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POSITIVE SOLUTIONS FOR A CLASS OF SINGULAR SECOND ORDER TWO POINT BOUNDARY VALUE PROBLEMS

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Abstract

By applying upper and lower solution method, this paper deals with the existence of positive solutions for a class of singular boundary value problems. Sufficient conditions are obtained that guarantee the existence of positive solutions. The interesting point is that the nonlinear term *f* is involved with the first-order derivative explicitly.

1. Introduction

In this paper, we consider the following singular boundary value problem:

$$\begin{cases} -x''(t) = \varphi(t) f(t, x(t), x'(t)), \;\; t \in (0, 1), \\ ax(0) - bx'(0) = 0, \;\; cx(1) + dx'(1) = 0, \end{cases} \tag{1.1}$$

where $f : [0, 1] \times [0, \infty)^2 \to [0, \infty)$ is continuous, $\varphi : (0, 1) \to [0, \infty)$ is continuous and may be singular with $t = 0$, and/or $t = 1$, $a, b, c, d \geq 0$ such that $\rho := ac + ad + bc > 0$.

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Boundary value problems arise from applied mathematical sciences, and they received a great deal of attention in the literature. Most of the available literature on singular boundary value problems (for instance [1, 3-6]) discuss the case where f is either continuous or a caratheodory function by using fixed point theorems and fixed point index theory and so on. Recently some papers such as [1, 3, 5, 6], by using approximating method or upper-lower solution approach, investigate the special case of (1.1).

The aim of this paper is to investigate the existence of positive solutions of BVP (1.1) in a Banach space. The interesting point is the nonlinear term f involved with the first-order derivative explicitly. The techniques used in this paper are a specially constructed cone, and the fixed point theorem of cone expansion and compression.

This paper is organized as follows: In Section 2, some preliminaries are given. In Section 3, we are devoted to our main results.

2. Preliminaries and Some Lemmas

Let $G(t, s) : [0, 1] \times [0, 1] \to [0, +\infty)$ is Green's function for

$$\begin{cases} -x''(t) = 0, \\ ax(0) - bx'(0) = 0, \quad cx(1) + dx'(1) = 0, \end{cases} \tag{2.1}$$

that is

$$G(t, s) = \frac{1}{\rho} \begin{cases} (b + as)(c + d - ct), & 0 \le s \le t \le 1, \\ (b + at)(c + d - cs), & 0 \le t \le s \le 1. \end{cases} \tag{2.2}$$

Denote

$$f^{\alpha} = \limsup_{|(x, y)| \to \alpha} \frac{f(t, x, y)}{|(x, y)|}, \quad \alpha = 0, \text{ or } \alpha = \infty,$$

$$f_{\beta} = \liminf_{|(x, y)| \to \beta} \frac{f(t, x, y)}{|(x, y)|}, \quad \beta = 0, \text{ or } \beta = \infty,$$

$$|(x, y)| = |x| + |y|.$$

Clearly

$$G(t, s) \geq 0, \quad (t, s) \in [0, 1] \times [0, 1]; \tag{2.3}$$

$$G(t, s) \leq G(s, s), \quad (t, s) \in [0, 1] \times [0, 1]; \tag{2.4}$$

$$G(t, s) \geq M_\theta G(s, s), \quad (t, s) \in [\theta, 1 - \theta] \times [0, 1]; \tag{2.5}$$

where

$$M_\theta = \min\left\{\frac{d + \theta c}{c + d}, \frac{b + \theta a}{a + b}\right\} < 1.$$

Let $X = C[0, 1]$ be a Banach space endowed with the norm

$$\| x \| = \max\{\max_{0 \leq t \leq 1} | x(t) |, \max_{0 \leq t \leq 1} | x'(t) |\}.$$

Definition 2.1. Let E be a real Banach space. Then a nonempty convex closed set $P \subset E$ is said to be a *cone* provided that

(i) $\lambda u \in P$ for all $u \in P$ and all $\lambda \geq 0$ and

(ii) $u, - u \in P$ implies $u = 0$.

Note that every cone $P \subset E$ induces an ordering in E given by $x \leq y$ if $y - x \in P$.

