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POSITIVE SOLUTIONS OF FOURTH-ORDER BOUNDARY VALUE PROBLEMS WITH PARAMETERS

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Abstract

In this paper, the existence results of positive solutions are obtained for a fourth-order boundary value problem with parameters, then by using the fixed point theorem in cone and the system of integral equation, an existence theorem is established.

1. Introduction

In recent years, there have been many papers studying the existence of positive solutions of the Dirichlet boundary value problem. For example, in paper [1, 2], Li studied the operator equation $u = Kfu$ and obtained the existence results for positive solutions and multiplicity of solutions. In particular, in [3], Liu considered the existence of positive solutions of the Dirichlet boundary value problem by using the critical point theory. In this paper, by employing the cone expansion or compression fixed point theorem we study the existence of the positive solution of the following boundary value problem:

$$u^4(t) + \beta u''(t) - \alpha u(t) = f(t, u(t)),\ t \in [0, 1], \tag{1.1}$$

$$u(0) = u'(1) = u''(0) = u'''(1) = 0 \tag{1.2}$$

where $f : [0, 1] \times [0, +\infty) \to [0, +\infty)$ is continuous.

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We suppose

(H_0) $\alpha, \beta \in \mathbb{R}$ and $\beta < \frac{\pi^2}{2}$, $\alpha \geq -\frac{\beta^2}{4}$, $\frac{16}{\pi^4}\alpha + \frac{4}{\pi^2}\beta < 1$.

For the sake of convenience, we note

$$\underline{f}_0 = \liminf_{u \to 0^+} \min_{t \in [0,1]} \frac{f(t, u)}{u}, \ \bar{f}_0 = \limsup_{u \to 0^+} \max_{t \in [0,1]} \frac{f(t, u)}{u},$$

$$\underline{f}_\infty = \liminf_{u \to +\infty} \min_{t \in [0, 1]} \frac{f(t, u)}{u}, \ \bar{f}_\infty = \limsup_{u \to +\infty} \max_{t \in [0, 1]} \frac{f(t, u)}{u}.$$

We are now going to state our main results.

Theorem 1.1. *Suppose* (H_0) *holds and satisfies assumptions*

(i) $\bar{f}_0 < \frac{\pi^4}{16} - \beta\frac{\pi^2}{4} - \alpha$, $\underline{f}_0 > \frac{\pi^4}{16} - \beta\frac{\pi^2}{4} - \alpha$; *or*

(ii) $\bar{f}_\infty > \frac{\pi^4}{16} - \beta\frac{\pi^2}{4} - \alpha$, $\bar{f}_\infty < \frac{\pi^4}{16} - \beta\frac{\pi^2}{4} - \alpha$.

Then problem (1.1) *and* (1.2) *has at least one positive solution.*

2. Auxiliary Results

Let λ_1, λ_2 be the roots of the polynomial $P(\lambda) = \lambda^2 + \beta\lambda - \alpha$, namely, $\lambda_1, \lambda_2 = \frac{-\beta \pm \sqrt{\beta^2 + 4\alpha}}{2}$. By the basic assumption (H_0), it is easy to see that $\lambda_1 \geq \lambda_2 > -\pi^2/4$.

Let G_i, $i = 1, 2$ be the Green's function of the linear boundary value problem $-u''(t) + \lambda_i u(t) = 0$, $t \in [0, 1]$, subject to $u(0) = u'(1) = 0$.

Lemma 2.1. G_i, $i = 1, 2$ *have the following results*:

(i) $G_i(t, s) > 0$, $t, s \in (0, 1)$;

(ii) $G_i(t, s) \leq C_i G_i(s, s)$, $t, s \in [0, 1]$, *where* $C_i > 0$ *is a constant*;

(iii) $G(t, s) \geq \delta_i G_i(t, t) G_i(s, s)t$, $s \in [0, 1]$, *where* $\delta_i > 0$ *is a constant.*

Let $C[0, 1]$ denote the usual real Banach space with the norm $\| u \| = \max_{0 \le t \le 1} |u(t)|$ and $C^+[0, 1]$ denote a cone in $C[0, 1]$ and $M_i = \max_{0 \le s \le 1} G_i(s, s)$, $m_i = \min_{\frac{1}{4} \le s \le \frac{3}{4}} G_i(s, s)\,(i = 1, 2)$, $C_0 = \int_0^1 G(\tau, \tau) G(\tau, \tau) d\tau$, then M_i, m_i, $C_0 > 0$.

Lemma 2.2. *Suppose* $f(t, u) = h(t) \in C^+[0, 1]$, *then the solution of BVP* (1.1), (1.2) *corresponding the linear boundary value problem satisfied*

$$u(t) \ge \frac{\delta_1 \delta_2 C_0}{C_1 C_2 M_1} G_1(t, t) \| u \|.$$

Define an operator $A : C^+[0, 1] \to C^+[0, 1]$

$$Au(t) = \int_0^1 \int_0^1 G_1(t, \tau) G_2(\tau, s) f(s, u(s)) ds d\tau.$$

Obviously $A : C^+[0, 1] \to C^+[0, 1]$ is completely continuous. It follows from Lemma 2.2 that a positive solution of the boundary value problem (1.1), (1.2) is equivalent to the nontrivial fixed point of operator A. Therefore, let

$$K = \{u \in C^+[0, 1] : u(t) \ge \sigma \| u \|,\ t \in [1/4, 3/4]\},$$

where $\sigma = \dfrac{\delta_1 \delta_2 C_0 m_1}{C_1 C_2 M_1}$, then K is a subcone in $C^+[0, 1]$.

