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MIXTURE PERIODIC AUTOREGRESSION WITH PERIODIC ARCH ERRORS

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Abstract

This paper explores some basic properties of mixture periodic autoregression with periodic *ARCH* errors (*MPAR-ARCH*), extending the *MAR-ARCH* model of Wong and Li [10], to capture the periodicity feature in the autocorrelation structure exhibited by many nonlinear time series. Our main focus is to provide the first and second moment periodic stationary conditions of this model. Furthermore, conditions for the existence of the fourth moments are established for some particular interesting cases. Closed-forms of these moments are obtained. *MLE* is carried out via the iterative *EM* algorithm, performance of which is shown via a simulation.

1. Introduction

It is recognized that autoregressive conditionally heteroskedastic (*ARCH*) models introduced by [5] and their generalized *GARCH* version [1] are the most used representations for modeling time-varying volatility exhibited by many financial time series. Various extensions of *ARCH*-type models have been proposed in the econometric literature in order to capture different additional features such as long memory and change in regime, while keeping stationarity. Other formulations intend to account for non stationarity by allowing the *ARCH* parameters to be time-varying [3]. In particular, the class of *GARCH* models with periodic time-varying

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parameters (*PGARCH*) introduced by [2] has shown to be appropriate for capturing periodicity in the stochastic conditional variance, a property that cannot be explained neither by the classical linear *ARMA* nor by the *GARCH* formulations. Franses and Paap [6] have combined the periodic *AR* (*PAR*) characterization with the *PGARCH* one to capture the periodicity in both the conditional mean and conditional variance, leading to the so-called *PAR-PGARCH* model which they successfully applied it to financial data. However, these models are shown to be inconsistent with other features exhibited by many time series with periodic structure, in particular the financial one, such as high kurtosis, outliers and extreme events. In order to capture these phenomena, Wong and Li [10] have proposed their mixture *AR-ARCH*. In the spirit of Wong and Li's formulation [10], our aim is to propose a model that is able to represent time series with periodicity as well as the mentioned features in both the conditional mean and conditional variance. Specifically, we propose a mixture of periodic autoregression with periodic *ARCH* error (*MPAR-ARCH*). Our model extends the mixture periodic autoregressive (*MPAR*) model proposed by [8] in which the conditional mean follows a mixture periodic *AR*. We then study some probabilistic properties of the proposed model such as second-order periodic stationarity, the existence of the finite moments and maximum likelihood estimation via the iterative *EM* algorithm.

The rest of this paper is organized as follows. In Section 2, we propose the class of mixture periodic *AR-ARCH* models. Section 3 studies the first and second moment stationary conditions. The explicit forms of these moments are, under these conditions, derived. In Section 4, conditions for the existence of the fourth moments are established for some particular interesting cases and the explicit forms of these fourth moments are obtained. In Section 5, parameter estimation is carried out by the maximum likelihood via the iterative *EM* algorithm. The performance of this algorithm is shown via simulation study in the last section.

2. Mixture Periodic *AR-ARCH* Model

Recall that a stochastic process $\{y_t;\, t \in \mathbb{Z}\}$ is said to have a *periodic AR representation* with periodic *ARCH* error of orders p and q and period

S (denoted by $PAR\text{-}ARCH_S(p, q)$), if it is given by

$$\begin{cases} y_t = \phi_{0,t} + \sum_{i=1}^{p} \phi_{i,t} y_{t-i} + \varepsilon_t, \quad t \in \mathbb{Z}, \\ \varepsilon_t = \sqrt{h_t}\, \zeta_t, \\ h_t := E(\varepsilon_t^2 / \mathcal{F}_{t-1}) = \beta_{0,t} + \sum_{j=1}^{q} \beta_{j,t} \varepsilon_{t-j}^2, \quad t \in \mathbb{Z}, \end{cases} \tag{2.1}$$

where $\{\zeta_t, t \in \mathbb{Z}\}$ is a sequence of independent and identically normally distributed (i.i.d.) random variables, with mean zero and unit variance, and where $\mathcal{F}_t$ denotes, as usual, the σ-algebra representing the available information up to time t. The parameters $\phi_{i,t}$ and $\beta_{j,t}$ are periodic in t, with period S, i.e., $\phi_{i,t+rS} = \phi_{i,t}$, $i = 0, \ldots, p$ and $\beta_{j,t+rS} = \beta_{j,t}$, $j = 0, \ldots, q$ such that $\beta_{0,t} > 0$ and $\beta_{j,t} \geq 0$, $j = 1, \ldots, q$, $\forall t$, $r \in \mathbb{Z}$.

The model *MPAR-ARCH* that we propose may be seen as mixture of models of type (2.1). The stochastic process $\{y_t; t \in \mathbb{Z}\}$ is then said to have a *mixture of K-component PAR-ARCH* with period S and orders $p_1, p_2, \ldots, p_K; q_1, q_2, \ldots, q_K$, denoted by $MPAR\text{-}ARCH_S(K; p_1, p_2, \ldots, p_K; q_1, q_2, \ldots, q_K)$, if it is given by

$$\begin{cases} F(y_t / \mathcal{F}_{t-1}) = \sum_{k=1}^{K} \lambda_k \Phi\left(\dfrac{\varepsilon_t^{(k)}}{\sqrt{h_t^{(k)}}} \right), \\ \varepsilon_t^{(k)} = y_t - \phi_{0,t}^{(k)} - \sum_{i=1}^{p_k} \phi_{i,t}^{(k)} y_{t-i}, \quad t \in \mathbb{Z}, \\ h_t^{(k)} = \beta_{0,t}^{(k)} + \sum_{j=1}^{q_k} \beta_{j,t}^{(k)} \varepsilon_{t-j}^{(k)2}, \quad t \in \mathbb{Z}, \end{cases} \tag{2.2}$$

where $\Phi(.)$ and $F(./\mathcal{F}_{t-1})$ are, respectively, the cumulative distribution function of the standard Gaussian distribution and the conditional cumulative distribution of y_t given the past information. As in (2.1), the parameters $\phi_{i,t}^{(k)}$ and $\beta_{j,t}^{(k)}$ are periodic in t, with period S, i.e., $\phi_{i,t+rS}^{(k)} = \phi_{i,t}^{(k)}$, $i = 0, \ldots, p_k$ and $\beta_{j,t+rS}^{(k)} = \beta_{j,t}^{(k)}$, $j = 0, \ldots, q_k$, $k = 1, \ldots, K$ and t, $r \in \mathbb{Z}$.

To avoid the possibility of zero or negative conditional variances, the *ARCH* parameters are set to be nonnegative, that is, $\beta_{0,t}^{(k)} > 0$ and $\beta_{j,t}^{(k)} \geq 0, j = 1, \ldots, q_k, k = 1, \ldots, K$ and $t \in \mathbb{Z}$. The constants λ_k, $k = 1, 2, \ldots, K$, are strictly positive real numbers such that $\sum_{k=1}^{K} \lambda_k = 1$.

For mathematical purposes p_k and q_k can be taken as constants in k merely set $p = \max_k p_k$, $q = \max_k q_k$ and take $\phi_{i,t}^{(k)} = 0$ for $i > p_k$ and $\beta_{j,t}^{(k)} = 0$ for $j > q_k$.

3. First and Second Moment Stationary Conditions

A standard problem which usually arises in studying the recurrence equations (2.2) is to search conditions for the existence of a first and/or second moment stationary solution. We first discuss conditions on the coefficients λ_k's and $\phi_{i,t}^{(k)}$'s which guarantee the existence of a stationary solution to equations (2.2) with finite first moments.

A. First moment stationary condition

It is easily seen that the unconditional mean, $E(y_t)$, of the process $\{y_t, t \in \mathbb{Z}\}$ satisfying the model (2.2), is given by

$$\mu_t := E(y_t) = a_0(t) + \sum_{i=1}^{p} a_i(t)\mu_{t-i}, \tag{3.1}$$

where

$$a_i(t) = \sum_{k=1}^{K} \lambda_k \phi_{i,t}^{(k)}, \; i = 0, 1, \ldots, p.$$

Thus, the unconditional periodic mean satisfies a periodic nonhomogeneous linear difference equation of order p.

In this subsection, we establish a necessary and sufficient condition for the linear difference equation (3.1) to have a finite solution. That is the first moment stationary condition for a general $MPAR\text{-}ARCH_S(K; p_1,$

$p_2, ..., p_K; q_1, q_2, ..., q_K)$ model. For this purpose, defining the following $p \times p$ matrices $A_{0,T}$ and $A_{1,T}$ given by

$$(A_{0,T})_{i,j} = \begin{cases} 1, & \text{if } i = j, \\ -a_{j-i}(pT - i + 1), & \text{if } i < j, \quad i, j = 1, 2, ..., p, \\ 0, & \text{if } i > j, \end{cases}$$

$$(A_{1,T})_{i,j} = \begin{cases} a_{p-i+j}(pT - i + 1), & \text{if } i \geq j, \\ 0, & \text{if } i < j \end{cases} \tag{3.2}$$

and letting $\mathbb{S}$ be a positive integer such that $p\mathbb{S}$ is the least common multiplier of p and the S, one can see that the matrices $A_{0,T}$ and $A_{1,T}$ are periodic, in T, with period $\mathbb{S}$. Using these notations, we are able to state the following result.

