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POPULATION MODELS WITH INDEFINITE WEIGHT AND CONSTANT YIELD HARVESTING

G. A. AFROUZI, S. H. RASOULI

and

R. SEDAGHAT

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Abstract

In this paper we study the existence of positive solution for the following reaction-diffusion equation

$$\begin{cases} -\Delta u = am(x)u - u^2 - ch(x), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases}$$

where a and c are positive constants, Ω is a smooth bounded domain in $R^N (N \geq 3)$ with $\partial\Omega$ of class C^2 and connected. The weight m satisfies $m \in C(\Omega)$ and $m(x) \geq m_0 > 0$ for $x \in \Omega$, also $\| m \|_\infty = l < \infty$ and $h : \overline{\Omega} \to R$ is a $C^\alpha(\overline{\Omega})$ function satisfying $h(x) \geq 0$ for $x \in \Omega$, $h(x) \not\equiv 0$, $\max h(x) = 1$ for $x \in \overline{\Omega}$ and $h(x) = 0$ for $x \in \partial\Omega$. We prove the existence of the positive solution under certain conditions.

1. Introduction

We consider the boundary value problem

$$\begin{cases} -\Delta u \equiv f(x, u) = am(x)u - u^2 - ch(x), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{1}$$

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where a and c are positive constants, Ω is a smooth bounded domain in $R^N (N \geq 3)$ with $\partial\Omega$ of class C^2 and connected. The weight m satisfies $m \in C(\Omega)$ and $m(x) \geq m_0 > 0$ for $x \in \Omega$, also $\| m \|_\infty = l < \infty$ and $h : \overline{\Omega} \to R$ is a $C^\alpha(\overline{\Omega})$ function satisfying $h(x) \geq 0$ for $x \in \Omega$, $h(x) \not\equiv 0$, $\max h(x) = 1$ for $x \in \overline{\Omega}$ and $h(x) = 0$ for $x \in \partial\Omega$. We denote by λ_k the k-th eigenvalue of

$$\begin{cases} -\Delta\phi + \lambda m(x)\phi = 0, & x \in \Omega, \\ \phi(x) = 0, & x \in \partial\Omega. \end{cases} \tag{2}$$

In particular, $\lambda_1 > 0$ is the principal eigenvalue with a positive eigenfunction ϕ_1 satisfying $\| \phi_1 \| = 1$ (see [2]).

Equation (1) arises in the study of population biology of one species with u representing the concentration of the species, $am(x)u - u^2$ represents the logistic growth and $ch(x)$ represents the rate of harvesting (see [6]). In [5] the author studied (1) when $c = 0$ (non-harvesting case) and without the weight function. However, the case $c > 0$ is a semipositone problem $(f(x, 0) < 0)$ and studying positive solutions in this case is significantly harder. More work on the diffusive logistic equation can be found in [1] and [3].

2. Preliminaries

We begin this section with some results on the dependence of solution on the parameter $a > 0$. First, we prove some nonexistence results:

Proposition 2.1. (i) *If* $a \leq \lambda_1$, *then* (1) *has no positive solution.*

(ii) *If* $a > \lambda_1$ *and*

$$c > \frac{al(a - \lambda_1)\int_\Omega m(x)\phi_1}{\int_\Omega h(x)\phi_1},$$

then (1) *has no positive solution.*

First we have following lemma [8].

Lemma 2.2. *Suppose that* $f : \Omega \times R^{+} \to R$ *is a continuous function such that* $f(x, s)/s$ *is strictly decreasing for* $s > 0$ *at each* $x \in \Omega$.

Let $w, v \in C(\overline{\Omega}) \cap C^2(\Omega)$ *satisfy*:

(a) $\Delta w + f(x, w) \leq 0 \leq \Delta v + f(x, v)$ *on* Ω,

(b) $w, v > 0$ *on* Ω *and* $w \geq v$ *on* $\partial\Omega$,

(c) $\Delta v \in L^1(\Omega)$.

Then $w \geq v$ *in* $\overline{\Omega}$.

Proof of Proposition 2.1. (i) Suppose otherwise, i.e., assume that there exists a positive solution u of (1). We calculate $((1)\phi_1 + (2)u)$ and integrate over Ω which yields

$$\int_\Omega (-\Delta u)\phi_1 dx + \int_\Omega (-\Delta \phi_1) u dx$$

$$= \int_\Omega (a - \lambda_1) m(x) u \phi_1 dx - \int_\Omega u^2 \phi_1 dx - c \int_\Omega h \phi_1 dx. \tag{3}$$

But by Green's identity we have

$$\int_\Omega (-\Delta u)\phi_1 dx + \int_\Omega (-\Delta \phi_1) u dx$$

$$= \int_\Omega \nabla u \cdot \nabla \phi_1 dx - \int_\Omega \nabla u \cdot \nabla \phi_1 dx = 0. \tag{4}$$

By using (4) in (3) we get

$$(a - \lambda_1) \int_\Omega m(x) u \phi_1 dx = \int_\Omega u^2 \phi_1 dx + c \int_\Omega h \phi_1 dx \geq 0. \tag{5}$$

Since $u \geq 0$, $m(x) \geq m_0 > 0$ and $\phi_1 > 0$, this requires $a \geq \lambda_1$, which is a contradiction.

(ii) From above lemma we have $u(x) \le al$ for any positive solution u. Hence from (5), we obtain

$$c\int_{\Omega} h\phi_1 dx \le (a-\lambda_1)\int_{\Omega} m(x)u\phi_1 dx \le al(a-\lambda_1)\int_{\Omega} m(x)\phi_1 dx, \qquad (6)$$

a contradiction.

