Far East Journal of Mathematical Sciences (FJMS)
Volume 29, Issue 1, 2008, Pages 21-34
Published online: June 10, 2008
This paper is available online at http://www.pphmj.com

THE DOUBLE ORLICZ SEQUENCE SPACES $\Gamma_M^2(p)$ AND $\wedge_M^2(p)$

N. SUBRAMANIAN and C. MURUGESAN

Department of Mathematics
SASTRA University
Tanjore-613 402, India
e-mail: nsmaths@yahoo.com

Department of Mathematics
Sathyabama University
Chennai-600 119, India
e-mail: murugaa23@sify.com

Abstract

In this paper we introduce and examine some properties of the double Orlicz sequence spaces $\Gamma_M^2(p)$ and $\wedge_M^2\ (p)$.

1. Introduction

Throughout w, Γ and $\wedge$ denote the classes of all, entire and analytic scalar valued single sequences, respectively.

We write w^2 for the set of all complex sequences (x_{mn}), where $m, n \in \mathbb{N}$ is the set of positive integers. Then w^2 is a linear space under the coordinatewise addition and scalar multiplication.

2000 Mathematics Subject Classification: 46A45, 40C05, 40D05.

Keywords and phrases: entire sequence, analytic sequence, double sequence, duals, paranorm.

Submitted by Animikh Biswas

Received July 6, 2007; Revised December 28, 2007

Some initial works on double sequence spaces are found in Bromwich [4]. Later on they were investigated by Hardy [8], Moricz [12], Moricz and Rhoades [13], Basarir and Sonalcan [2], Tripathy [20], Colak and Turkmenoglu [6], Turkmenoglu [22] and many others.

We need the following inequality in the sequel of the paper.

For $a, b \geq 0$ and $0 < p < 1$, we have

$$(a+b)^p \leq a^p + b^p. \tag{a}$$

The double series $\sum_{m,n=1}^{\infty} x_{mn}$ is called *convergent* if and only if the double sequence (S_{mn}) is called *convergent*, where

$$S_{mn} = \sum_{i,j=1}^{m,n} x_{ij} \quad (m, n = 1, 2, 3, \ldots)$$

(see [1]).

A sequence $x = (x_{mn})$ is said to be *double analytic* if $\sup_{mn} |x_{mn}|^{1/m+n} < \infty$. The vector space of all double analytic sequences will be denoted by $\wedge^2$. A sequence $x = (x_{mn})$ is called *double entire sequence* if $|x_{mn}|^{1/m+n} \to 0$ as $m, n \to \infty$. The double entire sequences will be denoted by Γ^2. Let $\Phi = \{\text{all finite sequences}\}$.

Consider a double sequence $x = (x_{ij})$. The (m, n)th section $x^{[m,n]}$ of the sequence is defined by $x^{[m,n]} = \sum_{i,j=0}^{m,n} x_{ij}\delta_{ij}$ for all $m, n \in \mathbb{N}$,

$$\delta_{mn} = \begin{pmatrix} 0, 0, \ldots, 0, 0, \ldots \\ 0, 0, \ldots, 0, 0, \ldots \\ . \\ . \\ 0, 0, \ldots, 1, 0, \ldots \\ 0, 0, \ldots, 0, 0, \ldots \end{pmatrix}$$

with 1 in the (m, n)th position and zero otherwise. An FK-space (or a metric space) X is said to have *AK property* if (δ_{mn}) is a Schauder basis for X or equivalently $x^{[m,n]} \to x$.

An FDK-space is a double sequence space endowed with a complete metrizable; locally convex topology under which the coordinate mappings $x = (x_k) \to (x_{mn})$ $(m, n \in \mathbb{N})$ are also continuous.

A double sequence $x = (x_{mn})$ is called a *Cauchy sequence* if and only if for every $\varepsilon > 0$ there exists a positive integer $n_0 = n_0(\varepsilon)$ such that $|x_{mn} - x_{pq}| < \varepsilon$, for all $m, n, p, q > n_0$.

It is known that a double sequence (x_{mn}) is a Cauchy sequence if and only if it is convergent [5].