Proposition 3.1. *The process* $\{y_t; t \in \mathbb{Z}\}$ *satisfying the*

$$MPAR\text{-}ARCH_S(K;\ p_1,\ p_2, ...,\ p_K;\ q_1,\ q_2, ...,\ q_K)$$

model is periodically stationary in the first moment if and only if the roots of the determinantal equation (*of degree* $p = \max_k p_k$)

$$|\, Iz - \Psi \,| = 0, \quad z \in \mathbb{C},$$

lie inside the unit disc, *where* $\Psi = A_{0,\mathbb{S}}^{-1} A_{1,\mathbb{S}} A_{0,\mathbb{S}-1}^{-1} A_{1,\mathbb{S}-1} \cdots A_{0,1}^{-1} A_{1,1}$. *Furthermore*, *the closed-form expression of the first moment*, *under this condition*, *is given by*

$$\underline{\mu}_T = (I - \Psi)^{-1} \sum_{r=0}^{S-1} \left(\prod_{j=0}^{r-1} B_{T-j} \right) C_{T-r}, \quad T \in \mathbb{Z}, \tag{3.3}$$

where

$$\underline{\mu}_T = (\mu_{pT}, \mu_{pT-1}, ..., \mu_{pT-(p-1)})', \quad C_T = A_{T,0}^{-1} c_T,$$

$$B_T = A_{0,T}^{-1} A_{1,T}, \quad c_T = (a_0(pT), a_0(pT-1), ..., a_0(pT-p+1))'$$

and where empty product is set equal to identity.

Proof. (a) **Necessary and sufficient condition.** Defining, for $T \in \mathbb{Z}$, the p-variate periodic vectors $\underline{\mu}_T = (\mu_{pT}, \mu_{pT-1}, \ldots, \mu_{pT-(p-1)})'$ and $c_T = (a_0(pT), a_0(pT-1), \ldots, a_0(pT-p+1))'$, one can rewrite equation (3.1) in the p-variate $\mathbb{S}$-periodic first order linear difference equation

$$\underline{\mu}_T = B_T \underline{\mu}_{T-1} + C_T, \quad T \in \mathbb{Z}, \tag{3.4}$$

where the matrix $B_T = A_{0,T}^{-1} A_{1,T}$ and the vector column $C_T = A_{0,T}^{-1} c_T$, with $A_{0,T}$ and $A_{1,T}$ are given by (3.2). It is worth noting that B_T and C_T are $\mathbb{S}$-periodic such that $1 \le \mathbb{S} \le S$. In the particular case where $\mathbb{S} = 1$, equation (3.4) becomes a classical nonhomogeneous difference equation $\underline{\mu}_T = B\underline{\mu}_{T-1} + C$, $T \in \mathbb{Z}$, where B is clearly equal to $\Psi = A_{0,1}^{-1} A_{1,1}$. A necessary and sufficient condition for this last nonhomogeneous difference equation to have a finite solution is that the roots of the determinantal equation (of degree p) $|Iz - \Psi| = 0$, lie inside the unit disc.

Defining, for the general case where $1 < \mathbb{S} \le S$, for each $T \in \mathbb{Z}$, the $p\mathbb{S}$-variate vectors columns

$$\underline{\boldsymbol{\mu}}_\tau = (\underline{\mu}'_{1+\mathbb{S}\tau}, \underline{\mu}'_{2+\mathbb{S}\tau}, \ldots, \underline{\mu}'_{\mathbb{S}+\mathbb{S}\tau})'$$

and

$$\underline{C}_\tau = (\underline{C}'_{1+\mathbb{S}\tau}, \underline{C}'_{2+\mathbb{S}\tau}, \ldots, \underline{C}'_{\mathbb{S}+\mathbb{S}\tau})', \quad \tau \in \mathbb{Z},$$

one can rewrite (3.4) in the equivalent form

$$\underline{V}\underline{\boldsymbol{\mu}}_\tau = \underline{W}\underline{\boldsymbol{\mu}}_{\tau-1} + \underline{C}_\tau, \quad \tau \in \mathbb{Z}, \tag{3.5}$$

where the $p\mathbb{S} \times p\mathbb{S}$ matrices $\underline{V}$ and $\underline{W}$ are given as follows:

$$\underline{V}_{ij} = \begin{cases} I, & \text{if } i = j, \\ -B_i, & \text{if } i = j+1, \quad i, j = 1, 2, \ldots, \mathbb{S} \\ \underline{0}, & \text{otherwise,} \end{cases}$$

and

$$\underline{W}_{ij} = \begin{cases} B_1, & \text{if } i = 1 \text{ and } j = \mathbb{S}, \\ \underline{0}, & \text{otherwise,} \end{cases} \qquad i, j = 1, 2, \ldots, \mathbb{S}.$$

Since the matrix $\underline{V}$ is regular, we can rewrite (3.5) in the following classical nonhomogeneous difference equation of order 1

$$\underline{\boldsymbol{\mu}}_\tau = \underline{V}^{-1}\underline{W}\underline{\boldsymbol{\mu}}_{\tau-1} + \underline{V}^{-1}\underline{C}_\tau, \quad \tau \in \mathbb{Z}.$$

Therefore, a necessary and sufficient condition for this nonhomogeneous difference equation to have a finite solution is that the roots of the determinantal equation $| Iz - \underline{V}^{-1}\underline{W} | = 0$, lie inside the unit disc. One can easily verify that the matrix $\underline{V}^{-1}\underline{W}$ is given by

$$(\underline{V}^{-1}\underline{W})_{ij} = \begin{cases} \prod_{s=0}^{i-1} B_{i-s}, & \text{if } j = \mathbb{S}, \\ \underline{0}, & \text{otherwise,} \end{cases} \qquad i, j = 1, 2, \ldots, \mathbb{S}$$

and hence we have $\left| Iz - \prod_{j=0}^{\mathbb{S}-1} B_{\mathbb{S}-j} \right| = | Iz - \Psi | = 0.$

It is worth noting that this condition is equivalent to that established, in Theorem 2.1, by [8], for a mixture *PAR* of order p, which corresponds to our particular model $MPAR\text{-}ARCH_S(K;\ p,\ p, \ldots,\ p;\ 0,\ 0, \ldots,\ 0)$.

(b) **Explicit moment expression.** By $S - 1$ consecutive replacements in (3.4) and taking into account the periodicity of the matrix B_T and the vector column C_T, we find

$$(I - \Psi)\underline{\mu}_T = \sum_{r=0}^{S-1}\left(\prod_{j=0}^{r-1} B_{T-j}\right)C_{T-r}, \quad T \in \mathbb{Z}.$$

Therefore, the periodic vector $\underline{\mu}_T$ is, under the condition established in Proposition 3.1, given by (3.3).

Corollary 3.1. *The process* $\{y_t;\ t \in \mathbb{Z}\}$ *satisfying the*

$$MPAR\text{-}ARCH_S(K;\ 1,\ 1, \ldots,\ 1;\ q_1,\ q_2, \ldots,\ q_K)$$

is periodically stationary in the first moment if and only if

$$\left|\prod_{s=1}^{S}\left(\sum_{k=1}^{K}\lambda_k\phi_{1,s}^{(k)}\right)\right| < 1.$$

Furthermore, *the closed-form of the first moment*, *under this condition*, *is given by*

$$\mu_s = \frac{\sum_{r=0}^{S-1}\left(\prod_{j=0}^{r-1}\left(\sum_{k=1}^{K}\lambda_k\phi_{1,s-j}^{(k)}\right)\right)\left(\sum_{k=1}^{K}\lambda_k\phi_{0,s-r}^{(k)}\right)}{1-\prod_{s=1}^{S}\left(\sum_{k=1}^{K}\lambda_k\phi_{1,s}^{(k)}\right)}, \quad s = 1, 2, \ldots, S. \quad (3.6)$$

Proof. In this particular case where $p = 1$, we have $\mathbb{S} = S$, and the matrices $A_{0,T}$ and $A_{1,T}$ reduce to the scalars 1 and $a_1(T)$, respectively. Consequently $B_T = a_1(T)$ is periodic with period S, hence the necessary and sufficient condition is reduced to $\left|\prod_{s=1}^{S}\left(\sum_{k=1}^{K}\lambda_k\phi_{1,s}^{(k)}\right)\right| < 1.$ It is the same condition obtained in Theorem 2.2 by [8], for a mixture *PAR* of order 1, which corresponds to our particular model *MPAR-ARCH*$_S$ (K; 1, 1, ..., 1; 0, 0, ..., 0). On the other hand, after $S - 1$ successive replacements in (3.1), with $p = 1$, we obtain (3.6).

