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THE PORTFOLIO PROBLEM UNDER THE VARIABLE RATE OF TRANSACTION COSTS

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Abstract

In the frictional market, contingent claims usually cannot be completely hedged, and this hedging problem can be regarded as a portfolio problem. Recently, people are studying the portfolio problem under the frictional market, but just studying the problem of the fixed rate of transaction costs. As far as we know, the research about the variable rate is still open. This paper poses a jump rate of transaction costs according to the real market, and based on this, we get an optimal model of minimizing risk. For this model, we use the Calculus of Variation to prove the existence of the optimal strategy.

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1. Introduction

In 1952, Markowitz [6] proposed the Mean-Variance Model under assumptions that the investors use the expect return to measure the real return, the variance of the return to measure the risk and the investors are risk aversion. In 1958, Modigliani and Miller [7] introduced the No-arbitrage Equilibrium method. In 1964, 1965 and 1966, Sharpe [9], Lintner [5] and Mossin [8] all proposed the CAPM model independently. In 1990, three American economics: Markowitz, Sharpe and Miller obtained the Economic Nobel Prize, and this also represented the Modern Portfolio Theory was coming maturity and accepted by the world's people. The classical models mentioned above do not consider the market frictions, but in the real market, the frictions exist everywhere, and the frictions make a long distance between the real market and the classical models. We introduce frictions to the financial problem to make the problem more close to the real market. There are all kinds of frictions such as tax, transaction costs, the bid-ask price and so on. Recently, many scholars are focusing on the portfolio problem of the transaction costs. In our country, Wang Shouyang and Li Zhongfei [4, 10-11] mainly study and character the No-arbitrage under the frictions' market. There are many scholars aboard paying much interesting on the portfolio problem with frictions, and proposed their views respectively. In 2002, Paolo Guasoni [1] obtained the risk minimization model under the transaction costs when the risk asset is a semi-martingales. In 2004, Paolo Guasoni and Walter Schachermayer [3] give the existent condition of solution of utility maximization problem under the transaction costs. In 2006, Paolo Guasoni [2] proposed the No-arbitrage definition of the friction Brown motion under the proportional transaction costs. But all these scholars only considered the proportional frictions, that is, the rate of transaction costs is a constant. In the real market, the rate of transaction costs usually is variable, for example, (1) in 1975, the corrected American Bond Rule cease the fixed transaction costs, and the middleman can decide the transaction costs by himself. (2) The bond trade may be having a favourable measurement just like the supermarket taking a promote sale. (3) Some economic events, the change of the rule and any other reasons can all make the rate of transaction costs alternative. For example, on

30th, May, 2007, the justification of the rate of the stamp-tax is a real example, and this justification lead to the alteration of the rate of transaction costs directly. So, consider the portfolio problem with variable rate of transaction costs is meaningfulness. In this paper, we mainly study the risk minimization problem with the variable rate of transaction costs according to the situation (2) and (3).

2. The Portfolio Value Model Under the Variable Rate of Transaction Costs

As usual in Mathematical finance, we consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, P)$, where the filtration $\mathcal{F}_t$ satisfies the usual assumptions, and $\mathcal{F} = \mathcal{F}_T$. $\mathcal{F}_t$ contains all informations, the investors can obtain at time t. In this market, we have a risk free asset and a risky asset. The risk free asset is used as numeraire, hence, it is assumed identical equal to 1. The price of the risky asset is given by a process X_t, adapted to the filtration $\mathcal{F}_t$. An agent starts with some initial capital c, and faces some contingent liability $H = (H_X, H_B)$, at time T, which requires the payment of H_X shares of the risky asset, and H_B units of the numeraire. Her goal is to set up a portfolio, which minimizes the total risk at time T. The self-financing condition implies that a trading strategy is uniquely determined by the number of shares θ_t invested in the risky asset at time t.

2.1. The gain process

In any reasonable market model, it is generally accepted that trading gains should be finite almost surely. Introducing the transaction costs, it is then natural to assume that the trading volume remains finite almost surely, in order to avoid the possibility of infinitely negative wealth.

