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ON THE ORDER OF CONVERGENCE OF MODIFIED PREDICTOR-CORRECTOR METHOD

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Abstract

An explicit numerical integration method is further developed. The method is based on the relationship that m-step Adams-Moulton method is linear convex combination of the $(m-1)$-step Adams-Moulton and m-step Adams-Bashforth method with a fixed weighting coefficients. A general form is used to evaluate the recurrence expressions using the different number of previous mesh points. The analytical upper bound of modified predictor-corrector method is given in the closed form. The expressed order of convergence is now $m+2$. The explicit numerical algorithm is given for modified 3-step predictor-corrector method. Some numerical examples, for the different kind of problems, first- and second-order nonlinear initial value problems, are used to demonstrate the efficiency and the accuracy of the proposed numerical method. The calculated numerical solutions show superiority of presented modified predictor-corrector method to the standard Adams-Bashforth-Moulton predictor-corrector method.

1. Introduction

According to the great advance in modern computer technology, computational mechanics has been applied for modelling almost all

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physical problems. Numerical methods are used to solve many problems in modern engineering and computational mechanics. Nonlinear differential equations are often used to describe and prepare for solving the most practical engineering problems. Analytical solutions of those equations are usually very complicated, and rarely we can find solutions in the closed form, what directly leads to necessary numerical integration methods. Predictor-corrector method is commonly used multi-step integration method to solve nonlinear ordinary differential equations. Modified predictor-corrector method as convex linear combination of predicted and corrected values calculated with traditional Adams-Bashforth-Moulton predictor-corrector method is given in [1].

In this paper, modified predictor-corrector method is applied on first- and second-order ordinary differential equations and systems of ordinary differential equations. The proposed method gives superior solution with little more computational effort at the each step of calculation what is shown on some numerical examples. Some first results of this investigation are proposed in [3].

2. Predictor-corrector Method

The expressions for traditional Adams-Bashforth-Moulton m-step predictor-corrector method are as follows, for predictor step

$$y^{p}(x_{i+1}) = y(x_i) + h \cdot \sum_{k=0}^{m-1} \left[\nabla^k f(x_i) \cdot (-1)^k \int_0^1 C_k^{-s} ds \right], \tag{1}$$

and for corrector step

$$\begin{aligned} y^{c}(x_{i+1}) = {} & y(x_i) + hf(x_{i+1}, \, y^{p}(x_{i+1}))(-1)^{m-1} \int_0^1 C_{m-1}^{-s} ds \\ & + h \cdot \sum_{k=0}^{m-1} \left\{ \nabla^k f(x_i) \left[(-1)^k \int_0^1 C_k^{-s} ds - (-1)^{m-1} \int_0^1 C_{m-1}^{-s} ds \right] \right\}, \end{aligned} \tag{2}$$

where h is mesh size and

$$C_k^{-s} = \frac{-s(-s-1)\cdots(-s-k+1)}{k!}. \tag{3}$$

The introduced back difference operators $\nabla^k f(x_i)$ are defined with recurrence relation as

$$\nabla f(x_i) = f(x_i) - f(x_{i-1}), \tag{4}$$

$$\nabla^k f(x_i) = \nabla(\nabla^{k-1} f(x_i)) \tag{5}$$

or by proposition given in [1]

$$\nabla^k f(x_i) = f(x_i) - \sum_{r=0}^{k-1} \nabla^r f(x_{i-1}). \tag{6}$$

The predicted value, y^p, gives first approximation of needed function at next mesh point. With corrector step, y^c, better approximated value is recalculated.

