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ON AN EIGENVALUE ASYMPTOTICS FOR A SCHRÖDINGER OPERATOR WITH THE DE GENNES EFFECT ASSOCIATED WITH SUPERCONDUCTIVITY

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Abstract

We study the eigenvalue asymptotics for a Schrödinger operator with a magnetic potential and with the de Gennes effect associated with the superconductivity near critical temperature. When the magnetic potential is depending on a parameter and the parameter tends to zero, we examine the asymptotics of the first eigenvalue and the corresponding eigenfunction. The result improves our previous paper Ando and Aramaki [2] and Pan [21].

1. Introduction

In the present paper, we consider the eigenvalue asymptotics for a magnetic Schrödinger operator associated with the superconductivity taking the de Gennes parameter into consideration. The superconductivity of the sample in a domain $\Omega \subset \mathbb{R}^3$ under the applied field $\mathbb{H}_{\text{appl}}$ is described by a minimizer $(\psi, \mathbb{A})$ of the Ginzburg-Landau functional

$$G[\psi, \mathbb{A}] = \int_{\Omega} \left\{ | \xi \nabla \psi - i\gamma\lambda^{-1} \mathbb{A} \psi |^2 + \frac{1}{2} (1 - | \psi |^2)^2 \right\} dx$$

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$$+\delta^2\int_{\mathbb{R}^3}|\operatorname{curl}\mathbb{A}-\mathbb{H}_{\text{appl}}|^2dx+\gamma\int_{\partial\Omega}|\psi|^2dS.$$

Here ψ is a complex valued function called an *order parameter* and $\mathbb{A}$ is a real valued vector field called a *magnetic potential*, and the penetration depth λ, the coherence length ξ, and δ is a positive parameter depending on materials and temperature and $\gamma \geq 0$ the de Gennes parameter. γ is very small for insulator, very large for magnetic material, and lying in between for non-magnetic material. If we put a new parameter $\mu = 1/\xi^2$, μ means physically,

$$\mu = \frac{1}{\xi^2} = \frac{4m\alpha^2 l^2(T_c - T)}{\hbar T_c},$$

where T is the temperature, T_c is the critical temperature under zero applied field, $\hbar$ is the Plank constant, l is a typical scale of the sample, m is the electron mass, α is a material constant independent of temperature. The Ginzburg-Landau parameter κ is defined by $\kappa = \lambda/\xi$. It is well known that if $\kappa > 1/\sqrt{2}$, the sample is of type II and if $0 < \kappa < 1/\sqrt{2}$, the sample is of type I. For these arguments, see Aramaki [5], Chapman et al. [6], Du et al. [9], Gunzburger and Ockendon [12], Lu and Pan [18, 19, 20], Helffer and Pan [17].

By a scaling

$$\mathcal{A} = \frac{\gamma\lambda^{-1}}{\xi}\mathbb{A}, \qquad \mathcal{H}_{\text{appl}} = \frac{\gamma\lambda^{-1}}{\xi}\mathbb{H}_{\text{appl}},$$

and put $\mathcal{H}_{\text{appl}} = \sigma\mathbf{H}$, where $\sigma > 0$ is a parameter which means the intensity of $\mathcal{H}_{\text{appl}}$ and $\mathcal{A} = \sigma\mathbf{A}$, the associated energy $G[\psi, \mathbb{A}]/\xi^2$ is written by

$$\mathcal{G}[\psi, \mathbf{A}] = \int_\Omega\left\{|\nabla_{\sigma\mathbf{A}}\psi|^2 + \frac{\mu}{2}(1-|\psi|^2)^2\right\}dx$$
$$+\frac{\kappa^2\sigma^2}{\mu}\int_{\mathbb{R}^3}|\operatorname{curl}\mathbf{A}-\mathbf{H}|^2dx+\gamma\int_{\partial\Omega}|\psi|^2dS, \qquad (1.1)$$

where dS denotes the surface element of $\partial\Omega$.

We assume that a given vector field $\mathbf{H}(x)$ is smooth and satisfies $\operatorname{div}\mathbf{H} = 0$ in $\mathbb{R}^3$. Then there exists a unique vector field $\mathbf{F}$ such that

$$\operatorname{curl}\mathbf{F} = \mathbf{H}, \quad \operatorname{div}\mathbf{F} = 0 \text{ in } \mathbb{R}^3, \quad \int_\Omega \mathbf{F}dx = 0. \tag{1.2}$$

In the above and the following, we use the notations for any magnetic potential $\mathbf{A}$ and any function ψ,

$$\nabla_{\mathbf{A}}\psi = \nabla\psi - i\mathbf{A}\psi, \quad \nabla^2_{\mathbf{A}}\psi = \Delta\psi - i[2\mathbf{A}\cdot\nabla\psi + \psi\operatorname{div}\mathbf{A}] - |\mathbf{A}|^2\psi.$$

The minimizers $(\psi, \mathbf{A})$ of the functional $\mathcal{G}$ satisfy the following Euler equation, called the *Ginzburg-Landau system*:

$$\begin{cases} -\nabla^2_{\sigma\mathbf{A}}\psi = \mu(1 - |\psi|^2)\psi & \text{in } \Omega, \\ \operatorname{curl}^2(\mathbf{A} - \mathbf{F}) = \dfrac{\mu}{\sigma\kappa^2}\Im\{\overline{\psi}\nabla_{\sigma\mathbf{A}}\psi\}\chi_\Omega & \text{in } \Omega, \\ (\nabla_{\sigma\mathbf{A}}\psi)\cdot\mathbf{v} + \gamma\psi = 0, \quad [\mathbf{v}\cdot\mathbf{A}] = 0, & \\ \quad [\mathbf{v}\times\operatorname{curl}\mathbf{A}] = 0, & \text{on } \partial\Omega, \\ \operatorname{curl}\mathbf{A} \to \mathbf{H} & \text{as } |x| \to \infty. \end{cases} \tag{1.3}$$

Here $\mathbf{v}$ is the unit outward normal vector at the boundary $\partial\Omega$ of Ω, $[\cdot]$ denotes the jump in the enclosed quantity across $\partial\Omega$, and χ_Ω is the characteristic function of Ω.

It is well known that if the applied field is strong, that is to say, if $\sigma > 0$ is large enough, $\mathcal{G}$ has only the trivial minimizer $(0, \mathbf{F})$ which corresponds with the normal state. Thus the critical field is defined by

$$H_c(\mathbf{H}, \mu, \kappa) = \inf\{\sigma > 0; (0, \mathbf{F}) \text{ is a global minimizer of } \mathcal{G}\}.$$

In order to find the asymptotics of H_c as $\mu \to 0$, we must consider the asymptotics of the first eigenvalue of the Schrödinger operator $-\nabla^2_{\varepsilon\mathbf{A}}$ with magnetic Robin type condition as $\varepsilon \to 0$. In this paper, we devote only the analysis for the asymptotics of the first eigenvalue and the corresponding eigenfunction of such a linear problem. For the asymptotics of H_c, we shall treat in the future work. Relatively, for the asymptotics as $\varepsilon \to \infty$, there are many articles, for example, see Aramaki [3, 4], Fournais and Helffer [10], Helffer [13], Helffer and Mohamed [14], Helffer and Morame [15, 16].

2. Asymptotics of the First Eigenvalue and the Corresponding Eigenfunction

In this section, we shall consider the asymptotic behavior of the first eigenvalue and the corresponding eigenfunction for a Schrödinger operator.