Definition of Gain Functions as follows:

$$G_t^c = \int_{[0,\, t]} \theta_s dX_s,$$

where θ represents the shares investor in the risky asset, H_X represents the price of the risky asset and θ_t is a cadlag function.

2.2. The costs process

In the real market, the rate of transaction costs is not a constant. When the economy is depressed, the country can reduce the rate of transaction costs to irritate the stock market, for example, in October, 1991, Shengzhen trading post reduced the stamp-tax from 0.6% to 0.3% in order to irritate the stock market. And when the economy inflates, then the country can raise the rate to control the stock market. On 30th, May, 2007, the stamp-tax raised from 0.3% to 0.6%, and this lead to stock point drop 500 points. This adjustment of stamp-tax is one kind of the variable rate. This measurement has an advantage that the effect is direct and quick. But the investor cannot know when the rate will change, and we assume that the change number satisfies the Poisson distribution according to the stochastic behavior of the change. And this jump-rate of transaction costs is an improvement of the constant rate. The jump-rate model:

(1) In the usual case, k_t is equal to k, so $dk_t = 0$;

(2) If the rate has a change, that is, the jump is occurred, then $dk_t = \Phi dN_t$, where $N(t)$ represents the total number that the rate has changed and it satisfies the Poisson distribution with the parameter $\lambda(t)$. When the jump occurs, the probability is $\lambda(t)dt$, and $dN_t = 1$; when it does not occur, the probability is $1 - \lambda(t)dt$, and $dN_t = 0$. Φ is a random variable representing the altitude of the jump. Assume that, $\Phi_1, \Phi_2, \ldots, \Phi_{N(t)}$, are independent to each other and have the same distribution to Φ. Φ and $N(t)$ are independent to each other, and $\Phi > -1$. Solving the jump-rate, we get an expression:

$$k_t = k \exp\left\{ \sum_{t=0}^{N(t)} \ln(1 + \Phi_t) \right\} = k \prod_{t=0}^{N(t)} \ln(1 + \Phi_t).$$

The jump-rate consider more widely than the constant rate. When in the time $[0, T]$, the jump does not occur, then the jump-rate is the same to the constant rate, and when at the moment τ_t, the jump occur, then the rate is changed into $k_{\tau_k} = k_{\tau_{k^-}}(1 + \Phi_k)$, then the investor pay the costs using the new rate until the next jump occur. Assume the investor can trade continually, and denoting by L_t, M_t respectively, the cumulative number of shares purchased and sold at time t, then we have:

$$\theta_t = L_t - M_t,$$

where θ_t is a finite variation and can be integrated path by path.

It is well known that any function of the bounded variation can be represented as a difference between two increasing functions, given by:

$$L_t = D\theta^+([0, t]),\ M_t = D\theta^-([0, t]),$$

where $D\theta^+$ and $D\theta^-$ are respectively, the positive and negative parts of the Radon measure $D\theta$. The increasing processes L_t and M_t are uniquely determined under the assumption that they do not simultaneously increase. From the financial point of view, this is a natural condition, since it prevents opposite transactions from taking place at the same time.

Assume that the rate associated to the purchase and sale of the risky asset are equal. With the convention that $\theta_t = 0$ when $t < 0$ and $\theta_T = \lim_{t \uparrow T} \theta_t$.

Definition of Cost Functions given by:

$$C_t^c = \int_{[0, t]} k \prod_{i=0}^{N(t)} \ln(1 + \Phi_i) X_s d|D\theta_s| = \int_{[0, t]} d|D\theta_s|.$$

Remark 2.1. The pointwise definition of variation can be modified into the following (much less intuitive), which is invariant up to sets of Lebesgue measure zero:

$$D|\theta|(\omega) = \sup_{\substack{\phi \in C_c^1(0, T) \\ \|\phi\|_\infty \leq 1}} \int_{[0, T]} \theta_s(\omega) \phi'(s) ds.$$