B. Second moment stationary condition

(1) Second moment stationary condition for *MPAR-ARCH*$_S$(K; 1, 1, ..., 1; $q_1, q_2, \ldots, q_K$) process: It is easy to show that the unconditional second moment of the process $\{y_t, t \in \mathbb{Z}\}$ satisfying (2.2), with $\max p_k = 1$, is given by

$$\begin{aligned} E(y_t^2) = & \sum_{k=1}^{K}\lambda_k(\beta_{0,t}^{(k)} + \phi_{0,t}^{(k)2} + 2\phi_{0,t}^{(k)}\phi_{1,t}^{(k)}\mu_{t-1}) \\ & + \sum_{k=1}^{K}\lambda_k\phi_{1,t}^{(k)2}E(y_{t-1}^2) \\ & + \sum_{k=1}^{K}\sum_{j=1}^{q}\lambda_k\beta_{j,t}^{(k)}E(\varepsilon_{t-j}^{(k)2}), \quad t \in \mathbb{Z}, \end{aligned} \quad (3.7)$$

where μ_t is the unconditional mean of y_t, and the expectation $E(\varepsilon_{t-j}^{(k)2})$ is given as

$$E(\varepsilon_{t-j}^{(k)2}) = \phi_{0,t-j}^{(k)2} - 2\phi_{0,t-j}^{(k)}\mu_{t-j} + 2\phi_{1,t-j}^{(k)}(\phi_{0,t-j}^{(k)} - a_0(t-j))\mu_{t-j-1} + E(y_{t-j}^2)$$

$$+ \phi_{1,t-j}^{(k)}[\phi_{1,t-j}^{(k)} - 2a_1(t-j)]E(y_{t-j-1}^2), \quad j = 1, ..., q \text{ and } t \in \mathbb{Z}.$$

Therefore, replacing $E(\varepsilon_{t-j}^{(k)2})$ by its expression in (3.7), we obtain

$$E(y_t^2) = \Lambda_{0,t} + \sum\nolimits_{j=1}^{q+1} \Lambda_{j,t} E(y_{t-j}^2), \quad t \in \mathbb{Z}, \tag{3.8}$$

where

$$\begin{aligned}\Lambda_{0,t} = {} & \sum_{k=1}^{K} \lambda_k (\beta_{0,t}^{(k)} + \phi_{0,t}^{(k)2}) + \sum_{k=1}^{K}\sum_{j=1}^{q} \lambda_k \beta_{j,t}^{(k)} \phi_{0,t-j}^{(k)2} \\ & + 2\sum_{k=1}^{K} \lambda_k \phi_{0,t}^{(k)} \phi_{1,t}^{(k)} \mu_{t-1} - 2\sum_{k=1}^{K}\sum_{j=1}^{q} \lambda_k \beta_{j,t}^{(k)} \phi_{0,t-j}^{(k)} \mu_{t-j} \\ & + 2\sum_{k=1}^{K}\sum_{j=1}^{q} \lambda_k \beta_{j,t}^{(k)} \phi_{1,t-j}^{(k)} (\phi_{0,t-j}^{(k)} - a_0(t-j))\mu_{t-j-1}\end{aligned}$$

and

$$\Lambda_{j,t} = \begin{cases} \displaystyle\sum_{k=1}^{K} \lambda_k (\phi_{1,t}^{(k)2} + \beta_{1,t}^{(k)}), & \text{if } j = 1, \\ \displaystyle\sum_{k=1}^{K} \lambda_k \left\{ \beta_{j,t}^{(k)} + \beta_{j-1,t}^{(k)} \phi_{1,t-j+1}^{(k)} \left[\phi_{1,t-j+1}^{(k)} - 2\sum_{l=1}^{K} \lambda_l \phi_{1,t-j+1}^{(l)} \right] \right\}, & \text{if } 2 \le j \le q, \\ \displaystyle\sum_{k=1}^{K} \lambda_k \beta_{q,t}^{(k)} \phi_{1,t-q}^{(k)} \left(\phi_{1,t-q}^{(k)} - 2\sum_{k=1}^{K} \lambda_k \phi_{1,t-q}^{(k)} \right), & \text{if } j = q+1. \end{cases}$$

Equation (3.8) is an S-periodic nonhomogeneous linear difference equation of order $q^* = q + 1$. To obtain a necessary and sufficient condition for this equation to have a finite solution, we follow the same way used in obtaining periodically first-order stationary condition (Proposition 3.1). Indeed, let $G_{0,T}$ and $G_{1,T}$ be $q^* \times q^*$, $\mathbb{S}$-periodic matrices, where $\mathbb{S}$ is such that $q^*\mathbb{S}$ is the least common multiplier of q^* and S, defined as follows:

$$(G_{0,T})_{i,j} = \begin{cases} 1, & \text{if } i = j, \\ -\Lambda_{j-i, q^*T-i+1}, & \text{if } i < j, \\ 0, & \text{if } i > j, \quad i, j = 1, 2, \dots, q^*. \end{cases}$$

$$(G_{1,T})_{i,j} = \begin{cases} \Lambda_{q^*-i+j, q^*T-i+1}, & \text{if } i \geq j, \\ 0, & \text{if } i < j, \end{cases}$$

Thus, a necessary and sufficient condition for a process satisfying an $MPAR\text{-}ARCH_S(K; 1, 1, \dots, 1; q_1, q_2, \dots, q_K)$ model to be periodically stationary in the second moment is given by the following proposition.

Proposition 3.2. *A necessary and sufficient condition for a periodically first-order stationary process* $\{y_t;\, t \in \mathbb{Z}\}$ *satisfying an* $MPAR\text{-}ARCH_S(K; 1, 1, \dots, 1; q_1, q_2, \dots, q_K)$ *model to be periodically stationary in the second moment is that the roots of the determinantal equation* (*of degree* $q^* = \max_k q_k + 1$)

$$| Iz - \Psi | = 0, \quad z \in \mathbb{C}$$

lie inside the unit disc, where $\Psi = G_{0,\mathbb{S}}^{-1} G_{1,\mathbb{S}} G_{0,\mathbb{S}-1}^{-1} G_{1,\mathbb{S}-1} \cdots G_{0,1}^{-1} G_{1,1}$. *Furthermore, the closed-form expression of the second moment, under this condition, is given by*

$$\underline{\pi}_T = (I - \Psi)^{-1} \sum_{r=0}^{S-1} \left(\prod_{j=0}^{r-1} D_{T-j} \right) H_{T-r}, \quad T \in \mathbb{Z}, \tag{3.9}$$

where

$$\underline{\pi}_T = (E(y^2_{q^*T}), E(y^2_{q^*T-1}), \dots, E(y^2_{q^*T-q^*+1}))',$$

$$H_T = G_{T,0}^{-1} c_T, \quad D_T = G_{0,T}^{-1} G_{1,T},$$

$$c_T = (\Lambda_{0,q^*T}, \Lambda_{0,q^*T-1}, \dots, \Lambda_{0,q^*T-q^*+1})'.$$

Proof. The proof is similar to that of Proposition 3.1, and will be omitted.

(2) Second moment stationary condition for $MPAR\text{-}ARCH_S(K;\, 2,\, 2,\, \ldots, 2; q_1, q_2, \ldots, q_K)$ process: In this paragraph, we establish a necessary and sufficient condition for the process $\{y_t,\, t \in \mathbb{Z}\}$ satisfying (2.2), with $p = \max_k p_k = 2$. In this case, it is easy to show that the unconditional second moment of y_t, is given by

$$E(y_t^2) = \sum_{k=1}^{K} \lambda_k (\phi_{0,t}^{(k)2} + 2\phi_{0,t}^{(k)}\phi_{1,t}^{(k)}\mu_{t-1} + 2\phi_{0,t}^{(k)}\phi_{2,t}^{(k)}\mu_{t-2} + \beta_{0,t}^{(k)})$$

$$+ \sum_{k=1}^{K} \lambda_k (\phi_{1,t}^{(k)2} E(y_{t-1}^2) + \phi_{2,t}^{(k)2} E(y_{t-2}^2) + 2\phi_{1,t}^{(k)}\phi_{2,t}^{(k)} E(y_{t-1}y_{t-2}))$$

$$+ \sum_{k=1}^{K} \sum_{j=1}^{q_k} \lambda_k \beta_{j,t}^{(k)} E(\varepsilon_{t-j}^{(k)2}), \tag{3.10}$$

where

$$E(\varepsilon_{t-j}^{(k)2}) = \phi_{0,t-j}^{(k)2} - 2\phi_{0,t-j}^{(k)}\mu_{t-j} + 2\phi_{0,t-j}^{(k)}\phi_{1,t-j}^{(k)}\mu_{t-j-1} + 2\phi_{0,t-j}^{(k)}\phi_{2,t-j}^{(k)}\mu_{t-j-2}$$

$$+ E(y_{t-j}^2) + \phi_{1,t-j}^{(k)2} E(y_{t-j-1}^2) + \phi_{2,t-j}^{(k)2} E(y_{t-j-2}^2) - 2\phi_{1,t-j}^{(k)} E(y_{t-j}y_{t-j-1})$$

$$+ 2\phi_{1,t-j}^{(k)}\phi_{2,t-j}^{(k)} E(y_{t-j-1}y_{t-j-2}) - 2\phi_{2,t-j}^{(k)} E(y_{t-j}y_{t-j-2}).$$

In this last equation, our aim is to eliminate

$$E(y_{t-j}y_{t-j-1}),\ E(y_{t-j}y_{t-j-2})$$

and

$$E(y_{t-j-1}y_{t-j-2}).$$

For this purpose, it is clear that

$$E(y_{t-j}y_{t-j-1}) = a_0(t-j)\mu_{t-j-1}$$

$$+ a_1(t-j)E(y_{t-j-1}^2) + a_2(t-j)E(y_{t-j-1}y_{t-j-2}),$$

and by $S-1$ consecutive replacements in this expression and taking into account the periodicity, we find

$$E(y_{t-j}y_{t-j-1})$$

$$= \frac{1}{1-\Phi_2}\sum_{r=0}^{S-1}\prod_{l=0}^{r-1}a_2(t-j-l)a_0(t-j-r)\mu_{t-j-r-1}$$

$$+\frac{1}{1-\Phi_2}\sum_{r=0}^{S-1}\prod_{l=0}^{r-1}a_2(t-j-l)a_1(t-j-r)E(y_{t-j-r-1}^2), \tag{3.11}$$

where

$$\Phi_2 = \prod_{s=1}^{S}a_2(s) = \prod_{s=1}^{S}\left(\sum_{k=1}^{K}\lambda_k\phi_{2,s}^{(k)}\right).$$