More precisely, let $\Omega \subset \mathbb{R}^3$ be a bounded, smooth and simply connected domain and $\mathbf{H} = \mathbf{H}(x)$ a given smooth vector field in $\mathbb{R}^3$ satisfying

$$\mathbf{H}(x) \neq 0 \text{ in } \Omega \quad \text{and} \quad \operatorname{div} \mathbf{H} = 0 \text{ in } \mathbb{R}^3. \tag{2.1}$$

Then there exists a unique, smooth vector field $\mathbf{F}(x)$ in $\mathbb{R}^3$ such that

$$\operatorname{curl} \mathbf{F} = \mathbf{H}, \quad \operatorname{div} \mathbf{F} = 0 \quad \text{in } \mathbb{R}^3 \quad \text{and} \quad \int_\Omega \mathbf{F} dx = 0. \tag{2.2}$$

Let $\mu(\varepsilon, \gamma)$ be the infimum of the following functional corresponding to the lowest eigenvalue of a Schrödinger operator with magnetic potential under some boundary condition:

$$\mu(\varepsilon, \gamma) = \inf_{\phi \in W^{1,2}(\Omega; \mathbb{C})} \frac{\int_\Omega |\nabla_{\varepsilon \mathbf{F}} \phi|^2 dx + \gamma \int_{\partial\Omega} |\phi|^2 dS}{\|\phi\|^2_{L^2(\Omega)}}, \tag{2.3}$$

where $\gamma \geq 0$ is a parameter. It is well known that $\mu(\varepsilon, \gamma)$ is achieved in $W^{1,2}(\Omega)$. Any minimizer of the functional (2.3) satisfies the Euler equation:

$$\begin{cases} -\nabla^2_{\varepsilon \mathbf{F}} \phi = \mu(\varepsilon, \gamma)\phi & \text{in } \Omega, \\ (\nabla_{\varepsilon \mathbf{F}} \phi) \cdot \mathbf{v} + \gamma\phi = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.4}$$

Taking (2.2) into consideration, we rewrite (2.4) into the form

$$\begin{cases} -\Delta\phi + 2i\varepsilon\mathbf{F}\cdot\nabla\phi + \varepsilon^{2}|\,\mathbf{F}\,|^{2}\phi = \mu(\varepsilon,\gamma)\phi & \text{in } \Omega, \\ \dfrac{\partial\phi}{\partial\mathbf{v}} - i\varepsilon\mathbf{F}\cdot\mathbf{v}\phi + \gamma\phi = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.5}$$

In the present paper, we consider the asymptotic behaviors of the first eigenvalue $\mu(\varepsilon, \gamma)$ and the corresponding eigenfunction $\phi_{\varepsilon,\gamma}$ as $\varepsilon \to 0.$

First, we consider the eigenvalue problem

$$\begin{cases} -\Delta\phi = \mu\phi & \text{in } \Omega, \\ \dfrac{\partial\phi}{\partial\mathbf{v}} + \gamma\phi = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.6}$$

It is well known that the first eigenvalue $\mu_0(\gamma)$ of (2.6) is simple, analytic with respect to γ and $\mu_0(0) = 0,$ and we can choose the corresponding eigenfunction ϕ_γ to be smooth and positive on $\overline{\Omega}.$ (See Gilberg and Trudinger [11, Theorem 8.21 and Lemma 3.4].

Next, we consider the problem

$$\begin{cases} -\Delta v - \mu_0(\gamma)v = -2\mathbf{F}\cdot\nabla\phi_\gamma & \text{in } \Omega, \\ \dfrac{\partial v}{\partial\mathbf{v}} + \gamma v = \mathbf{F}\cdot\mathbf{v}\phi_\gamma & \text{on } \partial\Omega. \end{cases} \tag{2.7}$$

We shall show that the problem (2.7) has a unique smooth solution $v_{1,\gamma}$ such that $\int_\Omega v_{1,\gamma}\phi_\gamma dx = 0.$

We are now in a position to state the main theorem.

Theorem 2.1. *Under the situations as above, we have the asymptotics of the first eigenvalue* $\mu(\varepsilon, \gamma)$ *and the corresponding eigenfunction* $\phi_{\varepsilon,\gamma}$ *as follows.*

$$\mu(\varepsilon,\gamma) = \mu_0(\gamma) + \varepsilon^2\mu_2(\gamma) + O(\varepsilon^3)$$

as $\varepsilon \to 0,$ *where*

$$\mu_2(\gamma) = \|\phi_\gamma\|^{-2}_{L^2(\Omega)}\Big[\int_\Omega \{|\,\nabla v_{1,\gamma} - \mathbf{F}\phi_\gamma\,|^2 + 2(\mathbf{F}\cdot\nabla\phi_\gamma)v_{1,\gamma}\}dx$$

$$- \mu_0(\gamma)\int_\Omega |v_{1,\gamma}|^2 dx + \gamma\int_{\partial\Omega} |v_{1,\gamma}|^2 dS\Bigg].$$

and

$$\phi_{\varepsilon,\gamma} = \alpha_\varepsilon\phi_\gamma + i\varepsilon\beta_\varepsilon v_{1,\gamma} + \varepsilon^2\psi_\gamma^{(2)} + \varepsilon^3\psi_\gamma^{(3)} + o(\varepsilon^3)$$

as $\varepsilon \to 0$, *where* $\alpha_\varepsilon \to 1$, $\beta_\varepsilon \to 1$ *as* $\varepsilon \to 0$ *and* $\psi_\gamma^{(2)}$, $\psi_\gamma^{(3)}$ *are smooth functions.*

Remark 2.2. Pan [21] got the similar asymptotics when the applied field $\mathbf{H}(x) =$ constant unit vector and, $\gamma = 0$. In this case, since $\mu_0(0) = 0$ and we can choose $\phi_\gamma = 1$, we see that

$$\mu_2(0) = |\Omega|^{-1}\int_\Omega |\nabla v_{1,\gamma} - \mathbf{F}\phi_\gamma|^2 dx.$$

Ando and Aramaki [2] considered the case where the applied field is non-constant and $\gamma = 0$. They got a more precise asymptotics of $\phi_{\varepsilon,0}$ than [21].

3. Proof of the Main Theorem

In this section we shall devote to the proof of Theorem 2.1.

We consider a functional

$$E_\gamma[\phi] = \int_\Omega \{|\nabla\phi - \mathbf{F}\phi_\gamma|^2 + 2(\nabla\phi_\gamma \cdot \mathbf{F})\phi\} dx + \gamma\int_{\partial\Omega} |\phi|^2 dS \tag{3.1}$$

on $W^{1,2}(\Omega)$.

It is easy to show that the functional (3.1) is strictly convex, continuous in $W^{1,2}(\Omega)$ and so weakly lower semi-continuous. We shall show that E_γ is bounded from below. When $\gamma = 0$, since ϕ_0 is constant, we have $E_\gamma[\phi] \geq 0$ for all $\phi \in W^{1,2}(\Omega)$. Therefore, let $\gamma > 0$. Then from the integration by parts and the Schwarz inequality, we have

$$E_\gamma[\phi] = \int_\Omega \{|\nabla\phi|^2 - 2(\nabla\phi\cdot\mathbf{F})\phi_\gamma + |\mathbf{F}\phi_\gamma|^2 + 2(\nabla\phi_\gamma\cdot\mathbf{F})\phi\}dx$$

$$+ \gamma\int_{\partial\Omega}|\phi|^2 dS$$

$$= \int_\Omega \{|\nabla\phi|^2 - 4(\nabla\phi\cdot\mathbf{F})\phi_\gamma + |\mathbf{F}\phi_\gamma|^2\}dx$$

$$+ 2\int_{\partial\Omega}\phi_\gamma\phi\mathbf{F}\cdot\mathbf{v}dS + \gamma\int_{\partial\Omega}|\phi|^2 dS$$

$$\geq \int_\Omega |\nabla\phi|^2 dx - 2\delta\int_\Omega|\nabla\phi|^2 dx - \frac{2}{\delta}\int_\Omega|\mathbf{F}\phi_\gamma|^2 dx$$

$$+ \int_\Omega|\mathbf{F}\phi_\gamma|^2 dx - \delta\int_{\partial\Omega}|\phi|^2 dS - \frac{1}{\delta}\int_{\partial\Omega}|\phi_\gamma\mathbf{F}\cdot\mathbf{v}|^2 dS$$

$$+ \gamma\int_{\partial\Omega}|\phi|^2 dS$$

for any $\delta > 0$. If we choose $\delta > 0$ so that $\delta < \min\{1/2, \gamma\}$, we see that E_γ is bounded from below.

Thus it follows from the standard variational theory that we see that $\inf_{\phi\in W^{1,2}(\Omega)} E_\gamma[\phi]$ is achieved by a unique, real valued function $w_\gamma \in W^{1,2}(\Omega)$ and taking the Euler equation, w_γ satisfies the equation

$$\begin{cases} -\Delta w_\gamma = -2\mathbf{F}\cdot\nabla\phi_\gamma & \text{in } \Omega, \\ \dfrac{\partial w_\gamma}{\partial \mathbf{v}} + \gamma w_\gamma = \mathbf{F}\cdot\mathbf{v}\phi_\gamma & \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

Now we shall show

Proposition 3.1. *Let* $\mu(\varepsilon, \gamma)$ *be the first eigenvalue as in* (2.3) *and* $\mu_0(\gamma)$ *be the lowest eigenvalue of* (2.6). *Then we have*

$$\mu(\varepsilon, \gamma) = \mu_0(\gamma) + O(\varepsilon^2)$$

as $\varepsilon \to 0$.

