Far East Journal of Mathematical Sciences (FJMS)
Volume 29, Issue 1, 2008, Pages 171-176
Published online: June 10, 2008
This paper is available online at http://www.pphmj.com

NEW SOLITON-LIKE SOLUTIONS TO THE TWO-DIMENSIONAL VARIABLE COEFFICIENT BURGERS EQUATION

YUAN XIAO

Department of Mathematics
Zhanjiang Normal University
Zhanjiang, 524048, P. R. China

Abstract

In this paper, the extended projective Riccati equations method is proposed and then the two-dimensional variable coefficient Burgers equation is chosen to illustrate the method. As a result, two families of exact soliton-like solutions for this equation are obtained.

1. Introduction

Finding more exact solutions of nonlinear partial differential equations (NPDEs) in mathematical physics plays an important role in mathematical physics. Recently, Conte et al. [3] presented an indirect method to seek more new solitary wave solutions of some nonlinear PDEs that can be expressed as a polynomial in two elementary functions which satisfy a project Riccati equations [1]. In this paper, we propose the extended projective Riccati equations method and then we choose the two-dimensional generalized Burgers equation to illustrate the method. As a result, two families of soliton-like solutions are found with the help of symbolic computation system-Maple.

2000 Mathematics Subject Classification: 35Qxx.

Keywords and phrases: symbolic computation, projective Riccati equations, soliton-like solutions, two-dimensional variable coefficient Burgers equation.

Received December 29, 2007

2. The Extended Projective Riccati Equations Method

Now we establish the improved projective Riccati system method as follows:

Given an NPDEs with three variables $\{x, y, t\}$,

$$P(u_t, u_x, u_y, u_{xt}, u_{yt}, u_{xx}, L) = 0, \tag{1}$$

Step 1. We assume that Eq. (1) has the following formal solutions:

$$u(x, y, t) = a_0 + \sum_{i=1}^{m} \sigma^{i-1}(\xi)[a_i\tau(\xi) + b_i\sigma(\xi)], \tag{2}$$

where $a_0 = a_0(x, y, t)$, $a_i = a_i(x, y, t)$, $b_i = b_i(x, y, t)$, $(i = 1, 2, \ldots, m)$ and $\xi = \xi(x, y, t)$ are all unknown functions of $\{x, y, t\}$, $\sigma(\xi)$ and $\tau(\xi)$ satisfy the following projective Riccati system [1-3, 6].

$$\sigma'(\xi) = \varepsilon\sigma(\xi)\tau(\xi), \quad \tau'(\xi) = R + \varepsilon\tau^2(\xi) - \mu\sigma(\xi), \quad \varepsilon \pm 1, \tag{3}$$

$$\tau^2(\xi) = -\varepsilon\left[R - 2\mu\sigma(\xi) + \frac{\mu^2 - 1}{R}\sigma^2(\xi)\right], \quad R \neq 0, \tag{4}$$

where R and μ are constants.

The parameter m is the balance constant [5], which is obtained by balancing the highest order derivative term and the nonlinear terms in (1) (m is usually a positive integer). If m is a fraction or a negative integer, we usually make the transformation $u(x, y, t) = \varphi^m(x, y, t)$.

Step 2. Substituting (2)-(4) into (1), we can obtain a set of algebraic polynomials for $\tau^i(\xi)\sigma^j(\xi)$ $(i = 0, 1;\ j = 0, 1, \ldots)$ from the resulting system's numerator. Setting the coefficients of these terms $\tau^i(\xi)\sigma^j(\xi)$ to zero, we get a system of over-determined partial differential equations (PDEs) with respect to unknown functions of a_0, a_i, b_i $(i = 1, 2, \ldots, m)$ and ξ.

Step 3. Using symbolic computation system-Maple, we would end up with the explicit expressions for μ, a_0, a_i, b_i $(i = 1, 2, \ldots, m)$ and ξ or the constraints among them.

We know that the projective Riccati equations (3) have the following solutions:

Case 1. When $\varepsilon = -1$, $R \neq 0$

$$\sigma_1(\xi) = \frac{R\,\mathrm{sech}(\sqrt{R}\xi)}{\mu\,\mathrm{sech}(\sqrt{R}\xi) + 1}, \quad \tau_1(\xi) = \frac{\sqrt{R}\tanh(\sqrt{R}\xi)}{\mu\,\mathrm{sech}(\sqrt{R}\xi) + 1},$$

$$\sigma_2(\xi) = \frac{R\,\mathrm{csch}(\sqrt{R}\xi)}{\mu\,\mathrm{csch}(\sqrt{R}\xi) + 1}, \quad \tau_2(\xi) = \frac{\sqrt{R}\coth(\sqrt{R}\xi)}{\mu\,\mathrm{csch}(\sqrt{R}\xi) + 1}. \tag{5}$$