To calculate $E(y_{t-j-1}y_{t-j-2})$, it is enough to replace t by $t-1$ in (3.11). On the other hand, we can easily show that

$$E(y_{t-j}y_{t-j-2})$$

$$= a_0(t-j)\mu_{t-j-2} + a_2(t-j)E(y_{t-j-2}^2)$$

$$+\frac{a_1(t-j)}{1-\Phi_2}\sum_{r=0}^{S-1}\prod_{l=0}^{r-1}a_2(t-j-l-1)a_0(t-j-r-1)\mu_{t-j-r-2}$$

$$+\frac{a_1(t-j)}{1-\Phi_2}\sum_{r=0}^{S-1}\prod_{l=0}^{r-1}a_2(t-j-l-1)a_1(t-j-r-1)E(y_{t-j-r-2}^2).$$

Thus, from (3.10) we find, after some algebraic manipulation and taking into account the periodicity, that

$$E(y_t^2)$$

$$= \varphi_{0,t} + b_{1,1}(t)E(y_{t-1}^2) + b_{2,2}(t)E(y_{t-2}^2)$$

$$+\sum_{j=1}^{q}d_j(t)E(y_{t-j}^2) + \sum_{j=1}^{q}v_{1,1,j}(t-j,\, t)E(y_{t-j-1}^2)$$

$$+\sum_{j=1}^{q}(v_{2,2,j}(t-j,\, t) - 2e_{2,j}(t-j,\, t)a_2(t-j))E(y_{t-j-2}^2)$$

$$+2\sum_{r=0}^{S-1}\prod_{j=0}^{r-1}a_2(t-j-1)\frac{b_{1,2}(t)a_1(t-r-1)}{1-\Phi_2}E(y_{t-r-2}^2)$$

$$-2\sum_{j=1}^{q}\sum_{r=0}^{S-1}\prod_{l=0}^{r-1}a_2(t-j-l)\frac{e_{1,j}(t-j,t)a_1(t-j-r)}{1-\Phi_2}E(y_{t-j-r-1}^2)$$

$$+2\sum_{j=1}^{q}\sum_{r=0}^{S-1}\prod_{l=0}^{r-1}a_2(t-j-l-1)\frac{v_{1,2,j}(t-j,t)a_1(t-j-r-1)}{1-\Phi_2}E(y_{t-j-r-2}^2)$$

$$-2\sum_{j=1}^{q}\sum_{r=0}^{S-1}\prod_{l=0}^{r-1}a_2(t-j-l-1)\frac{e_{2,j}(t-j,t)a_1(t-j)a_1(t-j-r-1)}{1-\Phi_2}E(y_{t-j-r-2}^2), \tag{3.12}$$

where

$$\varphi_{0,t}$$

$$= b_{0,0}(t)+d_0(t)+\sum_{j=1}^{q}v_{0,0,j}(t-j,t)+2b_{0,1}(t)\mu_{t-1}+2b_{0,2}(t)\mu_{t-2}$$

$$-2\sum_{j=1}^{q}e_{0,j}(t-j,t)\mu_{t-j}+2\sum_{j=1}^{q}v_{0,1,j}(t-j,t)\mu_{t-j-1}$$

$$-2\sum_{j=1}^{q}\sum_{r=0}^{S-1}\prod_{l=0}^{r-1}a_2(t-j-l)\frac{e_{1,j}(t-j,t)a_0(t-j-r)}{1-\Phi_2}\mu_{t-j-r-1}$$

$$+2\sum_{k=1}^{K}\sum_{j=1}^{q}\lambda_k\beta_{j,t}^{(k)}\phi_{0,t-j}^{(k)}\phi_{2,t-j}^{(k)}\mu_{t-j-2}-2\sum_{j=1}^{q}e_{2,j}(t-j,t)a_0(t-j)\mu_{t-j-2}$$

$$+2\sum_{r=0}^{S-1}\prod_{l=0}^{r-1}a_2(t-l-1)\frac{b_{1,2}(t)a_0(t-r-1)}{1-\Phi_2}\mu_{t-r-2}$$

$$+2\sum_{j=1}^{q}\sum_{r=0}^{S-1}\prod_{l=0}^{r-1}a_2(t-j-l-1)\frac{v_{1,2,j}(t-j,t)a_0(t-j-r-1)}{1-\Phi_2}\mu_{t-j-r-2}$$

$$-2\sum_{j=1}^{q}\sum_{r=0}^{S-1}\prod_{l=0}^{r-1}a_2(t-j-l-1)\frac{e_{2,j}(t-j,t)a_1(t-j)a_0(t-j-r-1)}{1-\Phi_2}\mu_{t-j-r-2}$$

and

$$b_{i,j}(t) = \sum_{k=1}^{K} \lambda_k \phi_{i,t}^{(k)} \phi_{j,t}^{(k)}, \quad i,\ j = 1, \ldots, p;\ k = 1, \ldots, K \text{ and } t \in \mathbb{Z},$$

$$d_j(t) = \sum_{k=1}^{K} \lambda_k \beta_{j,t}^{(k)}, \quad j = 1, \ldots, q;\ k = 1, \ldots, K \text{ and } t \in \mathbb{Z},$$

$$e_{i,j}(t, h) = \sum_{k=1}^{K} \lambda_k \phi_{i,t}^{(k)} \beta_{j,h}^{(k)}, \quad i = 1, \ldots, p;\ j = 1, \ldots, q;\ k = 1, \ldots, K \text{ and } t, h \in \mathbb{Z},$$

$$v_{i,j,l}(t, h) = \sum_{k=1}^{K} \lambda_k \phi_{i,t}^{(k)} \phi_{j,t}^{(k)} \beta_{l,h}^{(k)}, \quad i,\ j = 1, \ldots, p;\ l = 1, \ldots, q;\ k = 1, \ldots, K \text{ and } t, h \in \mathbb{Z}.$$

Therefore, we can rewrite the nonhomogeneous linear difference equation (3.12) as

$$(1 - \varphi_{S,t}) E(y_t^2) = \varphi_{0,t} + \sum_{i=1}^{S-1} \varphi_{i,t} E(y_{t-i}^2), \tag{3.12}$$

where the coefficients $\varphi_{i,t}$, $i = 1, \ldots, S$ are given according to values' of S and q. It is worth noting that these coefficients are periodic, in time, with period S. Consequently, equation (3.12) is an S-periodic nonhomogeneous linear difference equation of order $q^* = S - 1$.

Let $G_{0,T}$ and $G_{1,T}$ be $q^* \times q^*$, S-periodic matrices, defined as follows:

$$(G_{0,T})_{i,j} = \begin{cases} 1 - \varphi_{S,t}, & \text{if } i = j, \\ -\varphi_{j-i,q^*T-i+1}, & \text{if } i < j, \\ 0, & \text{if } i > j, \quad i,\ j = 1, 2, \ldots, q^*. \end{cases}$$

$$(G_{1,T})_{i,j} = \begin{cases} \varphi_{q^*-i+j,q^*T-i+1}, & \text{if } i \geq j, \\ 0, & \text{if } i < j, \end{cases}$$

To obtain a necessary and sufficient condition for this equation to have a periodically stationary solution, we are able to state the following result.

Proposition 3.3. *Suppose that the process* $\{y_t;\ t \in \mathbb{Z}\}$ *follows an* $MPAR\text{-}ARCH_S(K;\ 2, 2, \ldots, 2;\ q_1, q_2, \ldots, q_K)$ *model, with* $S \geq 2$, *is periodically first-order stationary.*

If $\prod_{s=1}^{S}(1-\varphi_{S,s}) \neq 0$ *and* $1 - \Phi_2 \neq 0$, *then a necessary and sufficient condition for the process to be periodically stationary in the second moment is that the roots of the determinantal equation* (*of degree* $q^* = S - 1$)

$$|Iz - \Psi| = 0, \quad z \in \mathbb{C}$$

lie inside the unit disc, *where* $\Psi = G_{0,S}^{-1}G_{1,S}G_{0,S-1}^{-1}G_{1,S-1} \cdots G_{0,1}^{-1}G_{1,1}$. *Furthermore*, *the closed-form expression of the second moment*, *under this condition*, *is given by*

$$\underline{\pi}_T = (I - \Psi)^{-1}\sum_{r=0}^{S-1}\left(\prod_{j=0}^{r-1} D_{T-j}\right)H_{T-r}, \quad T \in \mathbb{Z},$$

where

$$\underline{\pi}_T = (E(y_{q^*T}^2), E(y_{q^*T-1}^2), \ldots, E(y_{q^*T-q^*+1}^2))', \quad H_T = G_{T,0}^{-1}c_T,$$

$$D_T = G_{0,T}^{-1}G_{1,T}, \quad c_T = (\varphi_{0,q^*T}, \varphi_{0,q^*T-1}, \ldots, \varphi_{0,q^*T-q^*+1})'.$$

Proof. The proof is similar to that of Proposition 3.1, hence it is omitted.