Orlicz [16] used the idea of Orlicz function to construct the space (L^M). Lindenstrauss and Tzafriri [10] investigated Orlicz sequence spaces in more detail, and they proved that every Orlicz sequence space ℓ_M contains a subspace isomorphic to ℓ_p $(1 \leq p < \infty)$. Subsequently, different classes of sequence spaces were defined by Parashar and Choudhary [17], Mursaleen et al. [14], Bektas and Altin [3], Tripathy et al. [21], Rao and Subramanian [18], and many others. The Orlicz sequence spaces are the special cases of Orlicz spaces studied in [9].

Recall [16] and [9], an Orlicz function is a function $M : [0, \infty) \to [0, \infty)$ which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$, for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. If convexity of Orlicz function M is replaced by $M(x + y) \leq M(x) + M(y)$, then this function is called *modulus function*, defined by Nakano [15] and further discussed by Ruckle [19] and Maddox [11], and many others.

An Orlicz function M is said to satisfy the Δ_2-condition for all values of u if there exists a constant $K > 0$ such that $M(2u) \leq KM(u)$ $(u \geq 0)$. The Δ_2-condition is equivalent to $M(\ell u) \leq K\ell M(u)$, for all values of u and for $\ell > 1$.

Let (Ω, Σ, μ) be a finite measure space. We denote by $E(\mu)$ the space of all (equivalence classes of) Σ-measurable functions x from Ω into

$[0, \infty)$. Given an Orlicz function M, we define on $E(\mu)$ a convex functional I_M by

$$I_M(x) = \int_\Omega M(x(t))d\mu,$$

and an Orlicz space $L^M(\mu)$ by $L^M(\mu) = \{x \in E(\mu) : I_M(\lambda x) < +\infty$ for some $\lambda > 0\}$, (for detail see [9, 16]).

Lindenstrauss and Tzafriri [10] used the idea of Orlicz function to construct Orlicz sequence space

$$\ell_M = \left\{x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0\right\},$$

where $w = \{$all complex sequences$\}$.

The space ℓ_M with the norm

$$\| x \| = \inf\left\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \le 1\right\},$$

becomes a Banach space which is called an *Orlicz sequence space*. For $M(t) = t^p$ $(1 \le p < \infty)$, the space ℓ_M coincides with the classical sequence space ℓ_p.

If X is a sequence space, we give the following definitions:

(i) $X' =$ the continuous dual of X;

(ii) $X^\alpha = \left\{a = (a_{mn}) : \sum_{m,n=1}^{\infty} |a_{mn}x_{mn}| < \infty, \text{ for each } x \in X\right\}$;

(iii) $X^\beta = \left\{a = (a_{mn}) : \sum_{m,n=1}^{\infty} a_{mn}x_{mn} \text{ is convergent, for each } x \in X\right\}$;

(iv) $X^\gamma = \left\{a = (a_{mn}) : \sup_{m,n\ge 1}\left|\sum_{m,n=1}^{M,N} a_{mn}x_{mn}\right| < \infty, \text{ for each } x \in X\right\}$;

(v) let X be an FK-space $\supset \Phi$, then $X^f = \{f(\delta_{mn}) : f \in X'\}$;

(vi) $X^{\wedge} = \{a = (a_{mn}) : \sup_{(mn)} | a_{mn}x_{mn} |^{1/m+n} < \infty, \text{ for each } x \in X\}$;

X^{α}, X^{β}, X^{γ}, $X^{\wedge}$ are called α- (or *Köthe-Toeplitz*) *dual of* X, β- (or *generalized-Köthe-Toeplitz*) *dual of* X, γ-*dual of* X and $\wedge$-*dual of* X, respectively. X^{α} is defined by Gupta and Kamptan [7]. It is clear that $X^{\alpha} \subset X^{\beta}$ and $X^{\alpha} \subset X^{\gamma}$, but $X^{\beta} \subset X^{\gamma}$ does not hold, since the sequence of partial sums of a double convergent series need not to be bounded.