Definition 2.2. An operator is called *completely continuous* if it is continuous and maps bounded sets into precompact sets.

Definition 2.3. A function x is said to be a *solution* of BVP (1.1) if $x \in C^1[0, 1] \cap C^2(0, 1)$ satisfies BVP (1.1), in addition, x is said to be a *positive solution* if $x(t) > \theta$ for $t \in (0, 1)$ and x is a *solution* of BVP (1.1).

Definition 2.4. A function $\alpha(t)$ is called a *lower solution*, if $\alpha(t) \in C^2(0, 1) \cap C^1[0, 1]$ and satisfying

$$\begin{cases} - \alpha''(t) \leq \varphi(t) f(t, \alpha(t), \alpha'(t)), & t \in (0, 1), \\ a\alpha(0) - b\alpha'(0) \leq 0, & c\alpha(1) + d\alpha'(1) \leq 0. \end{cases} \tag{2.6}$$

Upper solution is defined by reversing the above inequality signs. If there exist a lower solution $\alpha(t)$ and an upper solution $\beta(t)$ for the BVP (1.1) such that $\alpha(t) \le \beta(t)$, then $(\alpha(t), \beta(t))$ is called a *couple* of upper and lower solutions for the BVP (1.1).

For convenience, we list the following assumptions:

($\mathbf{A}_1$) $\varphi \in C((0, 1), [0, +\infty))$ and $0 < \int_0^1 G(s, s)\varphi(s)ds < +\infty$;

($\mathbf{A}_2$) $f \in C([0, 1] \times [0, +\infty) \times [0, +\infty), [0, +\infty))$;

($\mathbf{A}_3$) $0 \le f^0 < L_1$, $L_1 = \left(2\int_0^1 G(s, s)\varphi(s)ds\right)^{-1}$;

($\mathbf{A}_4$) There exist a positive constant α_1 and Lebesgue measurable sets $E_1 \subset [0, 1]$ such that $\alpha_1 < f_\infty \le +\infty$, uniformly for $t \in E_1 \subset [0, 1]$ a.e. and $M_\theta \alpha_1 \int_{E_1} G\left(\frac{1}{2}, s\right)\varphi(s)ds \ge 1$, $\forall E_1 \subset [0, 1]$;

($\mathbf{A}_5$) $0 \le f^\infty < L_1$;

($\mathbf{A}_6$) There exist a positive constant β_1 and Lebesgue measurable sets $E_2 \subset [0, 1]$ such that $\beta_1 < f_0 \le +\infty$, uniformly for $t \in E_2 \subset [0, 1]$ a.e. and $M_\theta \beta_1 \int_E G\left(\frac{1}{2}, s\right)\varphi(s)ds \ge 1$, $\forall E \subset E_2$.

Remark. By (A_1), (A_2), there exists $t_1 \in (0, 1)$ such that $\varphi(t_1) > 0$. Thus there exists $\theta \in \left(0, \frac{1}{2}\right)$ such that $\int_\theta^{1-\theta} G\left(\frac{1}{2}, s\right)\varphi(s)ds > 0$.

Let

$$K = \{x \in X : x(t) \ge 0,\ t \in [0, 1],\ \min_{t \in [\theta, 1-\theta]} x(t) \ge M_\theta \| x \|\}.$$

It is easy to know that K is a cone of X. Define an operator $B : K \to C[0, 1]$ by

$$Bx(t) = \int_0^1 G(t, s)\varphi(s) f(s, x(s), x'(s))ds, \quad t \in [0, 1].$$

Obviously, the existence of positive solutions for BVP (1.1) is equivalent to the existence of fixed points of the operator equation $Bx = x, \ x \in C[0, 1]$.