3. Modified Predictor-corrector Method

A modified predictor-corrector method is introduced with idea to improve the numerical solution evaluated by standard predictor-corrector method. An expression for modified predictor-corrector method as convex linear combination of predicted, y^p, and corrected, y^c, values calculated with traditional Adams-Bashforth-Moulton predictor-corrector method is taken from [1]

$$y^m(x_{i+1}) = \frac{1}{W_1 + W_2}(W_1 y^p(x_{i+1}) + W_2 y^c(x_{i+1})), \tag{7}$$

where the weighting coefficients and the coefficients of convex linear combination are given as follows:

$$W_1 = -(-1)^m \int_0^1 C_m^{-s+1} ds, \tag{8}$$

$$W_2 = (-1)^m \int_0^1 C_m^{-s} ds, \tag{9}$$

$$W_1 + W_2 = (-1)^{m-1} \int_0^1 C_{m-1}^{-s} ds. \tag{10}$$

4. Order of the Convergence of Modified Predictor-corrector Method

The errors for predictor and corrector steps are respectively

$$err_p = h^{m+1} f^{(m)}(\mu_i,\ y(\mu_i))(-1)^m \int_0^1 C_m^{-s} ds, \tag{11}$$

$$err_c = h^{m+1} f^{(m)}(\zeta_i,\ y(\zeta_i))(-1)^m \int_0^1 C_m^{-s+1} ds, \tag{12}$$

where $\mu_i, \zeta_i \in [x_i, x_{i+1}]$. According to the definition of modified predictor-corrector algorithm, we can derive the error of modified predictor-corrector numerical solution

$$\begin{aligned}
err_{mpc} &= \frac{1}{W_1 + W_2}(W_1 \cdot err_p + W_2 \cdot err_c) \\
&= \frac{-(-1)^m \int_0^1 C_m^{-s+1} ds}{(-1)^{m-1} \int_0^1 C_{m-1}^{-s} ds} (-1)^m \int_0^1 C_m^{-s} ds \cdot h^{m+1} f^{(m)}(\mu_i,\ y(\mu_i)) \\
&\quad + \frac{(-1)^m \int_0^1 C_m^{-s} ds}{(-1)^{m-1} \int_0^1 C_{m-1}^{-s} ds} (-1)^m \int_0^1 C_m^{-s+1} ds \cdot h^{m+1} f^{(m)}(\zeta_i,\ y(\zeta_i)) \\
&= \frac{(-1)^m \int_0^1 C_m^{-s} ds (-1)^m \int_0^1 C_m^{-s+1} ds}{(-1)^{m-1} \int_0^1 C_{m-1}^{-s} ds} h^{m+1} [f^{(m)}(\zeta_i,\ y(\zeta_i)) \\
&\quad - f^{(m)}(\mu_i,\ y(\mu_i))] \\
&= \frac{(-1)^m \int_0^1 C_m^{-s} ds (-1)^m \int_0^1 C_m^{-s+1} ds}{(-1)^{m-1} \int_0^1 C_{m-1}^{-s} ds} h^{m+1} (\zeta_i - \mu_i) f^{(m+1)}(\xi_i) \\
&\le \left| \frac{(-1)^m \int_0^1 C_m^{-s} ds (-1)^m \int_0^1 C_m^{-s+1} ds}{(-1)^{m-1} \int_0^1 C_{m-1}^{-s} ds} \right| h^{m+2} M_{m+1, f},
\end{aligned} \tag{13}$$

where $\xi_i \in [x_i, x_{i+1}]$ and $M_{m+1, f}$ is a maximum of $(m+1)$-th derivative on $[x_i, x_{i+1}]$.

Calculated upper error bound shows that modified predictor-corrector algorithm has one order higher accuracy than standard predictor-corrector algorithm.

5. Application of Modified Predictor-corrector Method on Standard Adams-Bashforth-Moulton Predictor-corrector Method

The most involved multistep predictor-corrector method in numerical integration of nonlinear ordinary differential equations is standard Adams-Bashforth-Moulton predictor-corrector method. The expressions for numerical calculation of predicted, y_{i+1}^p, and corrected, y_{i+1}^c, values are given in standard form

$$y_{i+1}^p = y_i + \frac{h}{24}(55f_i - 59f_{i-1} + 37f_{i-2} - 9f_{i-3}), \tag{14}$$

$$y_{i+1}^c = y_i + \frac{h}{24}(9f_{i+1}^p + 19f_i - 5f_{i-1} + f_{i-2}), \tag{15}$$

where $f_k = f(x_k, y_k)$ and $f_{i+1}^p = f(x_{i+1}, y_{i+1}^p)$ are function values at according point.