2. Definitions and Preliminaries

Throughout the article w^2 denotes the spaces of all sequences. $\Gamma_M^2(p)$ and $\wedge_M^2(p)$ denote the Pringscheims sense of double Orlicz space of entire sequences and Pringscheims sense of double Orlicz space of bounded sequences, respectively.

Let w^2 denote the set of all complex double sequences $x = (x_{mn})_{m,n=1}^{\infty}$ and $M : [0, \infty) \to [0, \infty)$ be an Orlicz function, or a modulus function.

Given a double sequence, $x \in w^2$. If $p = (p_{mn})$ is a double sequence of strictly positive real numbers p_{mn}, then we write

$$\Gamma_M^2(p) = \left\{ x \in w^2 : \left(M\left(\frac{| x_{mn} |^{1/m+n}}{\rho} \right)^{p_{mn}} \right) \to 0 \text{ as } m, n \to \infty \text{ for some } \rho > 0 \right\}$$

and

$$\wedge_M^2(p) = \left\{ x \in w^2 : \sup_{m,n \geq 1} \left(M\left(\frac{| x_{mn} |^{1/m+n}}{\rho} \right)^{p_{mn}} \right) < \infty \text{ for some } \rho > 0 \right\}.$$

The space $\wedge_M^2(p)$ is a metric space with the metric

$$\widetilde{d}(x, y) = \inf\left\{ \rho > 0 : \sup_{m,n \geq 1} \left(M\left(\frac{| x_{mn} - y_{mn} |^{1/m+n}}{\rho} \right)^{p_{mn}} \right) \leq 1 \right\}$$

and the space $\Gamma_M^2(p)$ is a metric space with the metric

$$d(x,\ y) = \max\left\{\rho > 0 : \sup_{(m,n)}\left(M\left(\frac{|x_{mn} - y_{mn}|^{1/m+n}}{\rho}\right)^{p_{mn}}\right) : m, n = 1, 2, 3, \ldots\right\}.$$

Throughout the paper we write $\inf p_{mn}$, $\sup p_{mn}$ and $\sum_{mn}$ instead of $\inf\limits_{m,n\geq 1}$, $\sup\limits_{m,n\geq 1}$ and $\sum_{m,n=1}^{\infty}$, respectively.

3. Main Results

Theorem 1. *For every* $p = (p_{mn})$, $[\wedge_M^2(p)]^{\beta} = [\wedge_M^2(p)]^{\alpha} = [\wedge_M^2(p)]^{\gamma}$ $= \eta_M^2(p)$, *where*

$$\eta_M^2(p) = \bigcap_{N \in N-\{1\}}\left\{x = (x_{mn}) : \sum\nolimits_{m,n}\left(M\left(\frac{|x_{mn}|N^{m+n/p_{mn}}}{\rho}\right)\right) < \infty\right\}.$$

Proof. (a) To prove that $[\wedge_M^2(p)]^{\beta} = \eta_M^2(p)$. (1.1)

First we show that $\eta_M^2(p) \subset [\wedge_M^2(p)]^{\beta}$.

Let $x \in \eta_M^2(p)$ and $y \in \wedge_M^2(p)$. Then we can find a positive integer N such that $(|y_{mn}|^{1/m+n})^{p_{mn}} < \max(1, \sup\limits_{m,n\geq 1}(|y_{mn}|^{1/m+n})^{p_{mn}}) < N$, for all m, n.

Hence we may write

$$\left|\sum_{m,n} x_{mn}y_{mn}\right| \leq \sum_{m,n}|x_{mn}y_{mn}| \leq \sum_{m,n}\left(M\left(\frac{|x_{mn}y_{mn}|}{\rho}\right)\right)$$
$$\leq \sum_{m,n}\left(M\left(\frac{|x_{mn}|N^{m+n/p_{mn}}}{\rho}\right)\right).$$

Since $x \in \eta_M^2(p)$, the series on the right side of the above inequality is convergent, whence $x \in [\wedge_M^2(p)]^{\beta}$. Hence $\eta_M^2(p) \subset [\wedge_M^2(p)]^{\beta}$.

Now we show that $[\wedge_M^2(p)]^{\beta} \subset \eta_M^2(p)$.

