the slice category $\mathcal{U}/X$. Explicitly, an object of $U(X)$ is an unramified map into X, and a morphism is an unramified map over X, as in the following diagram:

$$\begin{tikzcd} Y \arrow[rr] \arrow[dr] & & Z \arrow[dl] \\ & X & \end{tikzcd}$$

A map between sheaf spaces is a sheaf space. In Section 2, we mentioned that similarly a map between cosheaf spaces is a cosheaf space. Thus, the categories of sheaf, respectively cosheaf, spaces over X are full subcategories of topological spaces over X. $U(X)$ is a full subcategory of both these categories.

Proposition 4.1. *Unramified maps are stable under pullback along an arbitrary map of locally connected spaces.*

Proof. Let $\psi : K \to X$ denote an unramified map. The pullback of ψ is a local homeomorphism, so if the codomain space of the pullback is locally connected, then so is its domain space. We may now apply Proposition 2.2.

Remark 4.1. Topos versions of Propositions 2.2 and 4.1 also hold, by the analogous concepts and tools for geometric morphisms [11, 13, 14, 18]. For instance, the topos version of Proposition 4.1 is as follows: *Complete spread objects are preserved under the inverse image functor of an arbitrary geometric morphism between locally connected toposes*. The proof of Proposition 2.2 depends on the fact that cosheaf spaces are closed under composition. Closure under composition of complete spread geometric morphisms is shown in [18].

Proposition 4.2. *For any locally connected space X, $U(X)$ has finite limits and small coproducts, which are created in the including category of sheaf spaces over X* (in turn, these are created in topological spaces over X).

Proof. The terminal unramified map is the identity $id : X \to X$. The sheaf space pullback of unramified maps is again an unramified map because cosheaf spaces pullback along local homeomorphisms (Corollary 2.2). It is not difficult to see that a small coproduct of cosheaf spaces (as topological spaces) is again a cosheaf space. If the maps are also sheaf spaces, then this coproduct is also the sheaf space coproduct.

Remark 4.2. Let X denote a locally connected space. A covering space is an unramified map, so by Proposition 4.2, $\prod_1(X)$ (Remark 3.1) is a full subcategory of $U(X)$.

$$\begin{tikzcd} \Pi_1(X) \arrow[r, tail] \arrow[dr, tail] & U(X) \arrow[d, tail] \\ & Sh(X) \end{tikzcd}$$

From [12], the inclusion of $\prod_1(X)$ in sheaves has both left and right adjoints. Therefore, the inclusion of $\prod_1(X)$ in $U(X)$ has both adjoints.

Remark 4.3. For the reader interested in indexed categories and fibrations (in the sense of categories) [4, 26], observe that $U(X)$ is a $\prod_1(X)$-indexed category when for $P \xrightarrow{p} X$ in $\prod_1(X)$, we define the fiber category $U(X)^p$ as the slice category

$$U(X)^p = U(X)/p = U(P).$$

This $\prod_1(X)$-indexed category has "Σ satisfying the BCC" because unramified maps compose. It is a locally small $\prod_1(X)$-category because the inclusion of $\prod_1(X)$ in $U(X)$ has a right adjoint.

A *discrete opfibration* is a functor $F : \mathbf{A} \to \mathbf{B}$ with the property that every morphism $F(A) \to B$ in $\mathbf{B}$ has a unique lifting $A \to A'$ in $\mathbf{A}$ [6, 20]. If $\mathbf{A}$ is a small category, then the category of functors $\mathbf{A} \to \mathit{Set}$ and natural transformations (denoted $\mathit{Set}^{\mathbf{A}}$) is canonically equivalent to the category of discrete opfibrations over $\mathbf{A}$.

Let $\pi_1(X)$ denote the fundamental groupoid of a space X. Then, the functor category $\mathit{Set}^{\pi_1(X)}$ is equivalent to the category of discrete opfibrations over the groupoid $\pi_1(X)$. Let $HLP(X)$ denote the full subcategory of sheaf spaces over X that have the HLP with respect to I.

We have a functor

$$\pi_1 : HLP(X) \to Set^{\pi_1(X)}$$

that carries $p : E \to X$ to the discrete opfibration $\pi_1(p) = p_* : \pi_1(E) \to \pi_1(X)$. (HLP with respect to I guarantees that p_* is a discrete opfibration.) For locally connected X, we have the following categories and functors:

$$\begin{tikzcd} \Pi_1(X) \arrow[r, tail] \arrow[dr, "\pi_1"'] & U(X) \arrow[r, tail] \arrow[d, "\pi_1"] & HLP(X) \arrow[dl, "\pi_1"] \\ & Set^{\pi_1(X)} & \end{tikzcd}$$

The horizontal arrows depict full subcategories (Remark 4.2 and Theorem 3.1). We may restrict π_1 to these full subcategories, as indicated in the diagram.