Lemma 2.1. *Let X be a Banach space, and let P be a cone in X. Assume that* $\Omega_1, \ \Omega_2$ *are open subsets of X with* $0 \in \Omega_1, \ \overline{\Omega}_1 \subset \Omega_2$. *Let* $A : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ *be a completely continuous operator, satisfying either*

(i) $\| Ax \| \leq \| x \|, \quad x \in P \cap \partial\Omega_1, \quad \| Ax \| \geq \| x \|, \quad x \in P \cap \partial\Omega_2,$

or

(ii) $\| Ax \| \geq \| x \|, \quad x \in P \cap \partial\Omega_1, \quad \| Ax \| \leq \| x \|, \quad x \in P \cap \partial\Omega_2.$

Then A has a fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Lemma 2.2. *Assume that* (A_1) *and* (A_2) *hold, then* $A(K) \subseteq K$ *and* $A : K \to K$ *is completely continuous.*

Proof. By (2.4), for all $t \in [0, 1]$, we get

$$(Ax)(t) = \int_0^1 G(t, s)\varphi(s)f(s, x(s), x'(s))ds$$

$$\leq \int_0^1 G(s, s)\varphi(s)f(s, x(s), x'(s))ds,$$

then

$$\| Ax \| \leq \int_0^1 G(s, s)\varphi(s)f(s, x(s), x'(s))ds.$$

For any $x \in K$, we know by (2.5) that

$$\min_{t\in[\theta, 1-\theta]}(Ax)(t) = \min_{t\in[\theta, 1-\theta]}\int_0^1 G(t, s)\varphi(s)f(s, x(s), x'(s))ds$$

$$\geq M_\theta \int_0^1 G(s, s)\varphi(s)f(s, x(s), x'(s))ds$$

$$\geq M_\theta \| Ax \|,$$

therefore, $A(K) \subseteq K$.

Now, let us prove $A : K \to K$ is completely continuous. For each $n \geq 1$ define the operator $A_n : K \to C[0, 1]$ by

$$(A_n x)(t) = \int_{\frac{1}{n}}^{\frac{n-1}{n}} G(t, s)\varphi(s)f(s, x(s), x'(s))ds, \quad x \in K, \ t \in [0, 1]. \tag{2.7}$$

By (A_1), (A_2) and the Arzela-Ascoli theorem, we know that $A_n : K \to C[0, 1]$ is completely continuous. By (2.4), we have

$$|(Ax)(t) - (A_n x)(t)|$$

$$= \int_0^{\frac{1}{n}} G(t, s)\varphi(s)f(s, x(s), x'(s))ds + \int_{\frac{n-1}{n}}^{1} G(t, s)\varphi(s)f(s, x(s), x'(s))ds$$

$$\leq \int_0^{\frac{1}{n}} G(s, s)\varphi(s)f(s, x(s), x'(s))ds + \int_{\frac{n-1}{n}}^{1} G(s, s)\varphi(s)f(s, x(s), x'(s))ds,$$

and so

$$\| Ax - A_n x \| \leq \int_0^{\frac{1}{n}} G(s, s)\varphi(s)f(s, x(s), x'(s))ds + \int_{\frac{n-1}{n}}^{1} G(s, s)\varphi(s)f(s, x(s), x'(s))ds.$$

Assumptions (A_1), (A_2) and the absolute continuity of integral imply that

$$\lim_{n\to\infty} \| Ax - A_n x \| = 0,$$

then A is completely continuous.

3. Main Results

3.1. One positive solution

Theorem 3.1. *Assume that* (A_1)-(A_4) *hold, then BVP* (1.1) *has at least one positive solution.*

Proof. By (A_3), there exist $\varepsilon_1 > 0$ and $r_1 > 0$ such that for all $t \in [0, 1]$,

$$f(t, x, y) \leq (L_1 - \varepsilon_1)(|x| + |y|), \quad 0 \leq |x| + |y| \leq r_1. \tag{3.1}$$

Let $\Omega_1 = \{x \in X : \|x\| < r_1\}$. Then, by (3.1), for any $x \in \partial\Omega_1 \cap K$, we get

$$\begin{aligned}
\|Ax\| &= \max_{t\in[0,1]} \int_0^1 G(t, s)\varphi(s)f(s, x(s), x'(s))ds \\
&\leq \int_0^1 G(s, s)\varphi(s)f(s, x(s), x'(s))ds \\
&\leq (L_1 - \varepsilon_1)\int_0^1 G(s, s)\varphi(s)(|x(s)| + |x'(s)|)ds \\
&\leq (L_1 - \varepsilon_1)\|x\| \int_0^1 2G(s, s)\varphi(s)ds \\
&= L_1 r_1 \int_0^1 2G(s, s)\varphi(s)ds - 2\varepsilon_1 r_1 \int_0^1 G(s, s)\varphi(s)ds \\
&< r_1 = \|x\|.
\end{aligned}$$