For this, let $x \in [\wedge_M^2(p)]^\beta$, and suppose that $x \notin \wedge_M^2(p)$. Then there exists a positive integer $N > 1$ such that $\sum_{m,n} \left(M\left(\frac{|x_{mn}| N^{m+n/p_{mn}}}{\rho} \right) \right) = \infty.$

If we define $y_{mn} = N^{m+n/p_{mn}} Sgn\, x_{mn}$, $m, n = 1, 2, \ldots$, then $y \in \wedge_M^2(p)$. But, since

$$\left| \sum_{m,n} x_{mn} y_{mn} \right| = \sum_{m,n} \left(M\left(\frac{|x_{mn} y_{mn}|}{\rho} \right) \right) = \sum_{m,n} \left(M\left(\frac{|x_{mn}| N^{m+n/p_{mn}}}{\rho} \right) \right) = \infty,$$

we get $x \notin [\wedge_M^2(p)]^\beta$, which contradicts to the assumption $x \in [\wedge_M^2(p)]^\beta$.

Therefore $x \in \eta_M^2(p)$. Therefore $[\wedge_M^2(p)]^\beta = \eta_M^2(p)$.

(b) $[\wedge_M^2(p)]^\alpha = \eta_M^2(p)$ and (c) $[\wedge_M^2(p)]^\gamma = \eta_M^2(p)$ can be shown in a similar way of (1.1). Therefore we omit it.

Theorem 2. *Let* $p = (p_{mn})$ *be an analytic double sequence of strictly positive real numbers* p_{mn}. *Then*

(i) $\wedge_M^2(p)$ *is a paranormed space with*

$$g(x) = \sup_{m,n \geq 1} \left(M\left(\frac{|x_{mn}|^{1/m+n}}{\rho} \right)^{p_{mn}/M} \right)$$

if and only if $h = \inf p_{mn} > 0$, *where* $M = \max(1, H)$ *and* $H = \sup p_{mn}$.

(ii) $\wedge_M^2(p)$ *is a complete paranormed linear metric space if the condition p in* (i) *is satisfied.*

Proof. (i) Sufficiency. Let $h > 0$. It is trivial that $g(\theta) = 0$ and $g(-x) = g(x)$.

The inequality $g(x + y) \leq g(x) + g(y)$ follows from the inequality (a), since $p_{mn}/M \leq 1$ for all positive integers m, n. We may also write $g(\lambda x) \leq \max(|\lambda|, |\lambda|^{h/M}) g(x)$, since $|\lambda|^{p_{mn}} \leq \max(|\lambda|^h, |\lambda|^M)$ for all positive integers m, n and for any $\lambda \in \mathbb{C}$, the set of complex numbers. Using this inequality, it can be proved that $\lambda x \to \theta$, when x is fixed and $\lambda \to 0$, or $\lambda \to 0$ and $x \to \theta$, or λ is fixed and $x \to \theta$.

Necessity. Let $\wedge_M^2(p)$ be a paranormed space with the paranorm $g(x) = \sup_{m,n\geq 1}\left(M\left(\frac{|x_{mn}|^{1/m+n}}{\rho}\right)^{p_{mn}/M}\right)$ and suppose that $h = 0$. Since $|\lambda|^{p_{mn}/M} \leq |\lambda|^{h/M} = 1$ for all positive integers m, n and $\lambda \in \mathbb{C}$ such that $0 < |\lambda| \leq 1$, we have $\sup_{m,n\geq 1}\left(M\left(\frac{|\lambda|^{p_{mn}/M}}{\rho}\right)\right) = 1$. Hence it follows that $g(\lambda x) = \sup_{m,n\geq 1}\left(M\left(\frac{|\lambda|^{p_{mn}/M}}{\rho}\right)\right) = 1$ for $x = (a) \in \wedge_M^2(p)$ as $\lambda \to 0$. But this contradicts to the assumption $\wedge_M^2(p)$ is a paranormed space with $g(x)$.

(ii) The proof is clear.

Corollary 1. $\wedge_M^2(p)$ *is a complete paranormed space with the natural paranorm if and only if* $\wedge_M^2(p) = \wedge_M^2$.