Now we turn to the classical case, i.e., $S = 1$, where the periodic coefficients difference equation (3.12) reduces to the following constant coefficient difference equation:

$$E(y_t^2) - \sum_{i=1}^{p+2} \varphi_i E(y_{t-i}^2) = \varphi_0,$$

where the coefficients φ_i, $i = 1, \ldots, p + 2$, are given by

$$\varphi_i = \begin{cases} b_{1,1} + d_1, & \text{if } i = 1, \\ b_{2,2} + v_{1,1,1} + d_2 + 2\dfrac{b_{1,2} - e_{1,1}}{1 - \Phi_2}a_1, & \text{if } i = 2, \\ d_i + v_{1,1,i-1} + v_{2,2,i-2} - 2e_{2,i-2}a_2 + 2\dfrac{v_{1,2,i-2} - e_{2,i-2}a_1 - e_{1,i-1}}{1 - \Phi_2}a_1, & \text{if } 3 \leq i \leq q, \\ v_{1,1,q} + v_{2,2,q-1} - 2e_{2,q-1}a_2 + 2\dfrac{v_{1,2,q-1} - e_{2,q-1}a_1 - e_{1,q}}{1 - \Phi_2}a_1, & \text{if } i = q + 1, \\ v_{2,2,q} - 2e_{2,q}a_2 + 2\dfrac{v_{1,2,q} - e_{2,q}a_1}{1 - \Phi_2}a_1, & \text{if } i = q + 2, \end{cases}$$

and

$$\varphi_0 = b_{0,0} + d_0 + \sum_{j=1}^{q} v_{0,0,j} + 2\left(b_{0,1} + b_{0,2} + \frac{b_{1,2}}{1-\Phi_2} a_0\right)\mu$$
$$+ 2\sum_{j=1}^{q}\left[v_{0,1,j} - e_{0,j} + v_{0,2,j} + \left(\frac{v_{1,2,j} - e_{2,j}a_1 - e_{1,j}}{1-\Phi_2} - e_{2,j}\right)a_0\right]\mu.$$

Corollary 3.2. *A necessary and sufficient condition for a first moment stationary process* $\{y_t; t \in \mathbb{Z}\}$ *satisfying the MAR-ARCH*$(K; 2, 2, \ldots, 2, q_1, q_2, \ldots, q_K)$ *model, such that* $\sum_{k=1}^{K} \lambda_k \phi_2^{(k)} \neq 1$, *to be second-order stationary is that the roots* $z_1, z_2, \ldots, z_{q+2}$ *of the equation*

$$z^{q+2} - \sum_{i=1}^{q+2} \varphi_i z^{q+2-i} = 0,$$

lie inside the unit disc, where $q = \max_k q_k$. *Furthermore, the closed-form expression of the second moment is, under this condition, given by*

$$E(y_t^2)$$
$$= \frac{\sum_{k=1}^{K} \lambda_k (\phi_0^{(k)2} + \beta_0^{(k)}) + \sum_{j=1}^{q}\sum_{k=1}^{K} \lambda_k \phi_0^{(k)2}\beta_j^{(k)}}{1 - \sum_{i=1}^{p+2} \varphi_i}$$
$$+ \frac{2\left(\sum_{j=1}^{q}\sum_{k=1}^{K} \lambda_k \phi_0^{(k)}(\phi_1^{(k)} + \phi_2^{(k)} - 1)\beta_j^{(k)}\right)\left(\sum_{k=1}^{K} \lambda_k \phi_0^{(k)}\right)}{\left(1 - \sum_{i=1}^{p+2} \varphi_i\right)\left(1 - \sum_{k=1}^{K} \lambda_k (\phi_1^{(k)} - \phi_2^{(k)})\right)}$$
$$+ \frac{2\left(\sum_{k=1}^{K} \lambda_k \phi_0^{(k)}(\phi_1^{(k)} + \phi_2^{(k)})\right)\left(\sum_{k=1}^{K} \lambda_k \phi_0^{(k)}\right) - 2\sum_{j=1}^{q}\sum_{k=1}^{K} \lambda_k \phi_2^{(k)}\beta_j^{(k)}\left(\sum_{k=1}^{K} \lambda_k \phi_0^{(k)}\right)^2}{\left(1 - \sum_{i=1}^{p+2} \varphi_i\right)\left(1 - \sum_{k=1}^{K} \lambda_k (\phi_1^{(k)} - \phi_2^{(k)})\right)}$$
$$+ \frac{2\left(\sum_{k=1}^{K} \lambda_k \phi_1^{(k)}\phi_2^{(k)}\right)\left(\sum_{k=1}^{K} \lambda_k \phi_0^{(k)}\right)^2 - 2\sum_{j=1}^{q}\sum_{k=1}^{K} \lambda_k \phi_1^{(k)}\beta_j^{(k)}\left(\sum_{k=1}^{K} \lambda_k \phi_0^{(k)}\right)^2}{\left(1 - \sum_{k=1}^{K} \lambda_k \phi_2^{(k)}\right)\left(1 - \sum_{i=1}^{p+2} \varphi_i\right)\left(1 - \sum_{k=1}^{K} \lambda_k (\phi_1^{(k)} - \phi_2^{(k)})\right)}$$
$$+ \frac{2\sum_{j=1}^{q}\left[\left(\sum_{k=1}^{K} \lambda_k \phi_1^{(k)}\phi_2^{(k)}\beta_j^{(k)}\right) - \left(\sum_{k=1}^{K} \lambda_k \phi_1^{(k)}\right)\left(\sum_{k=1}^{K} \lambda_k \phi_2^{(k)}\beta_j^{(k)}\right)\right]\left(\sum_{k=1}^{K} \lambda_k \phi_0^{(k)}\right)^2}{\left(1 - \sum_{k=1}^{K} \lambda_k \phi_2^{(k)}\right)\left(1 - \sum_{i=1}^{p+2} \varphi_i\right)\left(1 - \sum_{k=1}^{K} \lambda_k (\phi_1^{(k)} - \phi_2^{(k)})\right)}.$$

Proof. The second-order stationary condition is an immediate consequence of (3.21). Moreover, under this condition, an expression for the second-order moment is given by

$$E(y_t^2) = \varphi_0 \Big/ \left(1 - \sum_{i=1}^{q+2} \varphi_i\right).$$

Since, the mean of process, $\mu = E(y_t)$, can be obtained from equation (3.1) which reduces, for $p = 2$ and $a_i = \sum_{k=1}^{K} \lambda_k \phi_i^{(k)}$, $i = 0, 1, 2$, to a constant coefficients equation $\mu - a_1\mu - a_2\mu = a_0$, hence, we have $\mu = a_0/(1 - a_1 - a_2)$. Finally, the proof of the expression of $E(y_t^2)$ can be carried out by replacing μ by its value in the expression of φ_0.

Corollary 3.2 extends the result of the *MAR-ARCH*$(K; 1, 1, ..., 1; q_1, q_2, ..., q_K)$ case to the *MAR-ARCH*$(K;\ 2,\ 2,\ ...,\ 2;\ q_1, q_2, ..., q_K)$ case. It is worth noting that this corollary is reduced, for $\phi_2^{(k)} = 0$, $k = 1, ..., K$, to the well-known condition for the stationary of the second order given by [10] in Theorem 2.

4. Fourth Moment Stationary Condition

Like its counterpart for periodically second moment stationary condition, a necessary and sufficient condition for the existence of the fourth moment, $E(y_t^4)$, of a second moment stationary *MPAR-ARCH*$_S$ $(K; 0, 0, ..., 0; 2, 2, ..., 2)$ process is similar to the condition in Proposition 3.3 but with the periodic coefficients $\varphi_{i,t}$, $i = 0, ..., S$, of matrices $G_{0,T}$ and $G_{1,T}$ are replaced by

$$\varphi_{i,t} = \begin{cases} 3u_{0,0}(t) + 6u_{0,1}(t)E(y_{t-1}^2) + 6u_{0,2}(t)E(y_{t-2}^2) \\ \quad + 6\dfrac{u_{1,2}(t)\sum_{r=1}^{S}\left(\prod_{j=1}^{r-1} d_2(t-j)\right)d_0(t-r)E(y_{t-r-1}^2)}{1-\Psi_2}, & \text{if } i = 0, \\ 3\left[u_{1,1}(t) + 2\dfrac{u_{1,2}(t)d_1(t)\prod_{j=1}^{S-1} d_2(t-j)}{1-\Psi_2}\right], & \text{if } i = 1, \\ 3\left[u_{2,2}(t) + 2\dfrac{u_{1,2}(t)d_1(t-1)}{1-\Psi_2}\right], & \text{if } i = 2, \\ 6\dfrac{u_{1,2}(t)d_1(t-i+1)\prod_{j=1}^{i-2} d_2(t-j)}{1-\Psi_2}, & \text{if } 3 \le i \le S-1, \\ 6\dfrac{u_{1,2}(t)\left(\prod_{j=1}^{S-2} d_2(t-j)\right)d_1(t+1)}{1-\Psi_2}, & \text{if } i = S, \end{cases}$$

where

$$\Psi_2 = \prod_{s=1}^{S}\left(\sum_{k=1}^{K} \lambda_k \beta_{2,s}^{(k)}\right),$$

$$d_i(t) = \sum_{k=1}^{K} \lambda_k \beta_{i,t}^{(k)}, \quad i = 0, 1, 2$$

and

$$u_{i,j}(t) = \sum_{k=1}^{K} \lambda_k \beta_{i,t}^{(k)} \beta_{j,t}^{(k)}, \quad i, j = 0, 1, 2.$$

Using these notations, we are able to state the following result.

Proposition 4.1. *A necessary and sufficient condition for a second moment periodically stationary process* $\{y_t; t \in \mathbb{Z}\}$ *satisfying an* $MPAR\text{-}ARCH_S(K;\ 0, 0, \ldots, 0;\ 2, 2, \ldots, 2)$ *model, such that*

$$\prod_{s=1}^{S}(1 - \varphi_{S,s}) \neq 0, \quad \Psi_2 \neq 1 \text{ and } S > 2,$$

to have a finite fourth moment is that the roots of the determinantal equation (*of degree* $q^* = S - 1$)

$$|\, Iz - \Omega \,| = 0, \quad z \in \mathbb{C},$$

lie inside the unit disc, where $\Omega = G_{0,S}^{-1}G_{1,S}G_{0,S-1}^{-1}G_{1,S-1}\cdots G_{0,1}^{-1}G_{1,1}$.

