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ON SAMPLING PERIODIC FUNCTIONS

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Abstract

If the periodic function $f(x) = \sin x$ is sampled at integer points, then the sequence $(\sin n)_n$ obtained in this way satisfies a linear homogeneous recurrence relation of degree 2, with constant coefficients. In this note we use Kronecker's uniform distribution theorem to prove the following result. Let $f : \mathbb{R} \to \mathbb{R}$ be function with period $p > 0$ and let $\alpha/p \notin \mathbb{Q}$. Assume that f has a non-removable discontinuity x_0 and that f is continuous at $x_0 + t\alpha$ for all $t \in \mathbb{Z}, t \neq 0$. Then the sequence $a_n = f(n\alpha)$ does not satisfy any recurrence of the form $a_n = \phi(a_{n-1}, \ldots, a_{n-k})$, where $k \geq 1$ and $\phi : \mathbb{R}^k \to \mathbb{R}$ is continuous.

1. Introduction

The starting point of our observation is the following simple observation. If we sample the periodic function $f(x) = \sin x$ at integer points, then we get the sequence $a_n = \sin n$ which satisfies the following simple linear homogeneous recurrence relation of degree 2, with constant

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coefficients:

$$\sin n = 2\cos 1 \sin(n-1) - \sin(n-2).$$

Similarly, if we sample the cosine function, then we get

$$\cos n = 2\cos 1 \cos(n-1) - \cos(n-2).$$

It turns out that the situation is different for four other trigonometric sequences, $(\tan n)_n$, $(\cot n)_n$, $(\sec n)_n$ and $(\csc n)_n$, obtained by sampling the corresponding periodic functions at integer points. Not only do they not satisfy any linear homogeneous recurrence relation with constant coefficients, but they do not satisfy *any* kind of recurrence relation of the type

$$a_n = \phi(a_{n-1}, \ldots, a_{n-k}),$$

where $k \geq 1$ and $\phi : \mathbb{R}^k \to \mathbb{R}$ is continuous. This will be a consequence of a more general result, which will be addressed in the present note.

Our main result is the following.

Theorem 1. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a function with period* $p > 0$ *and let* α *be a real number such that* $\alpha/p \notin \mathbb{Q}$. *Assume that for some* $x_0 \in \mathbb{R}$, *the limits*

$$L_- := \lim_{x \to x_0^-} f(x)$$

and

$$L_+ := \lim_{x \to x_0^+} f(x)$$

exist as elements of $[-\infty, \infty]$, *with either one of them being infinite, or both being finite and distinct* (*that is,* x_0 *is a non-removable discontinuity of f of the first kind*). *Also assume that f is continuous at* $x_0 + t\alpha$ *for all* $t \in \mathbb{Z}$, $t \neq 0$.

Then the sequence $a_n = f(n\alpha)$ *does not satisfy any recurrence of the form* $a_n = \phi(a_{n-1}, \ldots, a_{n-k})$, *where* $k \geq 1$ *and* $\phi : \mathbb{R}^k \to \mathbb{R}$ *is continuous.*

2. Proof of the Main Result

The proof of Theorem 1 will be by contradiction. Let us assume that, for some $k \geq 1$ and some continuous function $\phi : \mathbb{R}^k \to \mathbb{R}$,

$$f(n\alpha) = \phi(f(n\alpha - \alpha), f(n\alpha - 2\alpha), \dots, f(n\alpha - k\alpha)) \tag{1}$$

holds for every large enough n. Let us pick a half-open interval $I = [a, a + p)$ containing x_0 as an interior point. For every real number x define, in a natural way, $x \bmod I$ to be the unique element of $\hat{x} \in I$ such that $\hat{x} - x$ is an integer multiple of p. Note that, since $\alpha/p \notin \mathbb{Q}$, according to the uniform distribution theorem of Kronecker [1, p. 390], the sequence $(n\alpha \bmod I)_n$ is uniformly distributed in $\bar{I} = [a, a + p]$. Also, since f has period p, we have $f(x) = f(x \bmod I)$ for all x.