Remark 4.4. If X is locally path-connected and semi-locally simply connected, then the four categories in the above diagram are equivalent. Indeed, it is well-known that if X is locally path-connected and semi-locally simply connected, then $\prod_1(X)$ is equivalent under π_1 to the topos $Set^{\pi_1(X)}$ (in this case a small coproduct of covering spaces is again a covering space). $HLP(X)$ is equal to $\prod_1(X)$ by Proposition 1.2. In particular, for such X, $U(X)$ is a Grothendieck topos.

One particular question about unramified maps that we have not answered comes to mind: Is $U(X)$ closed under coequalizers in the including topos of sheaf spaces over X?

5. U is a Homotopy Invariant

In this section, we describe in what sense U is a homotopy invariant. There is a notion of 2-cell between maps that naturally accompanies U, which we call a *U-homotopy*. Because U is a homotopy invariant, these U-homotopies may be regarded as generalized homotopies.

Definitions 5.1 and 5.2 below are in terms of what is called a *pseudo-functor*, or *homomorphism of bicategories* [4, 7, 26]. Let CAT denote the 2-category of locally small categories, functors, and natural isomorphisms. Briefly, *a pseudo-functor* in a category $\mathcal{C}$ is a functor

$$\mathcal{P} : \mathcal{C} \to CAT$$

that preserves the composition and identities of $\mathcal{C}$ only up to natural isomorphism in CAT. More precisely, a pseudo-functor $\mathcal{P}$ comes equipped with natural isomorphisms

$$\alpha^{f,\,g} : \mathcal{P}(f \cdot g) \cong \mathcal{P}(f) \cdot \mathcal{P}(g) \quad \text{and} \quad \alpha^{A} : \mathcal{P}(id_A) \cong id_{\mathcal{P}A}$$

subject to certain coherence conditions [7, 26].

By Proposition 4.1, a continuous map of locally connected spaces contravariantly induces a functor (by pullback) of unramified map categories. Let f^* denote the pullback functor along a map $f : X \to Y$ of spaces. When X and Y are locally connected, by Proposition 4.1, we have a functor

$$U(f) = f^* : U(Y) \to U(X) \quad \text{(Definition 4.1)},$$

where for any unramified map $K \xrightarrow{\psi} Y$, the (underlying set of the) domain space of the pullback $f^*(\psi)$ is the space

$$f^*(K) = \{(x,\, k) \in X \times K \mid f(x) = \psi(k)\}.$$

If $W \xrightarrow{g} X$ is another map (for locally connected W), then the natural isomorphism

$$\alpha^{f,\,g} : (f \cdot g)^* \cong g^* \cdot f^* \tag{1}$$

is given by

$$\alpha_{\psi}^{f,\, g}(w,\, k) = (w,\, (g(w),\, k)),$$

where $K \xrightarrow{\psi} Y$ is an unramified map. With these definitions, it is clear that we have a contravariant pseudo-functor:

$$U : Tsp^{\mathrm{op}} \to CAT,$$

where *Tsp* denotes the category of (locally connected) topological spaces and continuous functions (or maps).

Remark 5.1. A functor $U(f) = f^*$ is the restriction to unramified maps of the inverse image functor of the induced geometric morphism between corresponding sheaf toposes. For any map f between locally connected spaces, f^* preserves finite limits and small coproducts (Proposition 4.2).

For the following, let

$$\mathcal{P} : Tsp^{\mathrm{op}} \to CAT$$

denote an arbitrary pseudo-functor. For any map $f : X \to Y$, we denote the functor $\mathcal{P}(f) : \mathcal{P}(Y) \to \mathcal{P}(X)$ by f^*. We have coherence isomorphisms (1). We denote the identity natural isomorphism on the functor f^* by 1_{f^*}.

Definition 5.1. For a pseudo-functor $\mathcal{P}$ as above, a *$\mathcal{P}$-homotopy* $f \Rightarrow g : X \to Y$ is a natural isomorphism $f^* \Rightarrow g^*$. If $H : f \Rightarrow g$ is a $\mathcal{P}$-homotopy and ψ is an object of the category $\mathcal{P}(Y)$, then we call the isomorphism $H_\psi : f^*(\psi) \to g^*(\psi)$ in $\mathcal{P}(X)$ a *component isomorphism* of H.