So $a > \lambda_1$ is a necessary condition for the existence of positive solutions.

3. Existence of Solutions

In this section we prove the existence of solutions by comparison method. It is easy to see that any subsolution of

$$-\Delta u = am_0 u - u^2 - ch(x), \quad x \in \Omega, \qquad (7)$$

$$u(x) = 0, \quad x \in \partial\Omega, \qquad (8)$$

is a subsolution of (1), also any supersolution of

$$-\Delta u = alu - u^2 - ch(x), \quad x \in \Omega, \qquad (9)$$

$$u(x) = 0, \quad x \in \partial\Omega, \qquad (10)$$

is a supersolution of (1), where l is as defined before.

We denote by λ'_k, the k-th eigenvalue of

$$\begin{cases} \Delta\phi + \lambda'\phi = 0, & x \in \Omega, \\ \phi(x) = 0, & x \in \partial\Omega, \end{cases} \qquad (11)$$

with positive eigenfunction ϕ'_1 satisfying $\| \phi'_1 \| = 1$. Our main result is the following theorem.

Theorem 3.1. *Suppose that* $a > \lambda'_1/m_0$, *then there exists* $c_0 = c_0(a, m_0)$ *such that for* $0 < c < c_0$, (1) *has a positive solution u. Further, this solution u is such that*

$$u(x) \ge \frac{ch(x)}{\lambda'_1}.$$

Proof. We use the method of subsolution and supersolution. We recall the anti-maximum principle of Clement and Peletier (see [4]) in the following form: let λ_1' be as defined above. Then there exists a $\delta(\Omega) > 0$ such that the solution $z_{\lambda'}$ of

$$\Delta z + \lambda' z = 1, \quad x \in \Omega, \tag{12}$$

$$z = 0, \quad x \in \partial\Omega, \tag{13}$$

for $\lambda' \in (\lambda_1', \lambda_1' + \delta)$, is positive for $x \in \Omega$ and is such that $\frac{\partial z_{\lambda'}}{\partial n} < 0$ for $x \in \partial\Omega$. We construct the subsolution ψ of (9-10) using $z_{\lambda'}$ such that $\lambda_1'\psi \geq ch(x)$.

Fix $\lambda_*' \in (\lambda_1', \min\{a, \lambda_1' + \delta\})$. Let

$$\alpha = \| z_{\lambda_*'} \|_\infty,$$

$$K_0 = \inf\{K : \lambda_1' K z_{\lambda_*'} \geq h(x)\},$$

$$K_1 = \max\{1, K_0\}.$$

Note that $K_0 > 0$ exists, since $z_{\lambda_*'}(x)$ is positive for $x \in \Omega$ and is such that $\frac{\partial z_{\lambda'}}{\partial n} < 0$ for $x \in \partial\Omega$. Define $\psi(x) = Kcz_{\lambda_*'}$, where $K > 0$ is to be determined later. We will choose $K > 0$ and $c > 0$ properly so that ψ is a subsolution. First we require that $K \geq K_1$, then $\lambda_1'\psi \geq ch(x)$. We have

$$\begin{aligned}
&\Delta\psi + am_0\psi - (\psi)^2 - ch(x) \\
&= -cK(\lambda_*' z_{\lambda_*'} - 1) + acm_0 K z_{\lambda_*'} - (Kcz_{\lambda_*'})^2 - ch(x) \\
&\geq -cK(\lambda_*' z_{\lambda_*'} - 1) + acm_0 K z_{\lambda_*'} - (Kcz_{\lambda_*'})^2 - c \\
&= c[-c(Kz_{\lambda_*'})^2 + (am_0 - \lambda_*')(Kz_{\lambda_*'}) + (K - 1)].
\end{aligned}$$

Define

$$H(y) = -cy^2 + (am_0 - \lambda'_*)y + (K - 1).$$

Then $\psi(x)$ is a subsolution if $H(y) \geq 0$ for all $y \in [0, K\alpha]$. Notice that $H(0) = K - 1 \geq 0$, since $K \geq 1$, $H'(x) = (am_0 - \lambda'_*) > 0$, and $H''(0) = -2c < 0$. Hence $H(y) \geq 0$ for all $y \in [0, K\alpha]$ if

$$H(K\alpha) = c(K\alpha)^2 + (am_0 - \lambda'_*)(K\alpha) + (K - 1) \geq 0,$$

which is equivalent to

$$c \leq \frac{(am_0 - \lambda'_*)(K\alpha) + (K - 1)}{(K\alpha)^2}.$$

We define

$$c_0 = \sup_{K \geq K_1} \frac{(am_0 - \lambda'_*)(K\alpha) + (K - 1)}{(K\alpha)^2}.$$

For $c \in (0, c_0)$, there exists $\hat{K} \geq K_1$ such that

$$c \leq \frac{(am_0 - \lambda'_*)(\hat{K}\alpha) + (\hat{K} - 1)}{(\hat{K}\alpha)^2},$$

and hence $\psi(x) = \hat{K}cz_{\lambda'_*}$ turns out to be subsolution. It is easy to see that if any large positive constant C is a supersolution to (11-12), then this C is a supersolution to (1) for fixed a, $c > 0$. Thus from standard result of the sub-sup solution method (see [7]), for $c \in (0, c_0)$, there exists a solution u of (1) such that

$$u(x) \geq \frac{ch(x)}{\lambda_1 m_0}.$$

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Department of Mathematics
Faculty of Basic Sciences
Mazandaran University
Babolsar, Iran
e-mail: afrouzi@umz.ac.ir