By (A_4), there exist $r_2 > 0$ and $\varepsilon_2 > 0$ such that

$$f(t, x, y) \geq (\varepsilon_2 + \alpha_1)|(x, y)|, \quad |(x, y)| \geq r_2, \quad t \in E_1.$$

Let $r_2 > r_1$, $\Omega_2 = \{x \in X : \|(x, y)\| < r_2\}$. By (A_4), for any $x \in \partial\Omega_2 \cap K$, we have

$$\begin{aligned}
\|Ax\| \geq Ax\left(\frac{1}{2}\right) &= \int_0^1 G\left(\frac{1}{2}, s\right)\varphi(s)f(s, x(s), x'(s))ds \\
&\geq \int_{E_1} G\left(\frac{1}{2}, s\right)\varphi(s)f(s, x(s), x'(s))ds \\
&\geq (\varepsilon_2 + \alpha_1)\int_{E_1} G\left(\frac{1}{2}, s\right)\varphi(s)(|x(s)| + |x'(s)|)ds
\end{aligned}$$

$$\geq M_\theta(\varepsilon_2 + \alpha_1) r_2 \int_{E_1} G\left(\frac{1}{2}, s\right)\varphi(s)ds$$

$$= r_2 M_\theta \alpha_1 \int_{E_1} G\left(\frac{1}{2}, s\right)\varphi(s)ds + r_2 M_\theta \varepsilon_2 \int_{E_1} G\left(\frac{1}{2}, s\right)\varphi(s)ds$$

$$\geq r_2,$$

therefore, for any $x \in \partial\Omega_2 \cap K$. From (3.3), (3.4) and Lemma 2.1, A has a fixed point $x \in K$ such that $0 < r_1 < \| x \| < r_2$. It is clear that x is a positive solution of BVP (1.1).

Remark. From the above process of proof, we know that BVP (1.1) has a positive solution under the assumptions (A_1)-(A_4).

Corollary 3.2. *Assume that* (A_1), (A_2), (A_5), *and* (A_6) *hold, then BVP* (1.1) *has at least one positive solution.*

Proof. By (A_5), there exist $r_3' > 0$ and $\varepsilon_3 > 0$ such that $L_1 - \varepsilon_3 > 0$, thus for any $t \in [0, 1]$,

$$f(t, x, y) \leq (L_1 - \varepsilon_3)|(x, y)|, \quad |(x, y)| \geq r_3'.$$

Set

$$M_1 = \sup_{0 \leq |(x, y)| \leq r_3', t \in [0,1]} f(t, x, y),$$

then

$$f(t, x, y) \leq M_1 + (L_1 - \varepsilon_3)|(x, y)|, \quad (x, y) \in [0, \infty) \times [0, \infty), \quad t \in [0, 1]. \tag{3.2}$$

Let $r_3 > \max\left\{r_3', \frac{M_1}{2\varepsilon_3}\right\}$, $\Omega_3 = \{x : \| x \| < r_3\}$. By (3.2), for any $x \in \partial\Omega_3 \cap K$, $t \in [0, 1]$, we have

$$Ax(t) = \int_0^1 G(t, s)\varphi(s) f(s, x(s), x'(s))ds$$

$$\leq \int_0^1 G(s, s)\varphi(s)(M_1 + (L_1 - \varepsilon_3))(|x(s)| + |x'(s)|)ds$$

$$\le (L_1 - \varepsilon_3) r_3 \int_0^1 2G(s, s)\varphi(s)ds + M_1 \int_0^1 G(s, s)\varphi(s)ds$$

$$= r_3 L_1 \int_0^1 2G(s, s)\varphi(s)ds + (M_1 - 2r_3\varepsilon_3) \int_0^1 G(s, s)\varphi(s)ds$$

$$\le r_3 = \| x \|,$$

therefore, we get

$$\| Ax \| \le \| x \|.$$

By (A_6), there exist $r_4 > 0$ and $\varepsilon_4 > 0$ such that

$$f(t, x, y) \ge (\beta_1 + \varepsilon_4) |(x, y)|, \quad 0 \le |(x, y)| \le r_4, \quad t \in E_2.$$

Let $r_4 < r_3$, $\Omega_4 = \{x \in X : \| x \| < r_4\}$.