Furthermore, the closed-form expression of the fourth moment, under this condition, is given by

$$\underline{\xi}_T = (I - \Omega)^{-1} \sum_{r=0}^{S-1} \left(\prod_{j=0}^{r-1} F_{T-j} \right) D_{T-r}, \quad T \in \mathbb{Z},$$

where

$$\underline{\xi}_T = (\xi_{q^*T}, \xi_{q^*T-1}, \ldots, \xi_{q^*T-q^*+1})', \quad \xi_t := E(y_t^4),$$

$$D_T = G_{0,T}^{-1} \underline{C}_T, \quad \underline{C}_T = (\varphi_{0,q^*T}, \varphi_{0,q^*T-1}, \ldots, \varphi_{0,q^*T-q^*+1})'$$

and $F_T = G_{0,T}^{-1} G_{1,T}$.

Proof. The proof can be easily done in the same way as we did in the corresponding part of the proof of Proposition 3.1.

Remark. If $S = 2$, then it can be verified that

$$(1 - \varphi_{2,t}) E(y_t^4) = \varphi_{1,t} E(y_{t-1}^4) + \varphi_{0,t}, \tag{4.1}$$

where the 2-periodic coefficients $\varphi_{i,t}$ are given by

$$\varphi_{i,t} = \begin{cases} 3u_{0,0}(t) + 6\left[\dfrac{u_{1,2}(t) d_0(t-1)}{1 - \Psi_2} + u_{0,2}(t)\right] E(y_t^2) \\ \qquad + 6\left[\dfrac{u_{1,2}(t) d_2(t-1) d_0(t)}{1 - \Psi_2} + u_{0,1}(t)\right] E(y_{t-1}^2), & \text{if } i = 0, \\ \dfrac{6u_{1,2}(t) d_2(t-1) d_1(t)}{1 - \Psi_2} + 3u_{1,1}(t), & \text{if } i = 1, \\ \dfrac{6u_{1,2}(t) d_1(t-1)}{1 - \Psi_2} - 3u_{2,2}(t), & \text{if } i = 2. \end{cases}$$

From equation (4.1), the necessary and sufficient condition for a second moment periodically stationary $MPAR\text{-}ARCH_2(K;\ 0, 0, \ldots, 0;\ 2, 2, \ldots, 2)$ process to have a finite fourth moment, can be easily derived in the same way as we did in the proof of Proposition 3.1, where we get

$$|\, \varphi_{1,2} \varphi_{1,1} / (1 - \varphi_{2,2})(1 - \varphi_{2,1}) \,| < 1.$$

Hence, the fourth-order moment of $MPAR\text{-}ARCH_2(K; 0, 0, \ldots, 0; 2, 2, \ldots, 2)$ process is, under this condition, given by

$$E(y_t^4) = [\varphi_{1,t}\varphi_{0,t-1} + (1-\varphi_{2,t-1})\varphi_{0,t}]/[(1-\varphi_{2,t})(1-\varphi_{2,t-1}) - \varphi_{1,t}\varphi_{1,t-1}].$$

Corollary 4.1. *A necessary and sufficient condition for a second moment periodically stationary process* $\{y_t;\ t \in \mathbb{Z}\}$ *satisfying an* $MPAR\text{-}ARCH_S(K; 0, 0, \ldots, 0; 1, 1, \ldots, 1)$ *model to have a finite fourth moment is that*

$$\prod_{j=1}^{S}\left(\sum_{k=1}^{K}\lambda_k\beta_{1,j}^{(k)2}\right) < \left(\frac{1}{3}\right)^S. \tag{4.2}$$

Furthermore, the closed-form of the fourth moment is, under this condition, given by

$$E(y_t^4) = \frac{\sum_{r=0}^{S-1}3^{r+1}\left(\prod_{j=0}^{r-1}\sum_{k=1}^{K}\lambda_k\beta_{1,t-j}^{(k)2}\right)\left[\sum_{k=1}^{K}\lambda_k\beta_{0,t-r}^{(k)2} + \left(\sum_{k=1}^{K}\lambda_k\beta_{0,t-r}^{(k)}\beta_{1,t-r}^{(k)}\right)\gamma_{t-r-1}(0)\right]}{1-3^S\prod_{j=1}^{S}\left(\sum_{k=1}^{K}\lambda_k\beta_{1,j}^{(k)2}\right)}, \tag{4.3}$$

where $\gamma_t(0)$ *as defined in* (3.9).

Proof. In the simple particular case where $q = 1,$ we have

$$\frac{1}{3}E(y_t^4) = u_{0,0}(t) + 2u_{0,1}(t)E(y_{t-1}^2) + u_{1,1}(t)E(y_{t-1}^4).$$

After $S-1$ successive replacements, in the last equation, we obtain

$$\left[1 - \prod_{s=1}^{S}3u_{1,1}(s)\right]E(y_t^4)$$

$$= \sum_{r=0}^{S-1}3^{r+1}\left(\prod_{j=0}^{r-1}u_{1,1}(t-j)\right)[u_{0,0}(t-r) + u_{0,1}(t-r)E(y_{t-r-1}^2)]. \tag{4.4}$$

Using (4.4), we can easily show that a necessary and sufficient condition for a second moment periodically stationary $MPAR\text{-}ARCH_S(K; 0, 0, \ldots, 0; 1, 1, \ldots, 1)$ process to have a finite fourth moment is as given by (4.2). Therefore, under the condition (4.2) we can find (4.3).

The following corollary establishes in one side, a necessary and sufficient condition for a second moment stationary classical mixture $MAR\text{-}ARCH(K; 0, 0, \ldots, 0; 2, 2, \ldots, 2)$ process to have a finite fourth moment and in the other side the closed-form of this fourth moment.

Corollary 4.2. *A necessary and sufficient condition for a second moment periodically stationary process* $\{y_t;\, t \in \mathbb{Z}\}$ *satisfying the* $MAR\text{-}ARCH(K;\, 0,\, 0,\, \ldots,\, 0;\, 2,\, 2,\, \ldots,\, 2)$ *model, such that* $\sum_{k=1}^{K} \lambda_k \beta_2^{(k)} \neq 1$, *to have a finite fourth moment is that the roots* z_1 *and* z_2 *of the equation*

$$z^2 - 3\left(\sum_{k=1}^{K} \lambda_k \beta_1^{(k)2}\right) z - 3\left[\sum_{k=1}^{K} \lambda_k \beta_2^{(k)2} + \frac{2\left(\sum_{k=1}^{K} \lambda_k \beta_1^{(k)} \beta_2^{(k)}\right)\left(\sum_{k=1}^{K} \lambda_k \beta_1^{(k)}\right)}{1 - \sum_{k=1}^{K} \lambda_k \beta_2^{(k)}}\right] = 0$$

lie inside the unit disc. Furthermore, the closed-form expression of the fourth moment is, under this condition, given by

$$E(y_t^4) = \frac{3}{1 - 3\left(\sum_{k=1}^{K} \lambda_k (\beta_1^{(k)2} + \beta_2^{(k)2}) + 2\,\frac{\left(\sum_{k=1}^{K} \lambda_k \beta_1^{(k)}\right)\left(\sum_{k=1}^{K} \lambda_k \beta_1^{(k)} \beta_2^{(k)}\right)}{1 - \sum_{k=1}^{K} \lambda_k \beta_2^{(k)}}\right)}$$

$$\times \left\{ \sum_{k=1}^{K} \lambda_k \beta_0^{(k)2} + 2\,\frac{\left(\sum_{k=1}^{K} \lambda_k \beta_0^{(k)}\right)\left(\sum_{k=1}^{K} \lambda_k \beta_0^{(k)} (\beta_1^{(k)} + \beta_2^{(k)})\right)}{1 - \sum_{k=1}^{K} \lambda_k (\beta_1^{(k)} + \beta_2^{(k)})} \right.$$

$$\left. + 2\,\frac{\left(\sum_{k=1}^{K} \lambda_k \beta_0^{(k)}\right)^2 \left(\sum_{k=1}^{K} \lambda_k \beta_1^{(k)} \beta_2^{(k)}\right)}{\left(1 - \sum_{k=1}^{K} \lambda_k (\beta_1^{(k)} + \beta_2^{(k)})\right)\left(1 - \sum_{k=1}^{K} \lambda_k \beta_2^{(k)}\right)} \right\}.$$

Proof. The proof is similar to that of Corollary 4.1, where it suffices to write $E(y_t^4)$ as

$$E(y_t^4) = \varphi_1 E(y_{t-1}^4) + \varphi_2 E(y_{t-2}^4) + \varphi_0,$$

where the coefficients φ_r's are given by

$$\varphi_r = \begin{cases} 3u_{0,0} + 6\left[u_{0,1} + u_{0,2} + \dfrac{u_{1,2}d_0}{1-d_2}\right]\gamma(0), & \text{if } r = 0, \\ 3u_{1,1}, & \text{if } r = 1, \\ 3u_{2,2} + \dfrac{6d_1u_{1,2}}{1-d_2}, & \text{if } r = 2. \end{cases}$$

5. Expectation-maximization Algorithm

The parameters of *MPAR-ARCH* model can be easily estimated using the *EM* algorithm [4], which is a broadly applicable approach to the iterative computation of maximum likelihood estimates. Within the incomplete-data framework of the *EM* algorithm, we let $\underline{y} = (y_1, y_2, \ldots, y_{NS}, Z_1, Z_2, \ldots, Z_{NS})$ and $Z = (Z_1, Z_2, \ldots, Z_{NS})$ denote the vector containing, respectively, the complete data and the missing data, where $Z_t = (Z_{1,t}, Z_{2,t}, \ldots, Z_{K,t})$ is a K-dimensional vector with component $Z_{k,t}$ equal to 1 if the observation y_t comes from the k-th component of the conditional distribution function, and 0 otherwise.