Example 5.1. A U-homotopy $H : f \Rightarrow g : X \to Y$ is a natural isomorphism $f^* \Rightarrow g^*$. Such an H is given by a homeomorphism H_ψ :

$f^*(\psi) \to g^*(\psi)$ over X for every unramified map $F \xrightarrow{\psi} Y$. The component homeomorphisms H_ψ are natural in ψ in the sense that for any diagram

$$\begin{tikzcd} E \arrow[rr, "m"] \arrow[dr, "\varphi"'] & & F \arrow[dl, "\psi"] \\ & Y & \end{tikzcd}$$

over Y, the following diagram over X commutes:

$$\begin{tikzcd} f^*E \arrow[r, "f^*(m)"] \arrow[d, "H_\varphi"'] & f^*F \arrow[d, "H_\psi"] \\ g^*E \arrow[r, "g^*(m)"'] & g^*F \end{tikzcd}$$

Remark 5.2. We also have $\prod_1$-homotopies for the pseudo-functor $\prod_1$ (Remark 3.1). A U-homotopy defines a $\prod_1$-homotopy because $U(X)$ contains $\prod_1(X)$. Hence, the resulting classes of U-homotopy equivalent spaces are possibly smaller.

The next result shows that ordinary homotopies induce U-homotopies. We shall denote a homotopy $H : X \times I \to Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ by

$$H : f \Rightarrow g : X \to Y.$$

A homotopy H has an *involution* H^- such that $H^-(x, t) = H(x, 1 - t)$.

Proposition 5.1. *A homotopy* $H : f \Rightarrow g : X \to Y$, *where* X *is locally path-connected and* Y *is locally connected, induces by path-lifting a* U*-homotopy* $f \Rightarrow g$, *i.e.*, H *induces a natural isomorphism* $f^* \Rightarrow g^*$. (We shall denote the induced U-homotopy by $U(H)$. For any unramified map

$A \xrightarrow{\psi} Y$ we shall denote the component homeomorphism $U(H)_\psi$ of the natural isomorphism $U(H)$ simply by H_ψ, as above.)

Proof. Let $H : f \Rightarrow g : X \to Y$ be a homotopy, with X locally path-connected and Y locally connected. For any $A \xrightarrow{\psi} Y$, the homeomorphism $H_\psi : f^*(A) \to g^*(A)$ over X is defined by path-lifting as follows. Let $(x, a) \in f^*(A)$, so that $\psi(a) = f(x)$. We have a path $H(\psi(a), t) : f(x) \to g(x)$ in X, which may be lifted uniquely to a path $a \to b$ in A. We define $H_\psi(x, a) = (x, b)$. This defines a function H_ψ, which is easily seen to be a bijection. In fact, the inverse function $(H_\psi)^{-1}$ is equal to $(H^-)_\psi$. In order to show that H_ψ is continuous it suffices to show that $P_\psi : f^*(A) \to A$, $P_\psi(x, a) = b$, is continuous. We establish that P_ψ is continuous with the help of Lemma 3.1. First, we claim that P_ψ is path-continuous, meaning that for any path $I \xrightarrow{\gamma} f^*(A)$, $\gamma(s) = (x_s, a_s)$, $P_\psi \cdot \gamma$ is continuous. We have $f^*(\psi)(x, a) = x$. We have the homotopy $H(x_s, t) : I \times I \to Y$, and also the path $H(x_0, t), 0 \leq t \leq 1$. By the definition, the codomain of the unique lift of this path is $P_\psi(\gamma(0))$. We may lift $H(x_s, t)$ to a homotopy $\hat{H}$ in A. It follows that $P_\psi(\gamma(s)) = \hat{H}(s, 1)$, which is a continuous function.

We can now show that P_ψ is continuous. It suffices to show that $P_\psi^{-1}(V)$ is open for V with the property described in Lemma 3.1. Note that $f^*(A)$ is locally path-connected since X is (by assumption) and since $f^*(\psi)$ is a local homeomorphism. Let $(x, a) \in P_\psi^{-1}(V)$ so that $g(f^*(\psi)(x, a)) = \psi(P_\psi(x, a)) \in \psi(V)$. Hence $f^*(\psi)(x, a) \in g^{-1}(\psi(V))$. Since

$f^*(\psi)$ is continuous, there is a path-connected neighbourhood U of (x, a) such that $f^*(\psi)(U) \subseteq g^{-1}(\psi(V))$. Therefore,

$$\psi(P_\psi(U)) = g(f^*(\psi)(U)) \subseteq \psi(V),$$

and hence $P_\psi(U) \subseteq \psi^{-1}(\psi(V))$. $P_\psi(U)$ is path-connected because P_ψ is path-continuous. $P_\psi(U)$ meets the connected component V of $\psi^{-1}\psi(V)$ in a non-empty set (the point $P_\psi(x, a)$ is in this intersection), so $P_\psi(U) \subseteq V$. This concludes our argument that P_ψ, and hence H_ψ, is continuous. H_ψ is a homeomorphism because the same argument shows that $(H^-)_\psi$ is continuous. The naturality of the component isomorphisms H_ψ is readily verified.