We continue the proof of this proposition for some time. In order to estimate $\mu(\varepsilon, \gamma)$ from above, if we take $\phi = \phi_\gamma + i\varepsilon w_\gamma$ as a test function in (2.3), we have

$$\mu(\varepsilon, \gamma) \le \frac{\int_\Omega |\nabla_{\varepsilon\mathbf{F}}\phi|^2 dx + \gamma \int_{\partial\Omega} |\phi|^2 dS}{\int_\Omega |\phi|^2 dx}$$

$$= \left(\int_\Omega \{|\phi_\gamma|^2 + \varepsilon^2 |w_\gamma|^2\} dx \right)^{-1} \left[\int_\Omega \{|\nabla\phi_\gamma + \varepsilon^2 \mathbf{F} w_\gamma|^2 \right.$$

$$\left. + \varepsilon^2 |\nabla w_\gamma - \mathbf{F}\phi_\gamma|^2\} dx + \gamma \int_{\partial\Omega} \{|\phi_\gamma|^2 + \varepsilon^2 |w_\gamma|^2 dS \right]$$

$$\le \|\phi_\gamma\|_{L^2(\Omega)}^{-2} \left[\int_\Omega \{|\nabla\phi_\gamma|^2 + 2\varepsilon^2 (\mathbf{F} \cdot \nabla\phi_\gamma) w_\gamma + \varepsilon^4 |\mathbf{F} w_\gamma|^2 \right.$$

$$\left. + \varepsilon^2 |\nabla w_\gamma - \mathbf{F}\phi_\gamma|^2\} dx + \gamma \int_{\partial\Omega} \{|\phi_\gamma|^2 + \varepsilon^2 |w_\gamma|^2\} dS \right]$$

$$\le \mu_0(\gamma) + \varepsilon^2 \|\phi_\gamma\|_{L^2(\Omega)}^{-2} \left[\int_\Omega \{|\nabla w_\gamma - \mathbf{F}\phi_\gamma|^2 + 2(\nabla\phi_\gamma \cdot \mathbf{F}) w_\gamma\} dx \right.$$

$$\left. + \gamma \int_{\partial\Omega} |w_\gamma|^2 dS \right] + \varepsilon^4 \|\phi_\gamma\|_{L^2(\Omega)}^{-2} \int_\Omega |\mathbf{F} w_\gamma|^2 dx.$$

Thus if we put

$$W_\gamma = \|\phi_\gamma\|_{L^2(\Omega)}^{-2} \left[\int_\Omega \{|\nabla w_\gamma - \mathbf{F}\phi_\gamma|^2 + 2(\nabla\phi_\gamma \cdot \mathbf{F}) w_\gamma\} dx + \gamma \int_{\partial\Omega} |w_\gamma|^2 dS \right],$$

we see that

$$\mu(\varepsilon, \gamma) \le \mu_0(\gamma) + \varepsilon^2 W_\gamma + O(\varepsilon^4) \tag{3.3}$$

as $\varepsilon \to 0$.

In order to estimate $\mu(\varepsilon, \gamma)$ from below, we put $\phi_{\varepsilon,\gamma} = \alpha_\varepsilon \phi_\gamma + \varepsilon \psi_{\varepsilon,\gamma}$, where α_ε is chosen so that $\alpha_\varepsilon \int_\Omega \phi_\gamma^2 dx = \int_\Omega \phi_{\varepsilon,\gamma} \phi_\gamma dx$. Since $\phi_\gamma > 0$ on $\overline{\Omega}$,

α_ε is well defined. Then we note that $\int_\Omega \psi_{\varepsilon,\gamma}\phi_\gamma dx = 0$. If we substitute this function $\phi_{\varepsilon,\gamma}$ for (2.5) and use (2.6), we see that $\psi_{\varepsilon,\gamma}$ satisfies

$$\begin{cases} -\Delta\psi_{\varepsilon,\gamma} - \mu(\varepsilon,\gamma)\psi_{\varepsilon,\gamma} + 2i\varepsilon\mathbf{F}\cdot\nabla\psi_{\varepsilon,\gamma} + \varepsilon^{\mathbf{2}}|\,\mathbf{F}\,|^2\psi_{\varepsilon,\gamma} \\ = \dfrac{\mu(\varepsilon,\gamma)-\mu_0(\gamma)}{\varepsilon}\alpha_\varepsilon\phi_\gamma - \varepsilon\alpha_\varepsilon|\,\mathbf{F}\,|^2\phi_\gamma - 2i\alpha_\varepsilon\mathbf{F}\cdot\nabla\phi_\gamma & \text{in } \Omega \\ \dfrac{\partial\psi_{\varepsilon,\gamma}}{\partial\mathbf{v}} - i\varepsilon\mathbf{F}\cdot\mathbf{v}\psi_{\varepsilon,\gamma} + \gamma\psi_{\varepsilon,\gamma} = i\alpha_\varepsilon\mathbf{F}\cdot\mathbf{v}\phi_\gamma & \text{on } \partial\Omega. \end{cases} \tag{3.4}$$

We must prove that

$$\frac{\mu(\varepsilon,\gamma)-\mu_0(\gamma)}{\varepsilon^2} \text{ is bounded.} \tag{3.5}$$

As the first step, we shall show that $(\mu(\varepsilon,\gamma)-\mu_0(\gamma))/\varepsilon$ is bounded.

Lemma 3.2. *Under the situation as above, we see that* $(\mu(\varepsilon,\gamma)-\mu_0(\gamma))/\varepsilon$ *is bounded with respect to* $\varepsilon \in (0, 1]$.

Proof. By (2.3) and the Schwarz inequality,

$$\begin{aligned} \mu(\varepsilon,\gamma) &= \|\,\phi_{\varepsilon,\gamma}\,\|_{L^2(\Omega)}^{-2}\left[\int_\Omega |\,\nabla_{\varepsilon\mathbf{F}}\phi_{\varepsilon,\gamma}\,|^2 dx + \gamma\int_{\partial\Omega}|\,\phi_{\varepsilon,\gamma}\,|^2 dS\right] \\ &\geq \|\,\phi_{\varepsilon,\gamma}\,\|_{L^2(\Omega)}^{-2}\left[\int_\Omega |\,\nabla\phi_{\varepsilon,\gamma}\,|^2 dx - 2\varepsilon\int_\Omega |\,\nabla\phi_{\varepsilon,\gamma}\,||\,\mathbf{F}\phi_{\varepsilon,\gamma}\,| dx \right. \\ &\qquad \left. + \varepsilon^2\int_\Omega |\,\mathbf{F}\phi_{\varepsilon,\gamma}\,|^2 dx + \gamma\int_{\partial\Omega}|\,\phi_{\varepsilon,\gamma}\,|^2 dS\right] \\ &\geq (1-\varepsilon)\|\,\phi_{\varepsilon,\gamma}\,\|_{L^2(\Omega)}^{-2}\left[\int_\Omega |\,\nabla\phi_{\varepsilon,\gamma}\,|^2 dx + \gamma\int_{\partial\Omega}|\,\phi_{\varepsilon,\gamma}\,|^2 dS\right] \\ &\qquad - \varepsilon\|\,\phi_{\varepsilon,\gamma}\,\|_{L^2(\Omega)}^{-2}\int_\Omega |\,\mathbf{F}\phi_{\varepsilon,\gamma}\,|^2 dx \\ &\geq (1-\varepsilon)\mu_0(\gamma) - O(\varepsilon) \end{aligned}$$

as $\varepsilon \to 0$. Thus we see that $\mu(\varepsilon,\gamma) \geq \mu_0(\gamma) - O(\varepsilon)$. Taking (3.3) into consideration, the proof is completed. □

We return to the equation (2.5). Let $\phi_{\varepsilon,\gamma}$ be the normalized eigenfunction such that $\|\phi_{\varepsilon,\gamma}\|_{L^\infty(\Omega)} = 1$. Then by the elliptic estimate [11, Theorem 6.30], we see that $\|\phi_{\varepsilon,\gamma}\|_{C^{2+\alpha}(\overline{\Omega})} \le C(\alpha, \gamma) < \infty$ for any $\alpha \in (0, 1)$ and small $\varepsilon > 0$. Passing to a subsequence, we may assume that $\phi_{\varepsilon,\gamma} \to \phi_\gamma$ in $C^{2+\alpha}(\overline{\Omega})$. We remember that $\phi_{\varepsilon,\gamma} = \alpha_\varepsilon\phi_\gamma + \varepsilon\psi_{\varepsilon,\gamma}$.