Let $\underline{\boldsymbol{\theta}} = (\underline{\lambda}', \Phi'_{1,1}, \theta'_{1,1}, \ldots, \Phi'_{1,S}, \theta'_{1,S}, \Phi'_{2,1}, \theta'_{2,1}, \ldots, \Phi'_{2,S}, \theta'_{2,S}, \ldots, \Phi'_{K,1}, \theta'_{K,1}, \ldots, \Phi'_{K,S}, \theta'_{K,S})'$ such that

$$\Phi_{k,s} = (\phi_{0,s}^{(k)}, \phi_{1,s}^{(k)}, \ldots, \phi_{p_k,s}^{(k)})', \quad \theta_{k,s} = (\beta_{0,s}^{(k)}, \beta_{1,s}^{(k)}, \ldots, \beta_{q_k,s}^{(k)})$$

and

$$\underline{\lambda} = (\lambda_1, \ldots, \lambda_{K-1})',$$

let also $t = s + \tau S$ for $s = 1, \ldots, S$, $\tau = 0, 1, \ldots, N-1$ and $p + q + 1 = s_0 + \tau_0 S$ for $1 \leq s_0 \leq S$,

$$\tau_0 = \begin{cases} \left\lfloor \dfrac{p+q+1}{S} \right\rfloor, & \text{if } s_0 \neq S \\ \left\lfloor \dfrac{p+q+1}{S} \right\rfloor - 1, & \text{if } s_0 = S \end{cases} \quad \text{and} \quad \tau_1 = \begin{cases} \tau_0, & \text{if } s \geq s_0, \\ \tau_0 + 1, & \text{if } s < s_0, \end{cases}$$

where $\lceil x \rceil$ denotes the largest integer less than or equal to x.

Following [9], [10], [11] and [8], for a given realization $y = (y_1, y_2, ..., y_{NS})$ of model (2.2), the log conditional likelihood function of the parameter vector $\underline{\boldsymbol{\theta}}$ can be expressed as follows:

$$\mathcal{L}_n(\underline{\boldsymbol{\theta}}) = \frac{-(p+q-NS)\ln(2\pi)}{2} \sum_{s=s_0}^{S} \sum_{k=1}^{K} z_{k,s+S\tau_0} \left(\ln \lambda_k - \frac{1}{2} \left(\ln h_{s+S\tau_0}^{(k)} + \frac{\varepsilon_{s+S\tau_0}^{(k)2}}{h_{s+S\tau_0}^{(k)}} \right) \right)$$

$$+ \sum_{\tau=\tau_0+1}^{N-1} \sum_{s=1}^{S} \sum_{k=1}^{K} z_{k,s+S\tau} \left(\ln \lambda_k - \frac{1}{2} \left(\ln h_{s+S\tau}^{(k)} + \frac{\varepsilon_{s+S\tau}^{(k)2}}{h_{s+S\tau}^{(k)}} \right) \right). \tag{5.1}$$

Let $\tau_{k,s+S\tau}$ denote the conditional expectation of k-th component of $Z_{s+S\tau}$ given the past information, for the current estimate $\underline{\boldsymbol{\theta}}^{(i)}$. Then $\tau_{k,s+S\tau}$ can be expressed, for $k = 1, ..., K$, $s = 1, ..., S$ and $\tau \in \mathbb{Z}$ as

$$\tau_{k,s+S\tau} = \frac{\lambda_k^{(i)}}{\sqrt{h_{s+S\tau}^{(k)}(\underline{\boldsymbol{\theta}}^{(i)})}} \phi\left(\frac{\varepsilon_{s+S\tau}^{(k)}}{\sqrt{h_{s+S\tau}^{(k)}(\underline{\boldsymbol{\theta}}^{(i)})}} \right) \Bigg/ \sum_{k=1}^{K} \frac{\lambda_k^{(i)}}{\sqrt{h_{s+S\tau}^{(k)}(\underline{\boldsymbol{\theta}}^{(i)})}} \phi\left(\frac{\varepsilon_{s+S\tau}^{(k)}}{\sqrt{h_{s+S\tau}^{(k)}(\underline{\boldsymbol{\theta}}^{(i)})}} \right),$$

where $\phi(.)$ is the probability density function of the standard normal random variable. Now we turn to the second step of the *EM* algorithm, where we suppose that the messing data are known. Then the parameter estimates can be obtained by equating $\partial\mathcal{L}_n(\underline{\boldsymbol{\theta}})/\partial\lambda_k$, $\partial\mathcal{L}_n(\underline{\boldsymbol{\theta}})/\partial\phi_{i,s}^{(k)}$ and $\partial\mathcal{L}_n(\underline{\boldsymbol{\theta}})/\partial\beta_{j,s}^{(k)}$ to zero for $i = 0, 1, ..., p_k$, $j = 0, 1, ..., q_k$ and $k = 1, ..., K$.

The first-order derivatives of (5.1), with respect to λ_k, $\phi_{i,s}^{(k)}$ and $\beta_{j,s}^{(k)}$, constrained by $\sum_{k=1}^{K} \lambda_k = 1$, are, respectively, given for $s = 1, ..., S$ by

$$\frac{\partial\mathcal{L}_n(\underline{\boldsymbol{\theta}})}{\partial\lambda_k} = \sum_{t=q+1}^{NS} \left(\frac{z_{k,t}}{\lambda_k} - \frac{z_{K,t}}{\lambda_K} \right), \quad k = 1, ..., K-1,$$

$$\frac{\partial\mathcal{L}_n(\underline{\boldsymbol{\theta}})}{\partial\phi_{i,s}^{(k)}} = \sum_{\tau=\tau_1}^{N-1} \frac{z_{k,s+S\tau}}{2h_{s+S\tau}^{(k)}} \frac{\partial h_{s+S\tau}^{(k)}}{\partial\phi_{i,s}^{(k)}} \left(\frac{\varepsilon_{s+S\tau}^{(k)2}}{h_{s+S\tau}^{(k)}} - 1 \right) - \frac{z_{k,s+S\tau}\varepsilon_{s+S\tau}^{(k)}}{h_{s+S\tau}^{(k)}} \frac{\partial\varepsilon_{s+S\tau}^{(k)}}{\partial\phi_{i,s}^{(k)}},$$
$$k = 1, ..., K;\ i = 0, ..., p_k,$$

$$\frac{\partial\mathcal{L}_n(\underline{\boldsymbol{\theta}})}{\partial\beta_{i,s}^{(k)}} = -\frac{1}{2} \sum_{\tau=\tau_1}^{N-1} \frac{z_{k,s+S\tau}}{h_{s+S\tau}^{(k)}} \frac{\partial h_{s+S\tau}^{(k)}}{\partial\beta_{i,s}^{(k)}} \left(1 - \frac{\varepsilon_{s+S\tau}^{(k)2}}{h_{s+S\tau}^{(k)}} \right), \quad k = 1, ..., K;\ j = 0, ..., q_k,$$

where

$$\frac{\partial h_{s+S\tau}^{(k)}}{\partial \phi_{i,s}^{(k)}} = 2\sum_{j=1}^{q_k} \beta_{j,s}^{(k)} \frac{\partial \varepsilon_{s+S\tau-j}^{(k)}}{\partial \phi_{i,s}^{(k)}} \varepsilon_{s+S\tau-j}^{(k)} \quad \text{for } i = 1, \ldots, p_k,$$

$$\frac{\partial h_{s+S\tau}^{(k)}}{\partial \beta_{0,s}^{(k)}} = 1, \quad \frac{\partial \varepsilon_{s+S\tau}^{(k)}}{\partial \phi_{0,s}^{(k)}} = -1, \quad \frac{\partial \varepsilon_{s+S\tau}^{(k)}}{\partial \phi_{i,s}^{(k)}} = -y_{s+S\tau-i} \quad \text{for } i = 1, \ldots, p_k$$

and

$$\frac{\partial h_{s+S\tau}^{(k)}}{\partial \beta_{j,s}^{(k)}} = \varepsilon_{s+S\tau-j}^{(k)2} \quad \text{for } j = 1, \ldots, q_k.$$

Hence the parameter estimates of λ are then given by

$$\hat{\lambda}_k = \frac{1}{NS - p - q} \sum_{t=p+q+1}^{NS} z_{k,t}, \quad \text{for } k = 1, \ldots, K-1.$$