2.3. The value process model

The market value of the portfolio at time t is given by the initial capital, plus the trading gain, minus the transaction cost, namely:

$$V_t^c(\theta) = c + G_t(\theta) - C_t(\theta).$$

At the terminal date T, the payment of the liability H and the liquidation of the remaining portfolio will give a payoff equal to

$$\underline{V} = V_T^c(\theta) - k_T X_T |\, \theta_T - H_X \,| - X_T H_X - H_B.$$

3. The Problem of Risk Minimization Under the Variable Rate of Transaction Costs

3.1. Space of strategies

In order to make sure that the market is no-arbitrage, we should keep the gain process is coherent integral, the following spaces of strategies are often considered:

$$\Theta^p = \{\theta : \theta - \mathcal{F}_t \text{ predictable, } G_t(\theta) \in L^p(P)\}.$$

The presence of the transaction costs in face forces a much narrower set of admissible strategies than Θ^p. It leads us to define the following spaces:

$$\Theta_c^p = \{\theta \in \Theta^p,\, C_T(\theta) \in L^p(P)\},$$

endowed with the norm:

$$\|\cdot\|_{\Theta_c^p} : \theta \to \left(\left\| \int_0^T \theta_t dX_t \right\|_p^p + \left\| \int_{[0,\, T]} k_t d|\, D\theta \right\|_p^p \right)^{\frac{1}{p}}$$

$$= (\| G_T(\theta) \|_p^p + \| C_T(\theta) \|_p^p)^{\frac{1}{p}}.$$

Lemma 3.1.1. *The set $G_t(\Theta^p)$ is a linear subspace of L^p, and if X is a continuous martingale, then it is also closed.*

Theorem 3.1.1. *Let X be a continuous local martingale, and k be a continuous, adapted process, such that* $\overline{K}(\omega) = \min_{t \in [0, T]} k_t X_t > 0$ *for almost every* ω. *Then* Θ_c^p *is a Banach space for all* $p \geq 1$.

Lemma 3.1.2 [1]. *Let X be a continuous local martingale, and* $G_T(\theta^n) \to G_T(\theta)$ *in the* L^p *-norm. Then there hold*

(i) *If* $p > 1$, *up to a subsequence* θ^n *such that* $\theta^n \to \theta$ *for almost every* $d\langle X \rangle_t P(d\omega)$;

(ii) *If* $p = 1$, *there exists some convex combinations* η^n *of* θ^n, *such that* $\eta^n \to \theta$ *almost every* $d\langle X \rangle_t dP$.

Lemma 3.1.3 [1]. *For a fixed* ω, *let* $\theta^n(\omega)_t \to \theta(\omega)_t$ *for almost every* t, *and* $|D\theta(\omega)^n|([0, T]) < C$ *uniformly. Then* $D\theta(\omega)^n$ *converges to* $D\theta(\omega)$ *in the weak star topology of Radon measures.*

Theorem 3.1.2. *If* θ^n *is bounded in* Θ_c^p *and* $\theta_t^n \to \theta_t$ *almost every in dtdp, then*

$$C_T(\theta) \leq \liminf_{n \to \infty} C_T(\theta^n) \text{ a.s. } \omega,$$

and for all $p \geq 1$, *there holds*

$$\| C_T(\theta) \|_p \leq \liminf_{n \to \infty} \| C_T(\theta^n) \|_p.$$

Proof. By assumption, for almost every ω, $\theta_t^n \to \theta_t$ for almost every t. For all subsequences n_j for which $C_T(\theta^{n_j}(\omega))$ converges, we have

If $C_T(\theta^{n_j}(\omega)) \to \infty$, then $C_T(\theta(\omega)) \leq \lim_{j \to \infty} C_T(\theta^{n_j}(\omega))$ is trivial;

If not, then for all j, the following formula $C_T(\theta^{n_j}(\omega)) < M(\omega)$ is tenable. Since, $C_T(\theta^{n_j}(\omega)) = \int_{[0, T]} k_t X_t d|D(\theta^{n_j}(\omega))|_t < M(\omega)$, that is, there holds

$$|D(\theta^{n_j}(\omega))|([0, T]) < \frac{M(\omega)}{k(\omega)}.$$

By Lemma 3.1.3 and Theorem A.4, we obtain

$$C_T(\theta(\omega)) = \int_{[0, T]} k_t X_t d|D\theta(\omega)|_t$$

$$\leq \lim_{j\to\infty} \int_{[0, T]} k_t X_t d|D\theta^{n_j}(\omega)| = \lim_{j\to\infty} C_T(\theta^{n_j}(\omega)).$$