Case 2. When $\varepsilon = 1$, $R \neq 0$

$$\sigma_3(\xi) = \frac{R\sec(\sqrt{R}\xi)}{\mu\sec(\sqrt{R}\xi) + 1}, \quad \tau_3(\xi) = \frac{\sqrt{R}\tan(\sqrt{R}\xi)}{\mu\sec(\sqrt{R}\xi) + 1},$$

$$\sigma_4(\xi) = \frac{R\csc(\sqrt{R}\xi)}{\mu\csc(\sqrt{R}\xi) + 1}, \quad \tau_4(\xi) = \frac{\sqrt{R}\cot(\sqrt{R}\xi)}{\mu\csc(\sqrt{R}\xi) + 1}. \tag{6}$$

Step 4. Write the solutions of Eq. (1) from Eqs. (2), (5), (6) and the conclusions in Step 3.

3. New Soliton-like Solutions

Consider the two-dimensional generalized Burgers equation [4]

$$(u_t + uu_x - u_{xx})_x + s(t)u_{yy} = 0. \tag{7}$$

In [4], the author discussed symmetries and invariant solutions of Eq. (7). Now we apply the extend projective Riccati equations method to Eq. (7). By the balancing procedure, we assume the solutions of (7) in the following special form

$$u(x, t) = a_0 + b_0 x + a_1\tau(\xi) + b_1\sigma(\xi), \tag{8}$$

where $a_0 = a_0(y, t)$, $b_0 = b_0(t)$, $a_1 = a_1(t)$, $b_1 = b_1(t)$, $\xi = xp(t) + q(y, t)$ and $\tau(\xi)$, $\sigma(\xi)$ satisfy (1)-(2).

Substituting (3), (4) and (8) into (7), collecting coefficients of monomials of $\tau(\xi)$, $\sigma(\xi)$ and x of the resulting system's numerator (Notice that a_0, b_0, a_1, b_1, p and q are independent of x), then setting each coefficients to zero, setting $\varepsilon = -1$, we obtain the following over-determined system of 13 PDEs with respect to $\{a_0, b_0, a_1, b_1, s, p, q\}$ (Notice: for simplicity, we only consider the case with $\varepsilon = -1$)

$$-R^2(-2pa_1\mu b_0 - Rsq_y^2 b_1 - Rp^2 a_0 b_1 + Rp^3 a_1 \mu - sq_{yy} a_1 \mu + Ra_1^2 p^2 \mu$$
$$- Rpb_1 q_t - p_t a_1 \mu - pa_{1t}\mu) = 0 \tag{9}$$

$$R^3 pb_1(b_0 p + p_t) = 0 \tag{10}$$

$$-R(-7a_1^2 p^2 \mu + 7a_1^2 p^2 \mu^3 + 5Rp^2 b_1^2 \mu + 2sq_y^2 b_1 - 2pb_1 q_t \mu^2 - 12p^3 a_1 \mu$$
$$+ 2p^2 a_0 b_1 + 12p^3 a_1 \mu^3 - 2sq_y^2 b_1 \mu^2 + 2pb_1 q_t - 2p^3 a_0 b_1 \mu^2) = 0, \tag{11}$$

$$R^2(sa_{0yy} + b_0^2 + b_{0t}) = 0, \tag{12}$$

$$2Rpb_1(\mu - 1)(\mu + 1)(b_0 p + p_t) = 0, \tag{13}$$

$$-2R[3Rp^3 b_1 \mu + pa_1 q_t - sq_y^2 a_1 \mu^2 + sq_y^2 a_1 - pa_1 q_t \mu^2$$
$$+ 3Ra_1 \mu p^2 b_1 + p^2 a_0 a_1 (1 - \mu^2)] = 0, \tag{14}$$

$$2Rpa_1(\mu - 1)(\mu + 1)(b_0 p + p_t) = 0, \tag{15}$$

$$3p^2(\mu - 1)(\mu + 1)(2pa_1\mu^2 + \mu^2 a_1^2 - 2pa_1 - a_1^2 + Rb_1^2) = 0, \tag{16}$$

$$6Rp^2 b_1(\mu - 1)(\mu + 1)(p + a) = 0, \tag{17}$$

$$R^2(-sq_{yy} b_1 - 2pb_1 b_0 + Ra_1 p^2 b_1 - pb_{1t} + Rp^3 b_1$$
$$- p^3 a_0 a_1 \mu - sq_y^2 a_1 \mu - pa_1 \mu q_t - b_1 p_t) = 0, \tag{18}$$

$$-R^2 \mu pa_1(b_0 p + p_t) = 0, \tag{19}$$

$$R(5Ra_1^2\mu^2p^2 - 3Rpb_1q_t\mu + 2pa_1b_0 - 3Rsq_y^2b_1\mu - 4p^3a_1R$$
$$- 3Rp^2a_0b_1\mu + 2R^2p^2b_1^2 - p_ta_1\mu^2 + sq_{yy}a_1 - 2a_1^2Rp^2$$
$$+ p_ta_1 - sq_{yy}a_1\mu^2 + pa_{1t} - pa_{1t}\mu^2 - 2pa_1b_0\mu^2) = 0, \qquad (20)$$

$$-3R^2\mu pb_1(b_0p + p_1) = 0, \qquad (21)$$

where $s = s(t)$, $q_y = \dfrac{dq(y, t)}{dy}$, $a_{1t} = \dfrac{da_1(t)}{dt}$ and so on. Using Maple to solve the system of PDEs (9)-(21), we obtained the following results.