For $\phi_{k,s}$'s and $\beta_{k,s}$'s, there are no explicit solutions of $\partial \mathcal{L}_n(\underline{\boldsymbol{\theta}})/\partial \phi_{k,s} = 0$ and $\partial \mathcal{L}_n(\underline{\boldsymbol{\theta}})/\partial \beta_{k,s} = 0$. For this reason, we propose to use the Newton-Raphson method, where we need to evaluate the second derivative of the log-likelihood $\mathcal{L}_n(\underline{\boldsymbol{\theta}})$. Similarly to [10], it is easy to show that the second-order derivatives of the log-likelihood with respect to the parameters $\phi_{i,s}^{(k)}$ and $\beta_{j,s}^{(k)}$ can be approximated, for $k = 1, \ldots, K$ and $s = 1, \ldots, S$, by the following quantities:

$$\frac{\partial^2 \mathcal{L}_n(\underline{\boldsymbol{\theta}})}{\partial \phi_{i,s}^{(k)} \partial \phi_{j,s}^{(k)}} \approx -\sum_{\tau=\tau_1}^{N-1} z_{k,s+S\tau} \left[\frac{1}{2h_{s+S\tau}^{(k)2}} \frac{\partial h_{s+S\tau}^{(k)}}{\partial \phi_{i,s}^{(k)}} \frac{\partial h_{s+S\tau}^{(k)}}{\partial \phi_{j,s}^{(k)}} + \frac{1}{h_{s+S\tau}^{(k)}} \frac{\partial \varepsilon_{s+S\tau}^{(k)}}{\partial \phi_{i,s}^{(k)}} \frac{\partial \varepsilon_{s+S\tau}^{(k)}}{\partial \phi_{j,s}^{(k)}} \right],$$
$$i, j = 0, \ldots, p_k,$$

$$\frac{\partial^2 \mathcal{L}_n(\underline{\boldsymbol{\theta}})}{\partial \beta_{i,s}^{(k)} \partial \beta_{j,s}^{(k)}} \approx -\sum_{\tau=\tau_1}^{N-1} \frac{z_{k,s+S\tau}}{2h_{s+S\tau}^{(k)2}} \frac{\partial h_{s+S\tau}^{(k)}}{\partial \beta_{i,s}^{(k)}} \frac{\partial h_{s+S\tau}^{(k)}}{\partial \beta_{j,s}^{(k)}}, \quad i, j = 0, \ldots, q_k,$$

$$\frac{\partial^2 \mathcal{L}_n(\underline{\boldsymbol{\theta}})}{\partial \phi_{i,s}^{(k)} \partial \beta_{j,s}^{(k)}} \approx 0, \quad i = 0, \ldots, p_k, \; j = 0, \ldots, q_k.$$

This suggests that starting with an initial value $\Phi_{k,s}^{(0)}$ and $\theta_{k,s}^{(0)}$, the values of $\Phi_{k,s}$ and $\theta_{k,s}$ in the subsequent iterations can be given for $k = 1, 2, ..., K$ and $s = 1, 2, ..., S$ as

$$\Phi_{k,s}^{(i+1)} = \Phi_{k,s}^{(i)} + \left[\frac{\partial^2 \mathcal{L}_n}{\partial \Phi_{k,s} \partial \Phi'_{k,s}} \Bigg|_{\Phi_{k,s}^{(i)},\, \theta_{k,s}^{(i)}} \right]^{-1} \left[\frac{\partial \mathcal{L}_n}{\partial \Phi_{k,s}} \Bigg|_{\Phi_{k,s}^{(i)},\, \theta_{k,s}^{(i)}} \right] \tag{5.2}$$

and

$$\theta_{k,s}^{(i+1)} = \theta_{k,s}^{(i)} + \left[\frac{\partial^2 \mathcal{L}_n}{\partial \theta_{k,s} \partial \theta'_{k,s}} \Bigg|_{\Phi_{k,s}^{(i+1)},\, \theta_{k,s}^{(i)}} \right]^{-1} \left[\frac{\partial \mathcal{L}_n}{\partial \theta_{k,s}} \Bigg|_{\Phi_{k,s}^{(i+1)},\, \theta_{k,s}^{(i)}} \right], \tag{5.3}$$

where $\Phi_{k,s}^{(i)}$ and $\theta_{k,s}^{(i)}$ are the values in the i-th iteration. The parameter estimates $\hat{\Phi}_{k,s}$ and $\hat{\theta}_{k,s}$ in a particular *M*-step iteration are obtained by iterating (5.2) and (5.3) until convergence. In practice, the $z_{k,s+S\tau}$'s are set equal to the $\tau_{k,s+S\tau}$'s from the previous *E*-step of the *EM* procedure.

6. Simulation Studies

In this section, the performances of the *EM* algorithm are studied. We have assessed on many different time series, generated from $MPAR\text{-}PARCH_S$ models of different orders and periods for a variety of sample sizes. For each model, we consider 1000 Monte Carlo replications. We report here only two of these simulation studies. The true values of parameters of each of the considered *MPAR-ARCH* data-generating processes, the mean and standard errors of their estimates for the 1000 replications are reported in Tables 6.1 and 6.2.

From Tables 6.1 and 6.2, one can easily note that the *EM* estimation method has small bias. It can be also observed that the empirical standard errors, of parameters estimations, have reasonable values. Thus, the performances of the *EM* algorithm, in the *MAR-ARCH* model case [10], and in the *MPAR* model case [8], are also met in our *MPAR-ARCH* case.

Table 6.1. Results of simulation study for an $MPAR\text{-}ARCH_2(2; 1, 1; 1, 1)$ model, with sample size $NS = 1000$ and number of replications equal to 1000

True value		0.2500		.7500
Mean of estimates	λ_1	0.2509	λ_2	.7491
Empirical s.e.		0.0231		.0231
True value		.6000		1.6000
Mean of estimates	$\phi_{0,1}^{(1)}$	.6217	$\phi_{0,1}^{(2)}$	1.5996
Empirical s.e.		.2714		.0735
True value		.9000		−.9000
Mean of estimates	$\phi_{1,1}^{(1)}$	.8879	$\phi_{1,1}^{(2)}$	−.8994
Empirical s.e.		.1075		.0172
True value		1.0000		.6000
Mean of estimates	$\beta_{0,1}^{(1)}$	1.0805	$\beta_{0,1}^{2}$	.5926
Empirical s.e.		.5194		.0967
True value		.7000		.4000
Mean of estimates	$\beta_{1,1}^{(1)}$	.7126	$\beta_{1,1}^{(2)}$	.3958
Empirical s.e.		.2662		.0326
True value		.9000		2.0000
Mean of estimates	$\phi_{0,2}^{(1)}$	.9037	$\phi_{0,2}^{(2)}$	2.0023
Empirical s.e.		.3362		.0889
True value		.3000		.7000
Mean of estimates	$\phi_{1,2}^{(1)}$	.3015	$\phi_{1,2}^{(2)}$	.6994
Empirical s.e.		.1171		.0191
True value		1.4000		.9000
Mean of estimates	$\beta_{0,2}^{(1)}$	1.3690	$\beta_{0,2}^{(2)}$	.8890
Empirical s.e.		.7495		.1560
True value		.5000		.8000
Mean of estimates	$\beta_{1,2}^{(1)}$	.4986	$\beta_{1,2}^{(2)}$	.7999
Empirical s.e.		.1285		.0009

Table 6.2. Results of simulation study for an $MPAR\text{-}ARCH_4(2;1,1;1,1)$ model, with sample size NS = 2000 and number of replications equal to 1000

True value		0.3000						.7000
Mean of estimates	λ_1	0.3008					λ_2	.6992
Empirical s.e.		0.0226						.0226
True value		.9000		−.9000		.7500		1.5000
Mean of estimates	$\phi_{0,1}^{(1)}$	.9214	$\phi_{0,2}^{(1)}$	−.9028	$\phi_{0,3}^{(1)}$	.7282	$\phi_{0,4}^{(1)}$	1.4782
Empirical s.e.		.2731		.3216		.2781		.4248
True value		.9000		.7000		−.6000		.7500
Mean of estimates	$\phi_{1,1}^{(1)}$	.8904	$\phi_{1,2}^{(1)}$	.6982	$\phi_{1,3}^{(1)}$	.5998	$\phi_{1,4}^{(1)}$	.7577
Empirical s.e.		.0944		.0909		.1294		.1717
True value		1.0000		5.0000		1.4000		10.0000
Mean of estimates	$\beta_{0,1}^{(1)}$	1.1353	$\beta_{0,2}^{(1)}$	4.8693	$\beta_{0,3}^{(1)}$	1.3574	$\beta_{0,4}^{(1)}$	9.9534
Empirical s.e.		.6689		1.2082		.6867		2.0317
True value		.5000		.4000		.9000		.3200
Mean of estimates	$\beta_{1,1}^{(1)}$	.4742	$\beta_{1,2}^{(1)}$	.3926	$\beta_{1,3}^{(1)}$	.9063	$\beta_{1,4}^{(1)}$	.3084
Empirical s.e.		.1541		.1639		.2010		.1596
True value		−1.0000		.2500		1.2000		−.7000
Mean of estimates	$\phi_{0,1}^{(2)}$	−1.0083	$\phi_{0,2}^{(2)}$	.2482	$\phi_{0,3}^{(2)}$	1.2040	$\phi_{0,4}^{(2)}$	−.6995
Empirical s.e.		.1086		.1039		.0770		.1058
True value		.3000		−.7000		.6500		.5000
Mean of estimates	$\phi_{1,1}^{(2)}$	.3050	$\phi_{1,2}^{(2)}$	−.6989	$\phi_{1,3}^{(2)}$	.6498	$\phi_{1,4}^{(2)}$	.4996
Empirical s.e.		.0549		.0527		.0303		.0402
True value		.9000		1.0000		.4000		.7000
Mean of estimates	$\beta_{0,1}^{(2)}$	.9163	$\beta_{0,2}^{(2)}$	1.0026	$\beta_{0,3}^{(2)}$	.4062	$\beta_{0,4}^{(2)}$	.7039
Empirical s.e.		.2200		.1922		.0942		.1436
True value		.7500		.4500		.5000		.9000
Mean of estimates	$\beta_{1,1}^{(2)}$	.7153	$\beta_{1,2}^{(2)}$	.4508	$\beta_{1,3}^{(2)}$	.5000	$\beta_{1,4}^{(2)}$	.8997
Empirical s.e.		.1161		.0401		.0025		.0044

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