With weighting coefficients, taken from Chiou and Wu [1], the expression for modified predictor-corrector value, y_{i+1}^m, is given as

$$y_{i+1}^m = \frac{1}{270}(251y_{i+1}^c + 19y_{i+1}^p). \tag{16}$$

Analytical upper bound of the error for the given modified solution is

$$err_{mpc} \le \left| \frac{(-1)^3 \int_0^1 C_3^{-s} ds (-1)^3 \int_0^1 C_3^{-s+1} ds}{(-1)^2 \int_0^1 C_2^{-s} ds} \right| h^5 M_{4,f}$$

$$= \frac{\frac{251}{720}\frac{19}{720}}{\frac{3}{8}} h^5 M_{4,f} = \frac{4769}{194400} h^5 M_{4,f} \approx 0.024532 h^5 M_{4,f}. \tag{17}$$

6. Application of Modified Predictor-corrector Method on Second-Order Equations and System of Ordinary Differential Equations

For second-order differential equations, which could be written as

$$F(x,\ y,\ y',\ y'') = 0, \tag{18}$$

and associated initial conditions

$$y(x_0) = y_0,\ y'(x_0) = y_0', \tag{19}$$

we have to introduce new variable, denoted p, and defined as

$$p = y', \tag{20}$$

with its initial condition

$$p_0 = y_0'. \tag{21}$$

Then, we have the system of first-order ordinary differential equations

$$p' = g(x,\ y,\ p), \tag{22}$$

$$y' = p \tag{23}$$

with associated initial conditions

$$y(x_0) = y_0,\ p(x_0) = p_0 = y_0'. \tag{24}$$

The predicted and corrected values for each variable are now expressed with applied formerly mentioned expressions, for predicted values

$$y_{i+1}^{p} = y_i + \frac{h}{24}(55p_i - 59p_{i-1} + 37p_{i-2} - 9p_{i-3}), \tag{25}$$

$$p_{i+1}^{p} = p_i + \frac{h}{24}(55g_i - 59g_{i-1} + 37g_{i-2} - 9g_{i-3}), \tag{26}$$

and for corrected values

$$y_{i+1}^{c} = y_i + \frac{h}{24}(9p_{i+1}^{p} + 19p_i - 5p_{i-1} + p_{i-2}), \tag{27}$$

$$p_{i+1}^{c} = p_i + \frac{h}{24}(9g_{i+1}^{p} + 19g_i - 5g_{i-1} + g_{i-2}), \tag{28}$$

where $g_k = g(x_k,\ y_k,\ p_k)$ and $g_{i+1}^{p} = g(x_{i+1},\ y_{i+1}^{p},\ p_{i+1}^{p})$. The modified values for each variable could be easily expressed in following form:

$$y_{i+1}^{m} = \frac{1}{270}(251y_{i+1}^{c} + 19y_{i+1}^{p}), \tag{29}$$

$$p_{i+1}^{m} = \frac{1}{270}(251p_{i+1}^{c} + 19p_{i+1}^{p}). \tag{30}$$

7. Numerical Examples

Numerical examples are calculated for different kind of problems, first-order nonlinear differential equation, second-order differential equation and system of ordinary differential equations. The starting values in all examples are evaluated by using standard RK4 method.

First example is taken as the verification model for calculated error bounds for modified predictor-corrector method. The first order differential equation is taken with its associated initial condition

$$y' - e^{x} = 0, \; y(0) = 1. \tag{31}$$

Analytical solution of this problem is easy to find as

$$y(x) = e. \tag{32}$$

Numerical solutions are calculated by standard Adams-Bashforth-Moulton predictor-corrector method and modified predictor-corrector method with different integration step size. The evaluated numerical solutions are compared with calculated analytical bound of numerical error. The results are expressed in Table 1.

As second example, first-order nonlinear differential equation is taken with its associated initial condition

$$y' - e^{x-y} + e^{x} = 0, \; y(0) = \ln\frac{1+e}{e}. \tag{33}$$

Analytical solution of this problem is taken from [3]

$$y(x) = \ln[1 + e^{(-e^{x})}]. \tag{34}$$

Numerical solutions are calculated by standard Adams-Bashforth-Moulton predictor-corrector method and modified predictor-corrector method with different integration step size. The evaluated results are given in Table 2.