Theorem 3. *Let* $N_1 = \min\left\{n_0 : \sup_{m,n\geq n_0}\left(M\left(\frac{|x_{mn}|^{1/m+n}}{\rho}\right)^{p_{mn}}\right) < \infty\right\}$, $N_2 = \min\{n_0 : \sup_{m,n\geq n_0} p_{mn} < \infty\}$ *and* $N = \max(N_1, N_2)$.

(i) $\Gamma_M^2(p)$ *is a paranormed space with*

$$g(x) = \lim_{N\to\infty}\sup_{m,n\geq N}\left(M\left(\frac{|x_{mn}|^{1/m+n}}{\rho}\right)^{p_{mn}/M}\right) \tag{3.1}$$

if and only if $\mu > 0$, *where* $\mu = \lim_{N\to\infty}\inf_{m,n\geq N} p_{mn}$ *and* $M = \max(1, \sup_{m,n\geq N} p_{mn})$.

(ii) $\Gamma_M^2(p)$ *is complete with the paranorm* (3.1).

Proof. (i) Necessity. Let $\Gamma_M^2(p)$ be a paranormed space with (3.1) and suppose that $\mu = 0$.

Then $\alpha = \inf_{m, n \geq N} p_{mn} = 0$ for all $N \in \mathbb{N}$, and hence we obtain $g(\lambda x)$ $= \lim_{N \to \infty} \sup_{m, n \geq N} |\lambda|^{p_{mn}/M} = 1$ for all $\lambda \in (0, 1]$, where $x = (a) \in \Gamma_M^2(p)$. Whence $\lambda \to 0$ does not imply $\lambda x \to \theta$, when x is fixed. But this contradicts to (3.1) to be a paranorm.

Sufficiency. Let $\mu > 0$. It is trivial that $g(\theta) = 0$, $g(-x) = g(x)$ and $g(x + y) \leq g(x) + g(y)$. Since $\mu > 0$ there exists a positive number β such that $p_{mn} > \beta$ for sufficiently large positive integer m, n. Hence for any $\lambda \in \mathbb{C}$, we may write $|\lambda|^{p_{mn}} \leq \max(|\lambda|^M, |\lambda|^\beta)$ for sufficiently large positive integers $m, n \geq N$. Therefore, we obtain that $g(\lambda x) \leq$ $\max(|\lambda|, |\lambda|^{\beta/M}) g(x)$ using this, one can prove that $\lambda x \to \theta$, whenever x is fixed and $\lambda \to 0$, or $\lambda \to 0$ and $x \to \theta$, or λ is fixed and $x \to \theta$.

(ii) Let (x^{kl}) be a Cauchy sequence in $\Gamma_M^2(p)$, where $x^{kl} =$ $(x_{mn}^{kl})_{mn \in N}$.

Then for every $\varepsilon > 0$ $(0 < \varepsilon < 1)$ there exists a positive integer s_0 such that

$$g(x^{kl} - x^{rt}) = \lim_{N \to \infty} \sup_{m, n \geq N} \left(M\left(\frac{|x_{mn}^{kl} - x_{mn}^{rt}|^{1/m+n}}{\rho} \right)^{p_{mn}/M} \right) < \varepsilon/2$$

for all $k, l, r, t > s_0$. (3.2)

By (3.2) there exists a positive integer n_0 such that $\sup_{m, n \geq N} \left(M\left(\frac{|x_{mn}^{kl} - x_{mn}^{rt}|^{1/m+n}}{\rho} \right)^{p_{mn}/M} \right) < \varepsilon/2$ for all $k, l, r, t > s_0$ and for $N > n_0$. Hence we obtain

$$\left(M\left(\frac{|x_{mn}^{kl} - x_{mn}^{rt}|^{1/m+n}}{\rho} \right)^{p_{mn}/M} \right) < \varepsilon/2 < 1 \tag{3.3}$$