Now we claim that $\|\psi_{\varepsilon,\gamma}\|_{L^2(\Omega)}$ is bounded.

In fact, if the claim does not hold, passing to a subsequence, we may assume that $C_\varepsilon := \|\psi_{\varepsilon,\gamma}\|_{L^2(\Omega)} \to \infty$ as $\varepsilon \to 0$. Put $\widetilde{\psi}_{\varepsilon,\gamma} = \psi_{\varepsilon,\gamma}/C_\varepsilon$. Then $\widetilde{\psi}_{\varepsilon,\gamma}$ satisfies the equation

$$\begin{cases} -\Delta\widetilde{\psi}_{\varepsilon,\gamma} - \mu(\varepsilon, \gamma)\widetilde{\psi}_{\varepsilon,\gamma} + 2i\varepsilon\mathbf{F}\cdot\nabla\widetilde{\psi}_{\varepsilon,\gamma} + \varepsilon^2|\mathbf{F}|^2\widetilde{\psi}_{\varepsilon,\gamma} & \\ = \dfrac{\mu(\varepsilon, \gamma) - \mu_0(\gamma)}{\varepsilon C_\varepsilon}\phi_\gamma - \dfrac{\varepsilon}{C_\varepsilon}|\mathbf{F}|^2\phi_\gamma - \dfrac{2i}{C_\varepsilon}\mathbf{F}\cdot\nabla\phi_\gamma & \text{in } \Omega \\ \dfrac{\partial\widetilde{\psi}_{\varepsilon,\gamma}}{\partial\mathbf{v}} - i\varepsilon\mathbf{F}\cdot\mathbf{v}\widetilde{\psi}_{\varepsilon,\gamma} + \gamma\widetilde{\psi}_{\varepsilon,\gamma} = \dfrac{i}{C_\varepsilon}\mathbf{F}\cdot\mathbf{v}\phi_\gamma & \text{on } \partial\Omega. \end{cases} \tag{3.6}$$

Since $\|\widetilde{\psi}_{\varepsilon,\gamma}\|_{L^2(\Omega)} = 1$, it follows from [11, Theorem 8.13] or Agmon et al. [1, Theorem 15.2] that $\|\widetilde{\psi}_{\varepsilon,\gamma}\|_{W^{k+2,2}(\Omega)} \le C(k, \gamma)$ for any $k \in \mathbb{N}$ (cf. Du [7]). By the Sobolev imbedding theorem, $\|\widetilde{\psi}_{\varepsilon,\gamma}\|_{C^{2+\alpha}(\overline{\Omega})} \le C(\alpha, \gamma)$ for any $\alpha \in (0, 1)$. Passing to a subsequence, we may assume that $\widetilde{\psi}_{\varepsilon,\gamma} \to \widetilde{\psi}_\gamma$ in $C^{2+\alpha}(\overline{\Omega})$. Letting $\varepsilon \to 0$ in (3.6), we see that $\widetilde{\psi}_\gamma$ satisfies

$$\begin{cases} -\Delta\widetilde{\psi}_\gamma - \mu_0(\gamma)\widetilde{\psi}_\gamma = 0 & \text{in } \Omega, \\ \dfrac{\partial\widetilde{\psi}_\gamma}{\partial\mathbf{v}} + \gamma\widetilde{\psi}_\gamma = 0 & \text{on } \partial\Omega \end{cases}$$

and $\|\widetilde{\psi}_\gamma\|_{L^2(\Omega)} = 1$, $\int_\Omega \widetilde{\psi}_\gamma\phi_\gamma dx = 0$. Since the real part and the imaginary part of $\widetilde{\psi}_\gamma$ are non-zero constant signs, this leads to a contradiction. Thus $\|\psi_{\varepsilon,\gamma}\|_{L^2(\Omega)}$ is bounded.

Since $\|\psi_{\varepsilon,\gamma}\|_{L^2(\Omega)}$ is bounded, if we again apply the same arguments as above, we see that $\|\psi_{\varepsilon,\gamma}\|_{C^{2+\alpha}(\overline{\Omega})} \le C(\alpha, \gamma)$. Taking Lemma 3.2 into consideration, passing to a subsequence, we may assume that $\frac{\mu(\varepsilon, \gamma) - \mu_0(\gamma)}{\varepsilon} \to \mu_1(\gamma)$ and $\psi_{\varepsilon,\gamma} \to \phi_{1,\gamma}$ in $C^{2+\alpha}(\overline{\Omega})$ as $\varepsilon \to 0$. Letting $\varepsilon \to 0$ in (3.4), we have

$$\begin{cases} -\Delta\phi_{1,\gamma} - \mu_0(\gamma)\phi_{1,\gamma} = \mu_1(\gamma)\phi_\gamma - 2i\mathbf{F} \cdot \nabla\phi_\gamma & \text{in } \Omega \\ \dfrac{\partial\phi_{1,\gamma}}{\partial\mathbf{v}} + \gamma\phi_{1,\gamma} = i\mathbf{F} \cdot \mathbf{v}\phi_\gamma & \text{on } \partial\Omega. \end{cases}$$

Let $u_{1,\gamma}$ and $v_{1,\gamma}$ be the real part and imaginary part of $\phi_{1,\gamma}$, respectively. Then $u_{1,\gamma}$ is a solution of the problem

$$\begin{cases} -\Delta u_{1,\gamma} - \mu_0(\gamma)u_{1,\gamma} = \mu_1(\gamma)\phi_\gamma & \text{in } \Omega \\ \dfrac{\partial u_{1,\gamma}}{\partial\mathbf{v}} + \gamma u_{1,\gamma} = 0 & \text{on } \partial\Omega. \end{cases}$$

Since the boundary value problem $\left(-\Delta - \mu_0(\gamma), \dfrac{\partial}{\partial\mathbf{v}} + \gamma\right)$ is self adjoint, it follows from the Fredholm alternative theorem that "the orthogonality condition" $\mu_1(\gamma)(\phi_\gamma, \phi_\gamma)_{L^2(\Omega)} = 0$ holds. Thus we have $\mu_1(\gamma) = 0$. Since $u_{1,\gamma}$ has a constant sign in $\overline{\Omega}$ and $\int_\Omega u_{1,\gamma}\phi_\gamma dx = 0$, we see that $u_{1,\gamma} = 0$. Now $v_{1,\gamma}$ satisfies the equation

$$\begin{cases} -\Delta v_{1,\gamma} - \mu_0(\gamma)v_{1,\gamma} = -2\mathbf{F} \cdot \nabla\phi_\gamma & \text{in } \Omega \\ \dfrac{\partial v_{1,\gamma}}{\partial\mathbf{v}} + \gamma v_{1,\gamma} = \mathbf{F} \cdot \mathbf{v}\phi_\gamma & \text{on } \partial\Omega. \end{cases} \tag{3.7}$$

We note that the solution $v_{1,\gamma}$ of (3.7) satisfying $\int_\Omega v_{1,\gamma}\phi_\gamma dx = 0$ is unique.

Thus we can write $\phi_{\varepsilon,\gamma} = \phi_\gamma + i\varepsilon v_{1,\gamma} + \varepsilon\widetilde{\phi}_{\varepsilon,\gamma}$, where $\widetilde{\phi}_{\varepsilon,\gamma}$ is bounded in $C^{2+\alpha}(\overline{\Omega})$. Therefore, we have

$$\mu(\varepsilon,\gamma)\|\phi_{\varepsilon,\gamma}\|^2_{L^2(\Omega)} = \int_\Omega |\nabla\phi_{\varepsilon,\gamma} - i\varepsilon\mathbf{F}\cdot\nabla\phi_{\varepsilon,\gamma}|^2 dx + \gamma\int_{\partial\Omega}|\phi_{\varepsilon,\gamma}|^2 dS$$
$$= \int_\Omega \{|\nabla\phi_{\varepsilon,\gamma}|^2 - 2\varepsilon\Im\{(\mathbf{F}\cdot\nabla\phi_{\varepsilon,\gamma})\overline{\phi_{\varepsilon,\gamma}}\}$$
$$- \varepsilon^2|\mathbf{F}\phi_{\varepsilon,\gamma}|^2\}dx + \gamma\int_{\partial\Omega}|\phi_{\varepsilon,\gamma}|^2 dS.$$