As third example, second-order differential equation is taken with its associated initial conditions

$$y'' - y = 0$$

$$y(0) = 1, \ y'(0) = -1. \tag{35}$$

Analytical solution can be easily found as follows:

$$y(x) = e^{-x}. \tag{36}$$

Numerical solutions are calculated by standard Adams-Bashforth-Moulton predictor-corrector method and modified predictor-corrector method with different integration step size. The evaluated results are given in Table 3.

Forth example is the system of ordinary differential equations

$$tx' + y = 0$$

$$ty' + x = 0 \tag{37}$$

with associated initial conditions

$$x(1) = 2, \ y(1) = 0. \tag{38}$$

Analytical solution of this system of equations is taken from [3] as

$$x(t) = \frac{1 + t^2}{t}, \ y(t) = \frac{1 - t^2}{t}. \tag{39}$$

Numerical solutions are again calculated by standard Adams-Bashforth-Moulton predictor-corrector method and modified predictor-corrector method with different integration step size. The evaluated results are given in Table 4.

8. Conclusions

The modified predictor-corrector method is applied as numerical integration method for different kind of problems. It has been shown good quality of solution with relatively small modification and just a little increased computational effort. Numerical examples for nonlinear first-order differential equations, second-order differential equations and system of differential equations has been used to show accuracy of the presented numerical integration method. The rate of convergence of modified predictor-corrector method is one order higher than rate of convergence of standard predictor-corrector method.

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Table 1. Numerical results for the given problem (31) at $x = 1$

absolute error		error upper bound	
	p.c.	m.p.c	
$h = 0.2$	$3.28 \cdot 10^{-5}$	$4.67 \cdot 10^{-6}$	$2.13 \cdot 10^{-5}$
$h = 0.1$	$3.35 \cdot 10^{-6}$	$2.39 \cdot 10^{-7}$	$6.67 \cdot 10^{-7}$
$h = 0.05$	$2.47 \cdot 10^{-7}$	$8.93 \cdot 10^{-9}$	$2.08 \cdot 10^{-8}$

Table 2. Numerical results for the given problem (33)

error, %				
$N = 20$			$N = 40$	
	p.c.	m.p.c.	p.c.	m.p.c.
$y(1)$	$0.36 \cdot 10^{-3}$	$1.09 \cdot 10^{-4}$	$1.84 \cdot 10^{-5}$	$3.13 \cdot 10^{-6}$
$y(2)$	0.592	0.3987	$1.15 \cdot 10^{-2}$	$5.65 \cdot 10^{-3}$

Table 3. Numerical results for the given problem (35)

error, %				
	$N = 20$		$N = 40$	
	p.c.	m.p.c.	p.c.	m.p.c.
$y(2)$	$7.48 \cdot 10^{-6}$	$1.86 \cdot 10^{-6}$	$3.99 \cdot 10^{-7}$	$5.74 \cdot 10^{-8}$
$y(4)$	$3.84 \cdot 10^{-4}$	$1.57 \cdot 10^{-4}$	$1.66 \cdot 10^{-5}$	$4.36 \cdot 10^{-6}$

Table 4. Numerical results for the given problem (37)

error, %				
	$N = 20$		$N = 40$	
	p.c.	m.p.c.	p.c.	m.p.c.
$x(2)$	$0.74 \cdot 10^{-4}$	$0.97 \cdot 10^{-6}$	$5.19 \cdot 10^{-6}$	$3.52 \cdot 10^{-8}$
$y(2)$	$1.24 \cdot 10^{-4}$	$1.62 \cdot 10^{-6}$	$8.65 \cdot 10^{-6}$	$5.87 \cdot 10^{-8}$
$x(11)$	$0.99 \cdot 10^{-2}$	$7.16 \cdot 10^{-4}$	$1.12 \cdot 10^{-3}$	$5.08 \cdot 10^{-5}$
$y(11)$	$1.01 \cdot 10^{-2}$	$7.28 \cdot 10^{-4}$	$1.14 \cdot 10^{-3}$	$5.37 \cdot 10^{-5}$

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