In order to define what we mean by a homotopy invariant we briefly investigate what sort of 2-cell structure homotopies give the category *Tsp*. Homotopies may be composed vertically and horizontally. The *vertical composite KH* of $H : f \Rightarrow g$ and $K : g \Rightarrow h$ is defined using the map $I + I \to I(t_1 \mapsto \frac{t_1}{2}, t_2 \mapsto \frac{t_2 + 1}{2})$. The *horizontal composite* $K \cdot H$ of $H : f \Rightarrow g : X \to Y$ and $K : h \Rightarrow k : Y \to Z$ is given by

$$K \cdot H(x, t) = K(H(x, t), t).$$

For every map $f : X \to Y$, there is a distinguished homotopy denoted 1_f such that $1_f(x, t) = f(x)$, for every $(x, t) \in X \times I$. Any such homotopy 1_f is an idempotent, and we have $1_{f \cdot g} = 1_f \cdot 1_g$. The horizontal composite $H \cdot 1_h$ of homotopies $H : f \Rightarrow g : X \to Y$ and $1_h : h \Rightarrow h : W \to X$ is called "*whiskering H on the left by h*". We also denote $H \cdot 1_h$ by $H \cdot h$. We may also whisker H on the right by a map $Y \xrightarrow{h} Z$.

We mentioned previously that a homotopy H has an involution H^-.

The involution satisfies $(H^-)^- = H, (HK)^- = K^-H^-, (H \cdot K)^- = H^- \cdot K^-$, and $(1_f)^- = 1_f$.

The above definitions almost make *Tsp* into a 2-category. For instance, the interchange law holds: $(L \cdot J)(K \cdot H) = LK \cdot JH$, for homotopies $H : f \Rightarrow g : X \to Y, J : g \Rightarrow h : X \to Y, K : j \Rightarrow k : Y \to Z$, and $L : k \Rightarrow l : Y \to Z$. But the vertical composition is not associative, and the idempotent homotopies 1_f are not identities for vertical composition. A consequence of not having vertical identities is that we may not express horizontal composition in terms of whiskering and vertical composition: the following derivation, which is valid in a 2-category, is not valid for homotopies:

$$K \cdot H = (1_k K) \cdot (H 1_f) = (k \cdot H)(K \cdot f).$$

Even though *Tsp* with homotopies is not a bicategory we can still make sense of the notion of a homomorphism of bicategories from *Tsp*. We shall define a homotopy invariant as such a homomorphism. This is spelled out in the following definition:

Definition 5.2. A *homotopy invariant* is a pseudo-functor

$$\mathcal{H} : Tsp^{\mathrm{op}} \to CAT$$

that moreover associates with a homotopy $H : f \Rightarrow g$ a natural isomorphism $\mathcal{H}(H) : f^* \Rightarrow g^*$ (i.e., an $\mathcal{H}$-homotopy according to Definition 5.1) such that

(1) $\mathcal{H}(HK) = \mathcal{H}(H)\mathcal{H}(K)$,

(2) for $H : f \Rightarrow g : X \to Y$ and $K : h \Rightarrow k : Y \to Z$, we have

$$\alpha^{k,\, g}\mathcal{H}(K \cdot H) = (\mathcal{H}(H) \cdot \mathcal{H}(K))\,\alpha^{h,\, f}.$$

In the special case of whiskering this amounts to:

(a) $\alpha^{h,\, g}\mathcal{H}(h \cdot H) = (\mathcal{H}(H) \cdot h^*)\,\alpha^{h,\, f}$, for $h : Y \to Z$,

(b) $\alpha^{g,\,h}\mathcal{H}(H \cdot h) = (h^* \cdot \mathcal{H}(H))\,\alpha^{f,\,h}$, for $h : W \to X$,

(3) $\mathcal{H}(1_f) = 1_{f^*}$, and

(4) $\mathcal{H}(H^-) = \mathcal{H}(H)^{-1}$.

Remark 5.3. Let $\mathcal{H} : Tsp^{\mathrm{op}} \to CAT$ be a homotopy invariant. If X and Y are homotopy equivalent spaces, then $\mathcal{H}(X)$ and $\mathcal{H}(Y)$ are equivalent categories.