Here we note that since

$$\int_\Omega (\mathbf{F}\cdot\nabla\phi_{\varepsilon,\gamma})\overline{\phi_{\varepsilon,\gamma}}dx$$
$$= \int_\Omega \mathbf{F}\cdot\{\nabla\phi_\gamma + \varepsilon(i\nabla v_{1,\gamma} + \nabla\widetilde{\psi}_{\varepsilon,\gamma})\}(\phi_\gamma - i\varepsilon w_\gamma + \varepsilon\overline{\widetilde{\phi}_{\varepsilon,\gamma}})dx,$$

it follows that $\Im\int_\Omega (\mathbf{F}\cdot\nabla\phi_{\varepsilon,\gamma})\overline{\phi_{\varepsilon,\gamma}}dx = O(\varepsilon)$. Therefore, we have

$$\mu(\varepsilon,\gamma)\|\phi_{\varepsilon,\gamma}\|^2_{L^2(\Omega)} \geq \int_\Omega |\nabla\phi_{\varepsilon,\gamma}|^2 dx + \gamma\int_{\partial\Omega}|\phi_{\varepsilon,\gamma}|^2 dS - O(\varepsilon^2)$$
$$\geq \mu_0(\gamma)\|\phi_{\varepsilon,\gamma}\|^2_{L^2(\Omega)} - O(\varepsilon^2).$$

Summing up (3.3), we see that $\dfrac{\mu(\varepsilon,\gamma)-\mu_0(\gamma)}{\varepsilon^2}$ is bounded with respect to ε. That is to say, the claim (3.5) holds. This completes the proof of Proposition 3.1.

Thus if we put $\mu(\varepsilon,\gamma)-\mu_0(\gamma) = \varepsilon^2\lambda(\varepsilon,\gamma)$, passing to a subsequence, we may assume that $\lambda(\varepsilon,\gamma)\to\mu_2(\gamma)$ as $\varepsilon\to 0$. We remember that we can write $\phi_{\varepsilon,\gamma} = \alpha_\varepsilon\phi_\gamma + \varepsilon\psi^{(1)}_{\varepsilon,\gamma}$, where $\psi^{(1)}_{\varepsilon,\gamma}\to iv_{1,\gamma}$ in $C^{2+\alpha}(\overline{\Omega})$ as $\varepsilon\to 0$. We write $\psi^{(1)}_{\varepsilon,\gamma} = i\beta_\varepsilon v_{1,\gamma} + \varepsilon\psi^{(2)}_{\varepsilon,\gamma}$, where

$$\beta_\varepsilon = -i\frac{\int_\Omega v_{1,\gamma}\psi^{(1)}_{\varepsilon,\gamma}dx}{\int_\Omega |v_{1,\gamma}|^2 dx}.$$

Then we see that $\int_\Omega \psi^{(2)}_{\varepsilon,\gamma}\phi_\gamma dx = 0$ and $\int_\Omega \psi^{(2)}_{\varepsilon,\gamma}v_{1,\gamma}dx = 0$. Since $\psi^{(1)}_{\varepsilon,\gamma}\to$

$iv_{1,\gamma}$ in $C^{2+\alpha}(\overline{\Omega})$, it follows that $\beta_\varepsilon \to 1$ as $\varepsilon \to 0$. Taking (2.6) and (3.7) into consideration, $\psi^{(2)}_{\varepsilon,\gamma}$ satisfies the following equation

$$\begin{cases} -\Delta\psi^{(2)}_{\varepsilon,\gamma} - \mu_0(\gamma)\psi^{(2)}_{\varepsilon,\gamma} + 2i\varepsilon\mathbf{F}\cdot\nabla\psi^{(2)}_{\varepsilon,\gamma} + \varepsilon^2|\mathbf{F}|^2\psi^{(2)}_{\varepsilon,\gamma} \\ \quad - \varepsilon^2\lambda(\varepsilon,\gamma)\psi^{(2)}_{\varepsilon,\gamma} = f_{\varepsilon,\gamma} & \text{in } \Omega \\ \dfrac{\partial\psi^{(2)}_{\varepsilon,\gamma}}{\partial\mathbf{v}} + \gamma\psi^{(2)}_{\varepsilon,\gamma} - i\varepsilon\mathbf{F}\cdot\mathbf{v}\psi^{(2)}_{\varepsilon,\gamma} = i\dfrac{\alpha_\varepsilon - \beta_\varepsilon}{\varepsilon}\mathbf{F}\cdot\mathbf{v}\phi_\gamma - \beta_\varepsilon\mathbf{F}\cdot\mathbf{v}v_{1,\gamma} & \text{on } \partial\Omega, \end{cases} \tag{3.8}$$

where

$$f_{\varepsilon,\gamma} = 2\beta_\varepsilon\mathbf{F}\cdot\nabla v_{1,\gamma} - i\varepsilon\beta_\varepsilon|\mathbf{F}|^2 v_{1,\gamma} + i\varepsilon\beta_\varepsilon\lambda(\varepsilon,\gamma)v_{1,\gamma}$$
$$- 2i\frac{\alpha_\varepsilon - \beta_\varepsilon}{\varepsilon}\mathbf{F}\cdot\nabla\phi_\gamma - \alpha_\varepsilon|\mathbf{F}|^2\phi_\gamma + \alpha_\varepsilon\lambda(\varepsilon,\gamma)\phi_\gamma.$$

We shall show that $(\alpha_\varepsilon - \beta_\varepsilon)/\varepsilon$ is bounded with respect to ε.

Lemma 3.3. *If we define* $\delta_\varepsilon = (\alpha_\varepsilon - \beta_\varepsilon)/\varepsilon$, *then* $\{\delta_\varepsilon\}$ *is bounded with respect to* $\varepsilon \in (0, 1]$.

Proof. If the claim does not hold, passing to a subsequence, we may assume that $\delta_\varepsilon \to \infty$ as $\varepsilon \to 0$. If we define $\xi_{\varepsilon,\gamma} = \psi^{(2)}_{\varepsilon,\gamma}/\delta_\varepsilon$, it is clear that $\int_\Omega \xi_{\varepsilon,\gamma}\phi_\gamma dx = 0$ and $\int_\Omega \xi_{\varepsilon,\gamma}v_{1,\gamma}dx = 0$. From (3.8), $\xi_{\varepsilon,\gamma}$ satisfies the following equation

$$\begin{cases} -\Delta\xi_{\varepsilon,\gamma} - \mu_0(\gamma)\xi_{\varepsilon,\gamma} + 2i\varepsilon\mathbf{F}\cdot\nabla\xi_{\varepsilon,\gamma} + \varepsilon^2|\mathbf{F}|^2\xi_{\varepsilon,\gamma} \\ \quad - \varepsilon^2\lambda(\varepsilon,\gamma)\xi_{\varepsilon,\gamma} = -2i\mathbf{F}\cdot\nabla\phi_\gamma + \dfrac{1}{\delta_\varepsilon}g_{\varepsilon,\gamma} & \text{in } \Omega \\ \dfrac{\partial\xi_{\varepsilon,\gamma}}{\partial\mathbf{v}} + \gamma\xi_{\varepsilon,\gamma} - i\varepsilon\mathbf{F}\cdot\mathbf{v}\xi_{\varepsilon,\gamma} = i\mathbf{F}\cdot\mathbf{v}\phi_\gamma - i\dfrac{\beta_\varepsilon}{\delta_\varepsilon}\mathbf{F}\cdot\mathbf{v}v_{1,\gamma} & \text{on } \partial\Omega, \end{cases} \tag{3.9}$$

where

$$g_{\varepsilon,\gamma} = 2\beta_\varepsilon\mathbf{F}\cdot\nabla v_{1,\gamma} - i\varepsilon\beta_\varepsilon|\mathbf{F}|^2 v_{1,\gamma} + i\varepsilon\beta_\varepsilon\lambda(\varepsilon,\gamma)v_{1,\gamma}$$
$$- \alpha_\varepsilon|\mathbf{F}|^2\phi_\gamma + \alpha_\varepsilon\lambda(\varepsilon,\gamma)\phi_\gamma.$$

Case 1. $\|\xi_{\varepsilon,\gamma}\|_{L^2(\Omega)} \le C < \infty$.