Lemma 2.3. $A(K) \subset K$ *and* $A : K \to K$ *is completely continuous.*

The proof of Theorem 1.1 is based on the fixed point index theory in cone. Let E be a real Banach space and K be a closed convex cone in E. Suppose Ω is a bounded open subset of E, $\partial\Omega$ is the boundary Ω, $K \cap \Omega \neq \emptyset$. Let $K \cap \overline{\Omega} \to K$ is completely continuous. If $Au \neq u$, $u \in K \cap \partial\Omega$, then $i(A, K \cap \Omega, K)$ is defined.

3. Proof of The Main Result

We note $L = \frac{\pi^4}{16} - \beta\frac{\pi^2}{4} - \alpha.$

(i) $\bar{f}_0 < L$. By the definition of $\bar{f}_0$, there exist $\varepsilon \in (0, L)$ and $r_0 > 0$ so that $f(t, u) \le (L - \varepsilon)u, \forall t \in [0, 1], 0 \le u \le r_0.$

Let $r \in (0, r_0)$. We prove $\mu Au \ne u$, $u \in \partial K_r$, $0 < \mu \le 1$. In fact, we may assume that there exist $u_0 \in \partial K_r$ and $0 < \mu_0 \le 1$ such that $\mu_0 Au_0 = u_0$, then by the definition of A, $u_0(t)$ satisfies

$$u_0^{(4)} + \beta u_0''(t) - \alpha u_0(t) = \mu_0 f(t, u_0(t)), \; t \in [0, 1], \tag{3.1}$$

and boundary condition (1.2). By multiplying both sides of (3.1) by $\sin\frac{\pi}{2}tdt$ and integrating from 0 to 1 we have

$$L\int_0^1 u_0(t)\sin\frac{\pi}{2}tdt = \mu_0\int_0^1 f(t, u_0(t))\sin\frac{\pi}{2}tdt$$

$$\le (L - \varepsilon)\int_0^1 u_0(t)\sin\frac{\pi}{2}tdt, \tag{3.2}$$

and by Lemma 2.2, we have $\int_0^1 u_0(t)\sin\frac{\pi}{2}tdt > 0$. So it follows from inequality (3.2) which contradicts for $L \le L - \varepsilon$. Consequently, by using the fixed point index theory, we have

$$i(A, K_r, K) = 1. \tag{3.3}$$

On the other hand, applying $\underline{f}_\infty > L$ yields that there exist $\varepsilon > 0$ and $H > 0$ such that $f(t, u) \ge (L + \varepsilon)u, \forall t \in [0, 1], u \ge H$. And it follows that $f(t, u) - (L + \varepsilon)u, t \in [0, 1], u \ge H$ is continuous, then we have $f(t, u) \ge (L + \varepsilon)u - C, \forall t \in [0, 1], u \ge 0$. Choose $R > R_0 := \max\{H/\sigma, r_0\}$ and $e = \sin \pi t$, we will prove if R is so large, then $u - Au \ne \tau e$, $\forall u \in \partial K_R, \tau \ge 0$. In fact, if there exist $u_0 \in \partial K_R$ and $\tau_0 \ge 0$ such that

$u_0 - Au_0 = \tau_0 e,$ then by the definition of $A,$ $u_0(t)$ satisfies

$$u^4(t)_0 + \beta u_0''(t) - \alpha u_0(t) + \tau_0(\frac{\pi}{16}^4 \sin \pi t - \beta \frac{\pi}{4}^2 \sin \pi t - \alpha \sin \pi t)$$

$$= f(t, u_0(t)), \; 0 \leq t \leq 1 \tag{3.4}$$

and boundary condition (1.2). By multiplying both sides of (3.4) by $\sin \frac{\pi}{2} t$ and integrating from 0 to 1 we have

$$L \int_0^1 u_0(t) \sin \frac{\pi}{2} t dt + \frac{4}{3\pi} L\tau_0 = \int_0^1 f(t, u_0(t)) \sin \frac{\pi}{2} t dt$$

$$\geq (L + \varepsilon) \int_0^1 u_0(t) \sin \frac{\pi}{2} t dt - \frac{2}{\pi} C.$$

Therefore,

$$\int_0^1 u_0(t) \sin \frac{\pi}{2} t dt \leq \frac{4L\tau_0 + 6C}{3\pi\varepsilon}. \tag{3.5}$$

And by Lemma 2.2 it is easy to know

$$\int_0^1 u_0(t) \sin \frac{\pi}{2} t dt \geq \frac{\delta_1 \delta_2 C_0}{C_1 C_2 M_1} \int_0^1 G_1(t, t) \sin \frac{\pi}{2} t dt \| u_0 \|,$$

then it follows from the above equality and (3.5) we have

$$\| u_0 \| \leq \frac{4L\tau_0 + 6C}{3\pi\varepsilon} \frac{\delta_1 \delta_2 C_0}{C_1 C_2 M_1} \left(\int_0^1 G_1(t, t) \sin \frac{\pi}{2} t \right)^{-1} dt := \overline{R}.$$

Let $R > \max\{\overline{R}, R_0\}$. Then for any $u \in \partial K_R$ and $\tau \geq 0,$ $u - Au \neq \tau e.$ Consequently, by using the fixed point index theory we have

$$i(A, K_R, K) = 0. \tag{3.6}$$

And by using the addition fixed point index theory and (3.3), (3.6), we have

$$i(A, K_R \setminus \overline{K}_r, K) = i(A, K_R, K) - i(A, K_r, K) = -1.$$

Hence A has a fixed point in $K_R \setminus \overline{K}_r,$ namely, the boundary value problem (1.1), (1.2) has at least one positive solution.

(ii) The proof is the same as case (i), and hence is omitted.

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