For any $x \in \partial\Omega_4 \cap K$, we have

$$\| Ax \| \ge Ax\left(\frac{1}{2}\right) = \int_0^1 G\left(\frac{1}{2}, s\right)\varphi(s) f(s, x(s), x'(s))ds$$

$$\ge (\varepsilon_4 + \beta_1) \int_0^1 G\left(\frac{1}{2}, s\right)\varphi(s)(|x(s)| + |x'(s)|)ds$$

$$\ge (\varepsilon_4 + \beta_1) \int_{E_2} G\left(\frac{1}{2}, s\right)\varphi(s)(|x(s)| + |x'(s)|)ds$$

$$\ge M_\theta \beta_1 \| x \| \int_E G\left(\frac{1}{2}, s\right)\varphi(s)ds + M_\theta \varepsilon_4 \| x \| \int_E G\left(\frac{1}{2}, s\right)\varphi(s)ds$$

$$= r_4 \beta_1 M_\theta \int_E G\left(\frac{1}{2}, s\right)\varphi(s)ds + r_4 M_\theta \varepsilon_4 \int_E G\left(\frac{1}{2}, s\right)\varphi(s)ds$$

$$\ge r_4 = \| x \|,$$

hence

$$\| Ax \| \ge \| x \|,$$

then, by Lemma 2.1, A has a fixed point $x \in K$ such that $0 < r_4 < \| x \| < r_3$. It is clear that x is a positive solution of BVP (1.1).

3.2. A priori estimate

Next, we give a priori estimate for positive solutions of BVP (1.1).

Theorem 3.3. *If*

$$\lim_{|(x, x')| \to \infty} \max_{t \in [0,1]} \frac{f(t, x, x')}{|(x, x')|} = +\infty, \tag{3.3}$$

then there exists $C_0 > 0$ *such that* $\| x \| \le C_0$, *for all positive solutions* x *of BVP* (1.1).

Proof. Assume that by contradiction there exists a sequence of solution $\{x_n\} \subseteq K$ of BVP (1.1) such that $x_n \to \infty$. Without loss of generality, we may assume that $\| x_n \| \to \infty$. From the assumption (3.3), we can take a sequence of real numbers $\alpha_n \to \infty$ such that

$$\frac{f(s, x_n(s), x_n'(s))}{x_n(s) + x_n'(s)} \ge \alpha_n. \tag{3.4}$$

$$\begin{aligned} \| x_n \| \ge x_n\left(\frac{1}{2}\right) &= \int_0^1 G\left(\frac{1}{2}, s\right) \varphi(s) f(s, x_n(s), x_n'(s)) ds \\ &\ge \int_{1-\theta}^{\theta} G\left(\frac{1}{2}, s\right) \varphi(s) f(s, x_n(s), x_n'(s)) ds \\ &\ge \int_{1-\theta}^{\theta} G\left(\frac{1}{2}, s\right) \varphi(s) \frac{f(s, x_n(s), x_n'(s))}{x_n(s) + x_n'(s)} x_n(s) ds \\ &\ge M_\theta \| x_n \| \int_{1-\theta}^{\theta} G\left(\frac{1}{2}, s\right) \varphi(s) \frac{f(s, x_n(s), x_n'(s))}{x_n(s) + x_n'(s)} ds, \end{aligned}$$

which together with (3.4), implies that

$$\frac{1}{\alpha_n} \ge M_\theta \int_{1-\theta}^{\theta} G\left(\frac{1}{2}, s\right) \varphi(s) ds > 0,$$

which is a contradiction.

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