By building on Proposition 5.1, we shall show that U is a homotopy invariant of locally path-connected spaces. In particular, Theorem 5.1 confirms that a U-homotopy (Definition 5.1 and Example 5.1) is a generalized homotopy.

Theorem 5.1. *The pseudo-functor $U : Tsp^{\mathrm{op}} \to CAT$ is a homotopy invariant. In particular, if two locally path-connected spaces are homotopy equivalent, then their categories of unramified maps are equivalent.*

Proof. U is easily seen to respect vertical composition of homotopies. We shall show that U respects horizontal composition of homotopies (Definition 5.2(2)) by working with vertical composition and whiskering. Let $H : f \Rightarrow g : X \to Y$ and $K : k \Rightarrow h : Y \to Z$ be two homotopies. We shall show that $U(K \cdot H) = U((K \cdot g)(k \cdot H))$. Fix $x \in X$, and consider the homotopy

$$K(H(x,\, s),\, t) : I \times I \to Z,$$

depicted by the following diagram.

$$\begin{tikzcd}
K(H(x,0),0) \arrow[r] \arrow[d, "k\cdot H"'] \arrow[dr, "K\cdot H"] & K(H(x,0),1) \arrow[d] \\
K(H(x,1),0) \arrow[r, "K\cdot g"'] & K(H(x,1),1)
\end{tikzcd}$$

$U(K \cdot H)$ is defined by lifting the diagonal path in the above diagram, whilst $U((K \cdot g)(k \cdot H))$ is defined by lifting the left and bottom paths. These two paths are homotopic in Z, so we have $U(K \cdot H) = U((K \cdot g)(k \cdot H))$.

Now, we show that U respects whiskering on the left: $U(H \cdot h) = h^* \cdot U(H)$, for $H : f \Rightarrow g : X \to Y$ and $W \xrightarrow{h} X$. (We omit the coherence natural isomorphisms $\alpha^{f,\,h}$ from the notation since they play only a "background" role.) Let $A \xrightarrow{\psi} Y$ be an unramified map. We must show that $(H \cdot h)_\psi = h^*(H_\psi)$, where

$$\begin{tikzcd} f^*A \arrow[rr, "H_\psi"] \arrow[dr, "f^*(\psi)"'] & & g^*A \arrow[dl, "g^*(\psi)"] \\ & X & \end{tikzcd} \qquad \begin{tikzcd} h)^*A \arrow[rr, "(H\cdot h)_\psi"] \arrow[dr, "f\cdot h)^*(\psi)"'] & & (g\cdot h)^*A \arrow[dl, "(g\cdot h)^*(\psi)"] \\ & W & \end{tikzcd}$$

For instance, keep in mind that $(f \cdot h)^* A = \{(w, a) \in W \times A \mid f(h(w)) = \psi(a)\}$. We have only to show that

$$\begin{tikzcd} (f \cdot h)^*A \arrow[r, "p_1"] \arrow[d, "(H \cdot h)_\psi"'] & f^*A \arrow[d, "H_\psi"] \\ (g \cdot h)^*A \arrow[r, "p_2"'] & g^*A \end{tikzcd}$$

commutes, where the maps p_i are the canonical projections: $(w, a) \mapsto (h(w), a)$. Since g^*A is a pullback, it suffices to show that the square commutes when followed by the projection $p_0 : g^*A \to A$. We have the path

$$H(h(w), t) : \psi(a) = f(h(w)) \to g(h(w))$$

in Y. By the definition, $(p_0 \cdot p_2)((H \cdot h)_\psi(w, a))$ is equal to the codomain of the lift

$$a \to (H \cdot h)_\psi(w, a)$$

of the former path to A. But this is precisely the definition of $p_0(H_\psi(h(w), a))$. This shows that $(H \cdot h)_\psi = h^*(H_\psi)$. To show that U respects whiskering on the right we must show that $(h \cdot H)_\psi = H_{h^*\psi}$, for $Y \xrightarrow{h} W$ and an unramified map $A \xrightarrow{\psi} W$. We omit verification of this. We may now show that U respects horizontal composition with the following derivation (ignoring the α's):

$$\begin{aligned} U(K \cdot H) &= U((K \cdot g)(k \cdot H)) \\ &= U(K \cdot g)U(k \cdot H) \\ &= (g^* \cdot U(K))(U(H) \cdot k^*) \\ &= U(H) \cdot U(K), \end{aligned}$$

where the last step is by interchange in CAT. The remaining conditions of a homotopy invariant may be routinely verified.

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