Then applying the elliptic estimate as above, it can be seen that $\|\xi_{\varepsilon,\gamma}\|_{W^{k,2}(\Omega)} \le C(k)$ for any $k \in \mathbb{N}$. Therefore, by the Sobolev imbedding theorem, $\|\xi_{\varepsilon,\gamma}\|_{C^{2+\alpha}(\overline{\Omega})} \le C(\gamma, \alpha)$ for any $\alpha \in (0, 1)$. Passing to a subsequence, we may assume that $\xi_{\varepsilon,\gamma} \to \xi_\gamma$ in $C^{2+\alpha}(\overline{\Omega})$ as $\varepsilon \to 0$. Then we see that $\int_\Omega \xi_\gamma \phi_\gamma dx = 0$ and $\int_\Omega \xi_\gamma v_{1,\gamma} dx = 0$. Letting $\varepsilon \to 0$ in (3.9), we have the equation

$$\begin{cases} -\Delta\xi_\gamma - \mu_0(\gamma)\xi_\gamma = -2i\mathbf{F} \cdot \nabla\phi_\gamma & \text{in } \Omega \\ \dfrac{\partial \xi_\gamma}{\partial \mathbf{v}} + \gamma\xi_\gamma = i\mathbf{F} \cdot \mathbf{v}\phi_\gamma & \text{on } \partial\Omega. \end{cases}$$

Thus we have $\xi_\gamma = v_{1,\gamma}$. This leads to a contradiction.

Case 2. $\|\xi_{\varepsilon,\gamma}\|_{L^2(\Omega)}$ is unbounded.

In this case, passing to a subsequence, we may assume that $C_\varepsilon = \|\xi_{\varepsilon,\gamma}\|_{L^2(\Omega)} \to \infty$ as $\varepsilon \to 0$. If we put $\widetilde{\xi}_{\varepsilon,\gamma} = \xi_{\varepsilon,\gamma}/C_\varepsilon$, then we see that $\widetilde{\xi}_{\varepsilon,\gamma}$ satisfies the following equation

$$\begin{cases} -\Delta\widetilde{\xi}_{\varepsilon,\gamma} - \mu_0(\gamma)\widetilde{\xi}_{\varepsilon,\gamma} + 2i\varepsilon\mathbf{F} \cdot \nabla\widetilde{\xi}_{\varepsilon,\gamma} + \varepsilon^2|\mathbf{F}|^2\widetilde{\xi}_{\varepsilon,\gamma} & \\ \quad - \varepsilon^2\lambda(\varepsilon,\gamma)\widetilde{\xi}_{\varepsilon,\gamma} = -\dfrac{2i}{C_\varepsilon}\mathbf{F} \cdot \nabla\phi_\gamma + \dfrac{1}{\delta_\varepsilon C_\varepsilon} g_{\varepsilon,\gamma} & \text{in } \Omega \\ \dfrac{\partial\widetilde{\xi}_{\varepsilon,\gamma}}{\partial \mathbf{v}} + \gamma\widetilde{\xi}_{\varepsilon,\gamma} - i\varepsilon\mathbf{F} \cdot \mathbf{v}\widetilde{\xi}_{\varepsilon,\gamma} = \dfrac{i}{C_\varepsilon}\mathbf{F} \cdot \mathbf{v}\phi_\gamma - \dfrac{\beta_\varepsilon}{C_\varepsilon\delta_\varepsilon}\mathbf{F} \cdot \mathbf{v}v_{1,\gamma} & \text{on } \partial\Omega. \end{cases}$$

Similarly as Case 1, we may assume that $\widetilde{\xi}_{\varepsilon,\gamma} \to \widetilde{\xi}_\gamma$ in $C^{2+\alpha}(\overline{\Omega})$. Then $\|\widetilde{\xi}_\gamma\|_{L^2(\Omega)} = 1$ and $\int_\Omega \widetilde{\xi}_\gamma \phi_\gamma dx = 0$ and $\widetilde{\xi}_\gamma$ satisfies

$$\begin{cases} -\Delta\widetilde{\xi}_\gamma = \mu_0(\gamma)\widetilde{\xi}_\gamma & \text{in } \Omega \\ \dfrac{\partial\widetilde{\xi}_\gamma}{\partial \mathbf{v}} + \gamma\widetilde{\xi}_\gamma = 0 & \text{on } \partial\Omega. \end{cases}$$

Since any solution of this equation is constant sign on $\overline{\Omega}$, this leads to a contradiction. $\square$

Thus since $\delta_\varepsilon = (\alpha_\varepsilon - \beta_\varepsilon)/\varepsilon$ is bounded, we may assume that $\delta_\varepsilon \to \delta_0$ as $\varepsilon \to 0$. Since $\| \psi_{\varepsilon,\gamma}^{(2)} \|_{L^2(\Omega)} \le C$ in (3.8), as the similar arguments in Case 2, we have $\| \psi_{\varepsilon,\gamma}^{(2)} \|_{C^{2+\alpha}(\overline{\Omega})} \le C(\gamma, \alpha)$. Therefore, we may assume that $\psi_{\varepsilon,\gamma}^{(2)} \to \psi_\gamma^{(2)}$ in $C^{2+\alpha}(\overline{\Omega})$. Letting $\varepsilon \to 0$ in (3.8), we get the following equation

$$\begin{cases} -\Delta\psi_\gamma^{(2)} - \mu_0(\gamma)\psi_\gamma^{(2)} = 2\mathbf{F} \cdot \nabla v_{1,\gamma} - 2i\delta_0 \mathbf{F} \cdot \nabla\phi_\gamma - |\mathbf{F}|^2 \phi_\gamma & \\ \quad + \mu_2(\gamma)\phi_\gamma & \text{in } \Omega \\ \dfrac{\partial \psi_\gamma^{(2)}}{\partial \mathbf{v}} + \gamma\psi_\gamma^{(2)} = i\delta_0 \mathbf{F} \cdot \mathbf{v}\phi_\gamma - \mathbf{F} \cdot \mathbf{v} v_{1,\gamma} & \text{on } \partial\Omega. \end{cases} \tag{3.10}$$

Since $\psi_\gamma^{(2)}$ is a solution of (3.10), we have “the orthogonality condition”

$$\int_\Omega (2(\mathbf{F} \cdot \nabla v_{1,\gamma})\phi_\gamma - 2i\delta_0(\mathbf{F} \cdot \nabla\phi_\gamma)\phi_\gamma - |\mathbf{F}|^2\phi_\gamma^2 + \mu_2(\gamma)\phi_\gamma^2)dx$$

$$+ \int_{\partial\Omega} (i\delta_0 \mathbf{F} \cdot \mathbf{v}\phi_\gamma^2 - \mathbf{F} \cdot \mathbf{v} v_{1,\gamma}\phi_\gamma)dS = 0.$$

Thus we get

$$\begin{aligned} \mu_2(\gamma)\int_\Omega \phi_\gamma^2 dx &= \int_\Omega (-2(\mathbf{F} \cdot \nabla v_{1,\gamma})\phi_\gamma + |\mathbf{F}|^2\phi_\gamma^2)dx \\ &\quad + \int_{\partial\Omega} \mathbf{F} \cdot \mathbf{v} v_{1,\gamma}\phi_\gamma dS \\ &= \int_\Omega |\nabla v_{1,\gamma} - \mathbf{F}\phi_\gamma|^2 - \int_\Omega |\nabla v_{1,\gamma}|^2 dx \\ &\quad + \int_{\partial\Omega} \mathbf{F} \cdot \mathbf{v} v_{1,\gamma}\phi_\gamma dS. \end{aligned}$$

From integration by parts, we see that

$$\int_{\partial\Omega} \mathbf{F} \cdot \mathbf{v} v_{1,\gamma}\phi_\gamma dS = \int_{\partial\Omega} \left(\frac{\partial v_{1,\gamma}}{\partial \mathbf{v}} + \gamma v_{1,\gamma} \right) v_{1,\gamma} dS$$

$$= \int_\Omega |\nabla v_{1,\gamma}|^2 dx + \int_\Omega v_{1,\gamma}\Delta v_{1,\gamma} dx + \gamma\int_{\partial\Omega} |v_{1,\gamma}|^2 dS$$

$$= \int_\Omega |\nabla v_{1,\gamma}|^2 dx - \mu_0(\gamma)\int_\Omega |v_{1,\gamma}|^2 dx$$

$$+ 2\int_\Omega (\mathbf{F}\cdot\nabla\phi_\gamma)v_{1,\gamma} dx + \gamma\int_{\partial\Omega} |v_{1,\gamma}|^2 dS.$$

Thus we get

$$\mu_2(\gamma) = \|\phi_\gamma\|_{L^2(\Omega)}^{-2}\Big[\int_\Omega \{|\nabla v_{1,\gamma} - \mathbf{F}\phi_\gamma|^2 + 2(\mathbf{F}\cdot\nabla\phi_\gamma)v_{1,\gamma}\} dx$$

$$- \mu_0(\gamma)\int_\Omega |v_{1,\gamma}|^2 dx + \gamma\int_{\partial\Omega} |v_{1,\gamma}|^2 dS\Big].$$

In this stage, we got the asymptotics:

$$\mu(\varepsilon, \gamma) = \mu_0(\gamma) + \varepsilon^2\mu_2(\gamma) + o(\varepsilon^2),$$

$$\phi_{\varepsilon,\gamma} = \alpha_\varepsilon\phi_\gamma + i\varepsilon v_{1,\gamma} + \varepsilon^2\psi_\gamma^{(2)} + o(\varepsilon^2)$$

as $\varepsilon \to 0$.