so that

$$\left(M\left(\frac{|x_{mn}^{kl}-x_{mn}^{rt}|^{1/m+n}}{\rho}\right)\right)<\left(M\left(\frac{|x_{mn}^{kl}-x_{mn}^{rt}|^{1/m+n}}{\rho}\right)^{p_{mn}/M}\right)<\varepsilon/2 \quad (3.4)$$

for all $k, l, r, t > s_0$ and $m, n > n_0$. This implies that $(x_{mn}^{kl})_{kl\in N}$ is a Cauchy sequence in $\mathbb{C}$ for each fixed $m, n > n_0$. Hence the sequence $(x_{mn}^{kl})_{kl\in N}$ is convergent to x_{mn} say,

$$\lim_{k,l\to\infty} x_{mn}^{kl} = x_{mn} \text{ for each fixed } m, n > n_0. \quad (3.5)$$

Getting x_{mn}, we define $x = (x_{mn})$. From (3.2) we obtain

$$g(x^{kl}-x) = \lim_{N\to\infty}\sup_{m,n\geq N}\left(M\left(\frac{|x_{mn}^{kl}-x_{mn}|^{1/m+n}}{\rho}\right)^{p_{mn}/M}\right)<\varepsilon/2 \quad (3.6)$$

as $r, t\to\infty$, for $k, l > s_0$ by (3.5). This implies that $\lim_{kl\to\infty} x^{kl} = x$.

Now we show that $x = (x_{mn}) \in \Gamma_M^2(p)$. Since $x^{kl}\in\Gamma_M^2(p)$ for each $(k, 1)\in N\times N$, for every $\varepsilon > 0$ $(0 < \varepsilon < 1)$ there exists a positive integer $n_1 \in N$ such that

$$\left(M\left(\frac{|x_{mn}^{kl}|^{1/m+n}}{\rho}\right)^{p_{mn}/M}\right)<\varepsilon/2 \text{ for every } m, n > n_1. \quad (3.7)$$

By (3.6) and (3.7) and (a) we obtain

$$\begin{aligned}\left(M\left(\frac{|x_{mn}|^{1/m+n}}{\rho}\right)^{p_{mn}/M}\right) &\leq \left(M\left(\frac{|x_{mn}^{kl}|^{1/m+n}}{\rho}\right)^{p_{mn}/M}\right)\\ &\quad+\left(M\left(\frac{|x_{mn}^{kl}-x_{mn}|^{1/m+n}}{\rho}\right)^{p_{mn}/M}\right)\\ &\leq \varepsilon/2+\varepsilon/2=\varepsilon\end{aligned}$$

for all $k, l > \max(s_0, s_1)$ and $m, n > \max(n_0, n_1)$. This implies that $x \in \Gamma_M^2(p)$.

This completes the proof.

Theorem 4. *For every* $p = (p_{mn})$, *then* $\eta_M^2(p) \subset [\Gamma_M^2(p)]^\beta \underset{\neq}{\subset} \wedge^2$.

Proof. Case 1. First we show that $\eta_M^2(p) \subset [\Gamma_M^2(p)]^\beta$.

We know that $\Gamma_M^2(p) \subset \wedge_M^2(p)$.

$[\wedge_M^2(p)]^\beta \subset [\Gamma_M^2(p)]^\beta$. But $[\wedge_M^2(p)]^\beta = \eta_M^2(p)$, by Theorem 1.

Therefore

$$\eta_M^2(p) \subset \Gamma_M^2(p). \tag{4.1}$$

Case 2. Now we show that $(\Gamma_M^2(p))^\beta \underset{\neq}{\subset} \wedge^2$.

Let $y = \{y_{mn}\}$ be an arbitrary point in $(\Gamma_M^2(p))^\beta$. If y is not in $\wedge^2$, then for each natural number q, we can find an index $m_q n_q$ such that

$$M\left(\frac{|y_{m_q n_q}|^{1/m_q + n_q}}{\rho}\right)^{p_{mn}} > q, (1, 2, 3, \ldots).$$

Define $x = \{x_{mn}\}$ by $M\left(\frac{x_{mn}}{\rho}\right)^{p_{mn}} = \frac{1}{q^{m+n}}$ for $(m, n) = (m_q, n_q)$ for some $q \in \mathbb{N}$; and $M\left(\frac{x_{mn}}{\rho}\right)^{p_{mn}} = 0$ otherwise.