And so, $C_T(\theta(\omega)) \leq \lim_{j\to\infty} C_T(\theta^{n_j}(\omega))$ follows. By Fatou's Lemma and the boundedness of θ^n in Θ_c^p, we arrive at

$$\begin{aligned}
\| C_T(\theta(\omega)) \|_p^p &= E\left[\left(\int_{[0, T]} k_t X_t d|D\theta(\omega)|\right)^p\right] \\
&\leq E\left[\liminf_{n\to\infty}\left(\int_{[0, T]} k_t X_t d|D\theta^n(\omega)|\right)^p\right] \\
&= \liminf_{n\to\infty} E\left[\left(\int_{[0, T]} k_t X_t d|D\theta^n(\omega)|\right)^p\right] \\
&= \liminf_{n\to\infty} \|C_T(\theta^n(\omega))\|_p^p < \infty.
\end{aligned}$$

This ends the proof of Theorem 3.1.2.

3.2. The existence of optimal strategies

This chapter mainly contains the existence results of the optimal strategies with the transaction costs under the assumption that the risky asset is a martingale. In general, the existence of a minimum requires two basic ingredients: relative compactness of minimizing sequences, and lower semi-continuity of the functional.

Definition 3.2.1. We define a *convex decreasing risk functional* as a function $\rho : L^p \to \mathbb{R} \cup \{\infty\}$, satisfying the following properties:

(i) ρ is convex;

(ii) If $X(\omega) \leq Y(\omega)$ for a.e. ω, then $\rho(X) \geq \rho(Y)$, that is ρ is decreasing;

(iii) ρ has the Fatou property. Namely, if $X_n \to X$ a.e., then $\rho(X) \leq \liminf_{n\to\infty} \rho(X_n)$.

Theorem 3.2.1. *Let ρ be a convex decreasing functional, and $c > 0$. We denote by*

$$H(\theta) = k_T|\theta_T - H_X| + X_T H_X + H_B,$$

$$F : \theta \to \rho(V_T^c(\theta) - H(\theta)).$$

if $\theta^n \to \theta$ a.e. in $dtP(d\omega)$, then we have

(i) *F is convex;*

(ii) *F is l.s.c. with respect to a.s. convergence in $dtP(d\omega)$.*

Proof. It is not hard to derive that there holds the following:

$$\begin{aligned}
&V_T^c(\lambda_1\theta_1 + \lambda_2\theta_2) - H(\lambda_1\theta_1 + \lambda_2\theta_2) \\
&= c + \int_0^T (\lambda_1\theta_1 + \lambda_2\theta_2) dX_s - \int_{[0,\, T]} k_t d|D(\lambda_1\theta_1 + \lambda_2\theta_2)| \\
&\quad - k_T|\lambda_1\theta_{T1} + \lambda_2\theta_{T2} - (\lambda_1 + \lambda_2)H_X| - X_T H_X - H_B \\
&\geq c + \lambda_1\int_0^T \theta_1 dX_s + \lambda_2\int_0^T \theta_2 dX_s - \lambda_1\int_{[0,\, T]} k_t d|D\theta_1| \\
&\quad - \lambda_2\int_{[0,\, T]} k_t d|D\theta_2| - \lambda_1 k_T|\theta_{T1} - H_X| \\
&\quad - \lambda_2 k_T|\theta_{T2} - H_X| - X_T H_X - H_B \\
&= \lambda_1[V_T^c(\theta_1) - H(\theta_1)] + \lambda_2[V_T^c(\theta_2) - H(\theta_2)],\ (\lambda_1 + \lambda_2 = 1).
\end{aligned}$$