Case 1. Assume at least one side limit of f at x_0 is infinite. For example, say $L_- = \infty$. Taking into account Kronecker's theorem, and avoiding excessive notation, we may assume without loss of generality that the sequence $(n\alpha \bmod I)_n$ approaches x_0 from the left. Then the left hand side of (1) will have ∞ as limit:

$$\lim_{n\to\infty} f(n\alpha) = \lim_{n\to\infty} f(n\alpha \bmod I) = \lim_{x\to x_0^-} f(x) = \infty.$$

On the other hand, the right hand side of (1), due to the continuity of ϕ and the continuity of f at points of the form $x_0 + tp$ with $t \in \mathbb{Z}, t \neq 0$, will have a finite limit:

$$\lim_{n\to\infty} \phi(f(n\alpha - \alpha), f(n\alpha - 2\alpha), \dots, f(n\alpha - k\alpha))$$

$$= \phi(f(x_0 - \alpha), f(x_0 - 2\alpha), \dots, f(x_0 - k\alpha)) \in \mathbb{R}.$$

This is a contradiction.

Case 2. Assume both L_- and L_+ are finite and distinct. We will apply again Kronecker's theorem and will assume, without loss of generality, that the sequence $(n\alpha \bmod I)_n$ approaches x_0 from the left. Then the limit, when $n \to \infty$ of the left hand side of (1), will be L_-, while the limit of the right hand side will be $\phi(f(x_0 - \alpha), f(x_0 - 2\alpha), \ldots, f(x_0 - k\alpha))$. Thus

$$L_- = \phi(f(x_0 - \alpha), f(x_0 - 2\alpha), \ldots, f(x_0 - k\alpha)). \tag{2}$$

A similar argument, constructed for the limit from the right, will give

$$L_+ = \phi(f(x_0 - \alpha), f(x_0 - 2\alpha), \ldots, f(x_0 - k\alpha)). \tag{3}$$

From (2) and (3) we get $L_- = L_+$, again a contradiction. This concludes the proof of Theorem 1.

3. Further Comments

Note 2. The non-removability assumption cannot be discarded from the statement of Theorem 1. To see this, take for example $f(x)$ to be the characteristic function of $\mathbb{Z}$ viewed as a subset of $\mathbb{R}$: $f(x) = 1$ if $x \in \mathbb{Z}$ and zero otherwise. Then f has only removable discontinuities, is periodic of period $p = 1$, and for every irrational number α the sequence $a_n = f(n\alpha)$ is constant $(a_n = 0)$, so with $k = 1$ and $\phi : \mathbb{R} \to \mathbb{R}$, $\phi(x) = x$, we have $a_n = \phi(a_{n-1})$ for all n.

Corollary 3. *Let $f(x) = \{x\}$ (the fractional part of x). Then f is periodic with period $p = 1$ and $x_0 = 0$ is a non-removable first-kind discontinuity for f. Then, applying Theorem* 1, *we find out that for any irrational number α there is no $k \geq 1$ and no continuous function $\phi : \mathbb{R}^k \to \mathbb{R}$ such that the recurrence relation*

$$\{n\alpha\} = \phi(\{(n-1)\alpha\}, \{(n-2)\alpha\}, \ldots, \{(n-k)\alpha\})$$

holds.

Corollary 4. *Let $f(x) = \tan x$ if $x \notin \frac{\pi}{2} + \pi\mathbb{Z}$, and $f(x) = c$ (a fixed arbitrary constant) otherwise. Then f is periodic with period $p = \pi$ and $x_0 = \frac{\pi}{2}$ is a non-removable first-kind discontinuity for f. Then, applying Theorem* 1, *we find out that if α is an irrational multiple of π, there is no*

$k \geq 1$ and no continuous function $\phi : \mathbb{R}^k \to \mathbb{R}$ such that the recurrence relation

$$\tan(n\alpha) = \phi(\tan((n-1)\alpha), \tan((n-2)\alpha), \ldots, \tan((n-k)\alpha))$$

holds. Similar statements can be derived for the trigonometric sequences $(\cot n)_n$, $(\sec n)_n$ and $(\csc n)_n$.

Reference

[1] G. H. Hardy and E. M. Wright. An Introduction to the Theory of Numbers, Fourth Edition, Clarendon Press, Oxford, 1960.