Case 1. $\mu = b_1 = 0$, $s = s(t)$, $a_0 = F_1(t)y + F_2(t)$, $b_0 = \dfrac{1}{t + C_1}$, $a_1 = \dfrac{2C_2}{t + C_1}$, $p = \dfrac{C_2}{t + C_1}$,

$$q = -y(C_2\Phi - C_4) - \int \frac{s(C_2\Phi - C_4)^2(t + C_1)}{C_2}dt - \int \frac{C_2F_2(t)}{t + C_1}dt + C_3. \qquad (22)$$

Case 2. $s = s(t)$, $\mu = \pm\dfrac{\sqrt{C_3^2 + RC_2^2}}{C_3}$, $a_1 = -\dfrac{C_3}{t + C_1}$, $b_0 = \dfrac{1}{t + C_1}$, $a_0 = F_1(t)y + F_2(t)$, $p = \dfrac{C_3}{t + C_1}$, $b_1 = \dfrac{C_2}{t + C_1}$,

$$q = -y(C_3\Phi - C_3) - \int \frac{s(C_3\Phi - C_5)(t + C_1)}{C_3}dt - \int \frac{C_3F_2(t)}{t + C_1}dt + C_4, \qquad (23)$$

where $C_1, \ldots, C_4$ are arbitrary constants, $s = s(t)$ denotes that s is an arbitrary function of t, $F_1(t)$, $F_2(t)$ are arbitrary functions of t and $\Phi = \int \dfrac{F_1(t)}{t + C_1}dt.$

Therefore from (5), (6), (8), (22) and (23), we obtain two families of solutions for two-dimensional generalized Burgers equation (7) as follows:

Family 1

$$u_{11} = F_1(t)y + F_2(t) + \frac{x}{t + C_1} - \frac{2C_2}{t + C_1}\sqrt{R}\tanh\left[\sqrt{R}\left(\frac{xC_2}{t + C_1} + q\right)\right]$$

$$u_{12} = F_1(t)y + F_2(t) + \frac{x}{t + C_1} - \frac{2C_2}{t + C_1}\sqrt{R}\coth\left[\sqrt{R}\left(\frac{xC_2}{t + C_1} + q\right)\right],$$

where q is determined by (22).

Family 2

$$u_{21} = F_1(t)y + F_2(t) + \frac{x}{t + C_1} + a_1\frac{\sqrt{R}\tanh(\sqrt{R}\xi)}{\mu\operatorname{sech}(\sqrt{R}\xi) + 1} + b_1\frac{R\operatorname{sech}(\sqrt{R}\xi)}{\mu\operatorname{sech}(\sqrt{R}\xi) + 1},$$

$$u_{22} = F_1(t)y + F_2(t) + \frac{x}{t + C_1} + a_1\frac{\sqrt{R}\coth(\sqrt{R}\xi)}{\mu\operatorname{csch}(\sqrt{R}\xi) + 1} + b_1\frac{R\operatorname{csch}(\sqrt{R}\xi)}{\mu\operatorname{csch}(\sqrt{R}\xi) + 1},$$

where $\xi = xp + q$ and a_1, b_1, μ, p, q are determined by (23).

References

[1] T. C. Bountis, V. Papageorgiou and P. Winternitz, On the integrability of nonlinear ordinary differential equations with superposition principles, J. Math. Phys. 27 (1986), 1215-1224.

[2] Y. Chen and B. Li, Chaos, General projective Riccati equation method and exact solutions for generalized Kdv-type and Ktv-Burgers-type equations with nonlinear terms of any order, Solitons and Fractals (2004), 977-984.

[3] R. Conte and M. Musette, Link between solitary waves and projective Riccati equations, J. Phys. A (1992), 5609-5623.

[4] F. Gungor, Symmetries and invariant solutions of the two-dimensional variable coefficient Burgers equation, J. Phys. A (2001), 4313-4321.

[5] M. L. Wang, Application of a homogeneous balance method to exact solutions of nonlinear equations in mathematical physics, Phys. Lett. A (1996), 216-227.

[6] G. X. Zhang, Z. B. Li and Y. S. Duan, Exact solitary wave solutions of nonlinear wave equations, Sci. in China 44(3) (2001), 396-401.