This implies that $V_T^c - H$ is a concave functional. On the other hand, one arrives, by virtue of the fact, that ρ is a convex decreasing function, at

$$\rho(V_T^c(\lambda_1\theta_1 + \lambda_2\theta_2) - H(\lambda_1\theta_1 + \lambda_2\theta_2))$$

$$\leq \rho(\lambda_1 V_T^c(\theta_1) - \lambda_1 H(\theta_1) + \lambda_2 V_T^c(\theta_2) - \lambda_2 H(\theta_2))$$

$$\leq \lambda_1\rho(\theta_1) + \lambda_2\rho(\theta_2).$$

This means that F is of a convex function.

By using Theorem 3.1.2, one gets V_T^c is u.s.c (upper semicontinuity). At the same time, since ρ is decreasing and $H(\theta)$ is integrable, then we have

$$\rho(V_T^c(\theta) - H(\theta)) \leq \rho(\limsup_{n\to\infty}(V_T^c(\theta^n) - H(\theta^n)))$$

$$\leq \rho(\liminf_{n\to\infty}(V_T^c(\theta^n) - H(\theta^n))).$$

According to Fatou's Lemma, one has

$$\rho(V_T^c(\theta) - H(\theta)) \leq \liminf_{n\to\infty} \rho(V_T^c(\theta^n) - H(\theta^n)).$$

This ends the proof of Theorem 3.2.1.

Lemma 3.2.1 [1]**.** *For $C \in \mathbb{R}^+$ and $p > 1$, the set*

$$B_{C,D} = \{\theta : \| G_T(\theta) \|_p \leq C, \| C_T(\theta) \|_p \leq D\}$$

is Θ^p weak compact for $D \in \mathbb{R}^+ \cup \{+\infty\}$.

Theorem 3.2.2. *Let ρ be a convex decreasing functional, $c > 0$ and $(H_b + X_T H_X, k_T H_X) \in L^p(\Omega, \mathcal{F}_T, P)$ with $p > 1$. For any $M > 0$, let us denote by*

$$\Theta^p_{C,M} = \{\theta \in \Theta^p_C, \| G_T(\theta) \|_p \leq M\}.$$

Then the following minimum problem:

$$\min_{\theta\in\Theta^p_{C,M}} \rho(V_T^c(\theta) - H(\theta))$$

admits a solution.

Proof. Let θ^n be a minimizing sequence, so that $F(\theta^n) \to \inf_{\theta \in \Theta^p_{C,M}} F(\theta)$. Since $\Theta^p_{C,M}$ is a weakly compact by Lemma 3.2.1, up to a subsequence, we can assume that $\theta^n \rightharpoonup \theta \in \Theta^p_{C,M}$ weakly. Then by Theorem A.1, there exists a sequence of convex combinations $\eta^n = \sum_{k=n}^{\infty} \alpha_n^k \theta^k$, such that, $\eta^n \to \theta$ in the strong topology. By Lemma 3.1.2, we can assume up to a subsequence that $\eta^n \to \theta$ in the $d\langle X \rangle_t dP$ a.s., and hence, $dtdP$ -a.e. by hypotheses. By Jensen's inequality, we have

$$F(\eta^n) \le \sum_{k=n}^{\infty} \alpha_n^k F(\theta^k) \le \max_{n \le k} F(\theta^k).$$

Passing to the limit

$$\lim_{n \to \infty} F(\eta^n) \le \lim_{n \to \infty} \max_{n \le k} F(\theta^k) = \lim_{n \to \infty} F(\theta^n).$$

Finally, by the lower semicontinuity of F, we obtain

$$F(\theta) \le \lim_{n \to \infty} F(\theta^n).$$

Hence, θ is a minimizer. This completes the proof of Theorem 3.2.2.

We now turn to the case $p = 1$. That is to say, there holds the following:

Theorem 3.2.3. *Let* ρ *be a convex decreasing functional*, $c > 0$ *and* $(H_b + X_T H_X, k_T H_X) \in L^1(\Omega, \mathcal{F}_T, P)$. *For any* $M > 0$, *the following minimum problem*:

$$\min_{\theta \in \Theta^1_{C,M}} \rho(V_T^c(\theta) - H(\theta)),$$

admits a solution.