Then x is in $\Gamma_M^2(p)$, but for infinitely mn,

$$M\left(\frac{|y_{mn}x_{mn}|}{\rho}\right)^{p_{mn}} > 1. \tag{4.2}$$

Consider the sequence $z = \{z_{mn}\}$, where $M\left(\frac{z_{11}}{\rho}\right)^{p_{mn}} = M\left(\frac{x_{11}}{\rho}\right)^{p_{mn}} - s$ with $s = \sum M\left(\frac{x_{mn}}{\rho}\right)^{p_{mn}}$; and $M\left(\frac{z_{mn}}{\rho}\right)^{p_{mn}} = M\left(\frac{x_{mn}}{\rho}\right)^{p_{mn}}$ $(m, n = 1, 2, 3, \ldots)$.

Then z is a point of $\Gamma_M^2(p)$. Also $\sum M\left(\frac{z_{mn}}{\rho}\right)^{p_{mn}} = 0$. Hence z is in $\Gamma_M^2(p)$.

But, by the equation (4.2), $\sum M\left(\frac{z_{mn}y_{mn}}{\rho}\right)^{p_{mn}}$ does not converge.

$\Rightarrow \sum x_{mn}y_{mn}$ diverges.

Thus the sequence y would not be in $(\Gamma_M^2(p))^\beta$. This contradiction proves that

$$(\Gamma_M^2(p))^\beta \subset \wedge^2. \tag{4.3}$$

If we now choose $p = (p_{mn})$ is a constant, $M = id$, where id is the identity and $y_{1n} = x_{1n} = 1$ and $y_{mn} = x_{mn} = 0$ $(m > 1)$ for all n, then obviously $x \in \Gamma_M^2(p)$ and $y \in \wedge^2$, but $\sum_{m,n=1}^{\infty} x_{mn}y_{mn} = \infty$, hence

$$y \notin (\Gamma_M^2(p))^\beta. \tag{4.4}$$

From (4.3) and (4.4) we are granted

$$(\Gamma_M^2(p))^\beta \underset{\neq}{\subset} \wedge^2 . \tag{4.5}$$

Hence (4.1) and (4.5) we are granted $\eta_M^2(p) \subset [\Gamma_M^2(p)]^\beta \underset{\neq}{\subset} \wedge^2$.

This completes the proof.

Theorem 5. *Let M be an Orlicz function or modulus function which satisfies the Δ_2-condition. Then $\Gamma^2(p) \subset \Gamma_M^2(p)$.*

Proof. Let

$$x \in \Gamma^2(p). \tag{5.1}$$

Then $(|x_{mn}|^{1/m+n})^{p_{mn}} \leq \varepsilon$ for sufficiently large m, n and every $\varepsilon > 0$. But then by taking $\rho \geq 1/2$,

$$\left(M\left(\frac{|x_{mn}|^{1/m+n}}{\rho}\right)^{p_{mn}}\right) \leq \left(M\left(\frac{\varepsilon}{\rho}\right)\right) \text{ (because } M \text{ is non-decreasing)}$$

$$\leq (M(2\varepsilon))$$

$$\Rightarrow \left(M\left(\frac{|x_{mn}|^{1/m+n}}{\rho}\right)^{p_{mn}}\right) \leq KM(\varepsilon) \text{(by the } \Delta_2\text{-condition, for some } K > 0)$$

$$< \varepsilon \text{ (by defining } M(\varepsilon) < \varepsilon/K)$$

$$\Rightarrow \left(M\left(\frac{|x_{mn}|^{1/m+n}}{\rho}\right)^{p_{mn}}\right) \to 0 \text{ as } m, n \to \infty. \tag{5.2}$$

Hence

$$x \in \Gamma_M^2(p). \tag{5.3}$$

From (5.1) and (5.3) we get $\Gamma^2(p) \subset \Gamma_M^2(p)$.

This completes the proof.

Acknowledgement

The authors wish to thank the referee for his valuable comments and suggestions.

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