We shall continue further arguments.

If we put $\varphi_{\varepsilon,\gamma} = (\phi_{\varepsilon,\gamma} - \phi_\gamma)/\varepsilon$, then we get the following equation

$$\begin{cases} -\Delta\varphi_{\varepsilon,\gamma} - \mu_0(\gamma)\varphi_{\varepsilon,\gamma} = -2i\mathbf{F}\cdot\nabla\phi_{\varepsilon,\gamma} - \varepsilon|\mathbf{F}|^2\phi_{\varepsilon,\gamma} + \varepsilon\lambda(\varepsilon,\gamma)\phi_{\varepsilon,\gamma} & \text{in } \Omega \\ \dfrac{\partial\varphi_{\varepsilon,\gamma}}{\partial\mathbf{v}} + \gamma\varphi_{\varepsilon,\gamma} = i\mathbf{F}\cdot\mathbf{v}\phi_{\varepsilon,\gamma} & \text{on } \partial\Omega. \end{cases}$$

Again using the bootstrap argument, we see that $\{\varphi_{\varepsilon,\gamma}\}$ is bounded in $W^{k,2}(\Omega)$ for any $k \in \mathbb{N}$. Therefore, by the Sobolev imbedding theorem, $\{\varphi_{\varepsilon,\gamma}\}$ is bounded in $C^{2+\alpha}(\overline{\Omega})$ for any $\alpha \in (0, 1)$. Since

$$\frac{\phi_{\varepsilon,\gamma} - \alpha_\varepsilon\phi_\gamma}{\varepsilon} + \frac{(\alpha_\varepsilon - 1)\phi_\gamma}{\varepsilon} = \frac{\phi_{\varepsilon,\gamma} - \phi_\gamma}{\varepsilon},$$

if we multiply ϕ_γ to the both side and integrate over Ω, then we see that

$(\alpha_\varepsilon - 1)/\varepsilon$ is bounded with respect to ε. Moreover, since $\delta_\varepsilon = ((\alpha_\varepsilon - 1) - (\beta_\varepsilon - 1))/\varepsilon$, we also see that $(\beta_\varepsilon - 1)/\varepsilon$, is bounded with respect to ε. If we subtract (3.10) from (3.8), then we get the following equation for $\phi^{(3)}_{\varepsilon,\gamma} := (\psi^{(2)}_{\varepsilon,\gamma} - \psi^{(2)}_\gamma)/\varepsilon$

$$\begin{cases} -\Delta\phi^{(3)}_{\varepsilon,\gamma} - \mu_0(\gamma)\phi^{(3)}_{\varepsilon,\gamma} = -2i\varepsilon\mathbf{F}\cdot\nabla\psi^{(2)}_{\varepsilon,\gamma} \\ \quad - \varepsilon^2(|\mathbf{F}|^2 - \lambda(\varepsilon,\gamma))\psi^{(2)}_{\varepsilon,\gamma} + 2(\beta_\varepsilon - 1)\mathbf{F}\cdot\nabla v_{1,\gamma} \\ \quad - i\varepsilon\beta_\varepsilon(|\mathbf{F}|^2 - \lambda(\varepsilon,\gamma))v_{1,\gamma} - 2i(\delta_\varepsilon - \delta_0)\mathbf{F}\cdot\nabla\phi_\gamma \\ \quad - (\alpha_\varepsilon - 1)|\mathbf{F}|^2\phi_\gamma + (\alpha_\varepsilon - 1)\lambda(\varepsilon,\gamma)\phi_\gamma \\ \quad + (\lambda(\varepsilon,\gamma) - \mu_0(\gamma))\phi_\gamma & \text{in } \Omega \\ \dfrac{\partial\phi^{(3)}_{\varepsilon,\gamma}}{\partial\mathbf{v}} + \gamma\phi^{(3)}_{\varepsilon,\gamma} = i\varepsilon\mathbf{F}\cdot\mathbf{v}\psi^{(2)}_{\varepsilon,\gamma} + i(\delta_\varepsilon - \delta_0)\mathbf{F}\cdot\mathbf{v}\phi_\gamma \\ \quad - (\beta_\varepsilon - 1)\mathbf{F}\cdot\mathbf{v}v_{1,\gamma} & \text{on } \partial\Omega. \end{cases}$$

Using “the orthogonality condition”, we get

$$-2i(\delta_\varepsilon - \delta_0)\int_\Omega(\mathbf{F}\cdot\nabla\phi_\gamma)\phi_\gamma dx + (\lambda(\varepsilon,\gamma) - \mu_0(\gamma))\int_\Omega\phi_\gamma^2 dx$$

$$+ i(\delta_\varepsilon - \delta_0)\int_{\partial\Omega}\mathbf{F}\cdot\mathbf{v}\phi_\gamma^2 dS = O(\varepsilon).$$

Since by the integration by parts,

$$2\int_\Omega(\mathbf{F}\cdot\nabla\phi_\gamma)\phi_\gamma dx = \int_{\partial\Omega}\mathbf{F}\cdot\mathbf{v}\phi_\gamma^2 dS,$$

we get $(\lambda(\varepsilon,\gamma) - \mu_0(\gamma))\int_\Omega\phi_\gamma^2 dx = O(\varepsilon)$. That is to say, $\lambda(\varepsilon,\gamma) - \mu_0(\gamma) = O(\varepsilon)$. Therefore we obtain

$$\mu(\varepsilon,\gamma) = \mu(\gamma) + \varepsilon^2\mu_2(\gamma) + O(\varepsilon^3)$$

as $\varepsilon \to 0$. If we put $\psi^{(3)}_{\varepsilon,\gamma} = \dfrac{\psi^{(2)}_{\varepsilon,\gamma} - \psi^{(2)}_\gamma}{\varepsilon}$, then $\psi^{(3)}_{\varepsilon,\gamma}$ satisfies the following equation

$$\begin{cases} -\Delta\psi^{(3)}_{\varepsilon,\gamma} - \mu_0(\gamma)\psi^{(3)}_{\varepsilon,\gamma} = -2i\mathbf{F}\cdot\nabla\psi^{(2)}_{\varepsilon,\gamma} \\ \quad - \varepsilon(|\mathbf{F}|^2 - \lambda(\varepsilon,\gamma))\psi^{(2)}_{\varepsilon,\gamma} - 2\dfrac{\beta_\varepsilon - 1}{\varepsilon}\mathbf{F}\cdot\nabla v_{1,\gamma} \\ \quad - i\beta_\varepsilon(|\mathbf{F}|^2 - \lambda(\varepsilon,\gamma))v_{1,\gamma} \\ \quad - 2i\dfrac{\delta_\varepsilon - \delta_0}{\varepsilon}\mathbf{F}\cdot\nabla\phi_\gamma - \dfrac{\alpha_\varepsilon - 1}{\varepsilon}|\mathbf{F}|^2\phi_\gamma \\ \quad + \dfrac{\alpha_\varepsilon - 1}{\varepsilon}\lambda(\varepsilon,\gamma)\phi_\gamma + \dfrac{\lambda(\varepsilon,\gamma) - \mu_0(\gamma)}{\varepsilon}\phi_\gamma & \text{in } \Omega \\ \dfrac{\partial\psi^{(3)}_{\varepsilon,\gamma}}{\partial\mathbf{v}} + \gamma\psi^{(3)}_{\varepsilon,\gamma} = i\mathbf{F}\cdot\mathbf{v}\psi^{(2)}_{\varepsilon,\gamma} + i\dfrac{\delta_\varepsilon - \delta_0}{\varepsilon}\mathbf{F}\cdot\mathbf{v}\phi_\gamma \\ \quad - \dfrac{\beta_\varepsilon - 1}{\varepsilon}\mathbf{F}\cdot\mathbf{v}v_{1,\gamma} & \text{on } \partial\Omega. \end{cases} \tag{3.11}$$