Proof. Let θ^n be a minimizing sequence. By A.2, there exists a convex combination $\eta^n = \sum_{k=n}^{M_n} \alpha_k^n \theta^k$ such that $G_T(\eta^n) \to \xi$ in L^1. By A.3, there exists $\theta \in \Theta^1$ such that $\xi = G_T(\theta)$. To see that, $\xi \in \Theta^1$, we apply firstly, Lemma 3.1.2 to obtain $\xi^n = \sum_{j=n}^{M_n} \beta_j^n (\eta^j)^{T_{j,n}}$ such that $\xi^n \to \theta$ a.e., then by Theorem 3.1.2, we have

$$C_T(\theta) \in L^1(P).$$

By Jensen's inequality, we have

$$F(\xi^n) \leq \sum_{j=n}^{M_n} \beta_j^n F((\eta^j)^{T_{j,n}}) \leq \max_{n \leq j} F((\eta^j)^{T_{j,n}}),$$

$$F(\eta^n) \leq \sum_{j=n}^{M_n} \alpha_k^n F(\theta^k) \leq \max_{n \leq k} F(\theta^k).$$

Passing to the limit, there hold

$$\lim_{n\to\infty} F(\xi^n) \leq \lim_{n\to\infty} \max_{n \leq j} F((\eta^j)^{T_{j,n}}) = \lim_{n\to\infty} F(\eta^n),$$

$$\lim_{n\to\infty} F(\eta^n) \leq \lim_{n\to\infty} \max_{n \leq k} F(\theta^k) = \lim_{n\to\infty} F(\theta^n).$$

Finally, by the lower semicontinuity of F, we obtain

$$F(\theta) \leq \lim_{n\to\infty} F(\theta^n).$$

Hence, θ is a minimizer. This ends the proof of Theorem 3.2.3.

Example 3.2.1 (Utility maximization)**.** Let U be a concave bounded increasing function. The utility maximization problem

$$\max_{\theta \in \Theta_C^1} E[U(V_T^c(\theta)) - H(\theta)]$$

admits a solution.

Proof. We can change this maximization problem to the minimization problem as follows:

$$\min_{\theta \in \Theta_C^1} - E[U(V_T^c(\theta) - H(\theta))] = \min_{\theta \in \Theta_C^1} E[-U(V_T^c(\theta) - H(\theta))],$$

and it satisfies Definition 3.2.1. At last, similar to the proof of Theorem 3.2.3 one can derive the existence of the optimal strategies.

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Appendix

Theorem A.1 [1]. *Let* x_n *be a relatively weakly compact sequence in a Banach space V. Then, there exists a sequence of convex combinations* $y_n = \sum_{i=n}^{\infty} \alpha_i^n x_n$ *converges in the norm topology of V.*

Theorem A.2 [1]. *Let* x_n *be a sequence of random variables, such that* $\sup_n E|x_n| < \infty$. *Then, there exists a subsequence* x'_n *and a random variable* $x \in L^1$ *such that for each subsequence* x''_n *of* x'_n, *we have*

$$\frac{1}{n}\sum_{i=1}^{n} x''_i \to x \ a.\ e.$$

Theorem A.3 [1]. *Let X be a continuous local martingale,* θ^n *is a sequence of X-integrable predictable stochastic processes such that the sequence* $\int_0^\infty \theta_s^n dX_s$ *converges to a random variable G in the norm topology of* L^1. *Then, there is an* $\mathcal{F}^X$*-predictable stochastic process* θ *such that* $\int_0^t \theta_s^n dX_s$ *is a bounded martingale, and there holds* $\int_0^t \theta_s^n dX_s = G$.

Theorem A.4 [1]. *Let* $\mu^n \to \mu$, *where* μ^n, μ *are Radon measures on I, respectively, and the convergence is meant in the weak star sense. Then, one arrives at*

$$|\mu| \le \liminf_{n\to\infty} |\mu^n|.$$

■