Here we claim

$$\gamma_\varepsilon = \frac{\delta_\varepsilon - \delta_0}{\varepsilon} \text{ is bounded.} \tag{3.12}$$

In fact, if (3.12) does not hold, passing to a subsequence, we may assume that $\gamma_\varepsilon \to \infty$ as $\varepsilon \to 0$. If we put $\widetilde{\psi}^{(3)}_{\varepsilon,\gamma} = \psi^{(3)}_{\varepsilon,\gamma}/\gamma_\varepsilon$, $\widetilde{\psi}^{(3)}_{\varepsilon,\gamma}$ satisfies the following equation

$$\begin{cases} -\Delta\widetilde{\psi}^{(3)}_{\varepsilon,\gamma} - \mu_0(\gamma)\widetilde{\psi}^{(3)}_{\varepsilon,\gamma} = -\dfrac{2i}{\gamma_\varepsilon}\mathbf{F}\cdot\nabla\psi^{(2)}_{\varepsilon,\gamma} \\ \quad - \dfrac{\varepsilon}{\gamma_\varepsilon}(|\mathbf{F}|^2 - \lambda(\varepsilon,\gamma))\psi^{(2)}_{\varepsilon,\gamma} - \dfrac{2(\beta_\varepsilon - 1)}{\varepsilon\gamma_\varepsilon}\mathbf{F}\cdot\nabla v_{1,\gamma} \\ \quad - \dfrac{i\beta_\varepsilon}{\gamma_\varepsilon}(|\mathbf{F}|^2 - \lambda(\varepsilon,\gamma))v_{1,\gamma} - 2i\mathbf{F}\cdot\nabla\phi_\gamma \\ \quad - \dfrac{\alpha_\varepsilon - 1}{\varepsilon\gamma_\varepsilon}|\mathbf{F}|^2\phi_\gamma + \dfrac{\alpha_\varepsilon - 1}{\varepsilon\gamma_\varepsilon}\lambda(\varepsilon,\gamma)\phi_\gamma + \dfrac{\lambda(\varepsilon,\gamma) - \mu_0(\gamma)}{\varepsilon\gamma_\varepsilon}\phi_\gamma & \text{in } \Omega \\ \dfrac{\partial\widetilde{\psi}^{(3)}_{\varepsilon,\gamma}}{\partial\mathbf{v}} + \gamma\widetilde{\psi}^{(3)}_{\varepsilon,\gamma} = \dfrac{i}{\gamma_\varepsilon}\mathbf{F}\cdot\mathbf{v}\psi^{(2)}_{\varepsilon,\gamma} + i\mathbf{F}\cdot\mathbf{v}\phi_\gamma \\ \quad - \dfrac{\beta_\varepsilon - 1}{\varepsilon\gamma_\varepsilon}\mathbf{F}\cdot\mathbf{v}v_{1,\gamma} & \text{on } \partial\Omega. \end{cases} \tag{3.13}$$

By the elliptic estimate as above, $\|\widetilde{\psi}^{(3)}_{\varepsilon,\gamma}\|_{C^{2+\alpha}(\overline{\Omega})} \le C(\alpha,\gamma)$. Thus we may assume that $\widetilde{\psi}^{(3)}_{\varepsilon,\gamma} \to \psi^{(3)}_\gamma$ in $C^{2+\alpha}(\overline{\Omega})$. Letting $\varepsilon \to 0$ in (3.13), we get

$$\begin{cases} -\Delta\psi_\gamma^{(3)} - \mu_0(\gamma)\psi_\gamma^{(3)} = -2i\mathbf{F}\cdot\nabla\phi_\gamma & \text{in } \Omega \\ \dfrac{\partial\psi_\gamma^{(3)}}{\partial\mathbf{v}} + \gamma\psi_\gamma^{(3)} = i\mathbf{F}\cdot\mathbf{v}\phi_\gamma & \text{on } \partial\Omega. \end{cases}$$

On the other hand, since $\int_\Omega \psi_\gamma^{(3)}\phi_\gamma dx = 0$ and $\int_\Omega \psi_\gamma^{(3)} v_{1,\gamma} dx = 0$, this leads to a contradiction.

Thus $(\delta_\varepsilon - \delta_0)/\varepsilon$ is bounded. If we return to (3.11), then by the similar arguments, we see that $\| \psi_{\varepsilon,\gamma}^{(3)} \|_{C^{2+\alpha}(\overline{\Omega})} \le C(\alpha, \gamma)$. Thus we see that $\psi_{\varepsilon,\gamma}^{(2)} = \psi_\gamma^2 + \varepsilon\psi_{\varepsilon,\gamma}^{(3)}$ and $\psi_{\varepsilon,\gamma}^{(3)} \to \psi_\gamma^{(3)}$ in $C^{2+\alpha}(\overline{\Omega})$ as $\varepsilon \to 0$. This completes the proof of Theorem 2.1.

References

[1] S. Agmon, A. Douglis and L. Nirenberg, Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions, I, Comm. Pure Appl. Math. 12 (1959), 623-727.

[2] N. Ando and J. Aramaki, A remark on the eigenvalue asymptotics associated with superconductivity near critical temperature, Int. J. Pure Appl. Math. 40(1) (2007), 123-134.

[3] J. Aramaki, Semiclassical asymptotics of the ground state energy for the Neumann problem associated with superconductivity, Int. J. Differ. Equ. Appl. 9(3) (2004), 239-271.

[4] J. Aramaki, Upper critical field and location of surface nucleation for the Ginzburg-Landau system in non-constant applied field, Far East J. Math. Sci. (FJMS) 23(1) (2006), 89-125.

[5] J. Aramaki, Asymptotics of the eigenvalues for the Neumann Laplacian with non-constant magnetic field associated with superconductivity, Far East J. Math. Sci. (FJMS) 25(3) (2007), 529-584.

[6] S. J. Chapman, S. D. Howison and J. R. Ockendon, Macroscopic models for superconductivity, SIAM Reviews 34 (1992), 529-560.

[7] Y. Du, Order Structure and Topological Method in Nonlinear Partial Differential Equations, Word Scientific, New Jersey, London, Singapore, Beijing, Shanghai, Hongkong, Taipei, Chennai, 2006.

[8] Y. Du and X.-B. Pan, Multiple states and hysteresis for type I superconductors, J. Math. Phys. 46 (2005), 073301-1-34.

[9] Q. Du, M. Gunzburger and J. Peterson, Analysis and approximation of the Ginzburg-Landau model of superconductivity, SIAM Reviews 34 (1992), 45-81.

[10] S. Fournais and B. Helffer, Accurate eigenvalues asymptotics for the magnetic Neumann Laplacian, Ann. Inst. Fourier, Grenoble 564(1) (2006), 1-67.

[11] D. Gilbarg and N. S. Trudinger, Elliptic Partial Differential Equations of Second Order, Springer, New York, 1983.

[12] M. Gunzburger and J. Ockendon, Mathematical models in superconductivity due to strong fields for the Ginzburg-Landau model, SIAM J. Math. Anal. 30 (1999), 341-359.

[13] B. Helffer, Bouteilles magnétiques et supraconductivité, (d'après Helffer-Morame, Lu and Pan et Helffer-Pan), Séminaire EDP de l'école Polytechnique 2001-2002.

[14] B. Helffer and A. Mohamed, Semiclassical analysis for the ground state energy of a Schrödinger operator with magnetic wells, J. Funct. Anal. 138 (1996), 40-81.

[15] B. Helffer and A. Morame, Magnetic bottles in connection with superconductivity, J. Funct. Anal. 185 (2001), 604-680.

[16] B. Helffer and A. Morame, Magnetic bottles for the Neumann problem: curvature effects in the case of dimension 3 (general case), Ann. Sci. École Norm. Sup. (4) 37 (2004), 105-170.

[17] B. Helffer and X.-B. Pan, Upper critical field and location of surface nucleation of superconductivity, Ann. Inst. H. Poincaré Anal. Non Linéaire 20(1) (2003), 145-181.

[18] K. Lu and X.-B. Pan, Estimates of upper critical field for the Ginzburg-Landau equations of superconductivity, Physica D 127 (1999), 73-104.

[19] K. Lu and X.-B. Pan, Eigenvalue problems of Ginzburg-Landau operator in bounded domains, J. Math. Physics 40 (1999), 2647-2670.

[20] K. Lu and X.-B. Pan, Surface nucleation of superconductivity in 3-dimension, J. Differential Equations 168 (2000), 386-452.

[21] X.-B. Pan, Superconductivity near the critical temperature, J. Math. Physics 44(6) (2003), 2639-2678.