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ON MULTIVARIATE ECONOMICAL QUALITY CONTROL

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Abstract

Methods are developed for economical quality control which are taken into account adjustment, inspection and deviation costs. A generalization of the method is given when a unit product is rejected if one or more characteristics are out of control under hypothesis that the characteristics are independent.

1. Introduction and Notation

The need for process regulation arises when the quality process is afflicted with disturbances that cause it to wander off target if no action is taken. In developing a model for economical quality control, therefore, we need a reasonably realistic representation of the deviation from target Z_t that would occur if no adjustment action is taken and a good approximation for the costs generated by the control process.

Consider a discrete production system where a multivariate quality process is monitored and where the control action is taken during a regular time interval.

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Given a quality degree ω, the quality of the tth observed item is denoted by $\mathbf{X}_t(w) = (X_{t1}(w), X_{t2}(w), \ldots, X_{tp}(w))$. It is a result of the quality of all the p characteristics, $X_{t1}(w), X_{t2}(w), \ldots, X_{tp}(w)$ simultaneously for which the target mean vector is $\mu = (\mu_1, \mu_2, \ldots, \mu_p)$. Centering this tth observation with respect to its target value defines a shift vector denoted by $\mathbf{Z}_t$ that affects all quality characteristics. The zero vector is the target vector that any quality process tries to realize, then

$$\mathbf{Z}_t = \mathbf{X}_t - \mu, \tag{1}$$

where $\mathbf{Z}_t$, $\mathbf{X}_t$ and μ are vectors of order $p \times 1$.

2. Disturbance Model

The need for process regulation is revealed when the system is exposed to disturbances that cause it to go off target if no action is taken. In developing a linear and multivariate control model, a realistic representation is needed for the shift off target $\mathbf{Z}_t$ of (1) with respect to its reference vector of values.

The expression of multivariate white noise $\mathbf{a}_t = (a_{t1}, a_{t2}, \ldots, a_{tp})$ is used to denote a sequence of random vectors iid of order $p \times 1$. The vector $\mathbf{a}_t$ specifies the innovation vector that is normally distributed, $\mathbf{a}_t \sim N(0, \mathbf{\Omega}_a)$ such that

$$\left.\begin{array}{l} E(\mathbf{a}_t) = 0, \\ E(\mathbf{a}_t \mathbf{a}'_s) = \mathbf{\Omega}_a \text{ pour } t = s, \\ E(\mathbf{a}_t \mathbf{a}'_s) = 0 \text{ pour } t \neq s, \end{array}\right\} \tag{2}$$

where $\mathbf{\Omega}_a$ is a positive definite and symmetrical matrix with order $p \times p$.

The simplest model for disturbances of a process in a state of control assumes that the shift vector of means is a sequence of vectorial white noise. Therefore,

$$\mathbf{Z}_t = \mathbf{a}_t. \tag{3}$$

A progressive tendency to go away from the reference vector implies dependence between successive shifts. A more general class of models for a disturbance, in which shifts are dependent can be written as weighted sum of these innovation vectors $\mathbf{Z}_t = \mathbf{a}_t + \psi_1\mathbf{a}_{t-1} + \psi_2\mathbf{a}_{t-2} + \cdots$ and ψ_i is the ith coefficient matrix of order $p \times p$. An important class of such models is stationary models for which the shift vector of means with respect to the reference is $\mathbf{0}$ and its covariance matrix is a finite matrix $\mathbf{\Omega}_Z$. Particular examples are vector autoregressive stationary models of order p, noted VAR(p) of the form $\mathbf{Z}_t = \varphi_1\mathbf{Z}_{t-1} + \varphi_2\mathbf{Z}_{t-2} + \cdots + \varphi_p\mathbf{Z}_{t-p} + \mathbf{a}_t$, in which the coefficients matrices φ_i are chosen to fulfill the preceding conditions of stationarity. These stationary disturbance models suffer from being impractical and unrealistic ones because if the quality processes are left to itself and no adjustments are made, then it would continue to vary about the same fixed reference vector of values. Conversely, if the process continuously moves away from the reference vector, such an assumption would generate unrealistic control procedures. A class of nonstationary models that can represent such shifting behavior is the vector autoregressive integrated moving average models noted VARIMA and discussed by Lutkepohl [5]. The simplest and most used of such models is the vector integrated moving average (VIMA) defined by the expression $(1 - B)\mathbf{Z}_t = (\boldsymbol{I} - \Theta B)\mathbf{a}_t$, where $\boldsymbol{I}$ is the identity matrix and B is the matrix lag operator. The eigenvalues, in absolute value, of the coefficient matrix Θ are smaller than the unit so that the process MA is invertible. As shown in Reinsel [6] and under the invertibility condition $\mathbf{Z}_t = (\boldsymbol{I} - \Theta)\sum_{i=1}^{+\infty}\Theta^{i-1}\mathbf{Z}_{t-i} + \mathbf{a}_t$, then

$$\mathbf{Z}_t = \hat{\mathbf{Z}}_t + \mathbf{a}_t, \tag{4}$$

where $\hat{\mathbf{Z}}_t$ is independent of the innovation vector $\mathbf{a}_t$ and is a vector EWMA-exponentially weighted moving average - of past observations like

$$\hat{\mathbf{Z}}_t = \Lambda(\mathbf{Z}_{t-1} + \Theta\mathbf{Z}_{t-2} + \Theta^2\mathbf{Z}_{t-3} + \cdots). \tag{5}$$

This vector is characterized by the matrix of nonstationarity coefficients $\Lambda = \boldsymbol{I} - \Theta$ defined on the basis of the smoothing matrix Θ.

Without loss of generality, Θ is a diagonal matrix whose entry values are such that $0 \leq \theta_j < 1$ and hence Λ is a diagonal matrix whose entries are taking the values $0 < \lambda_j \leq 1$. The coefficients $\Lambda, \Lambda\Theta, \Lambda\Theta^2, \ldots$ in equation (5) form a consistent series of diagonal matrices that sum to the identity matrix. This simplification is to avoid cointegration phenomena between the shift vector components $\mathbf{Z}_t$.

At time $(t-1)$ according to Reinsel [6], the value of $\hat{\mathbf{Z}}_t$ is an estimate of the minimum mean square error (MMSE) for one step. As a result $\hat{\mathbf{Z}}_t$ is an estimate of location of the series at time *t*. In particular, $\hat{\mathbf{Z}}_t$ is the MMSE forecast, at time $(t-1)$, of $\mathbf{Z}_t$. By algebraic manipulation of equations (4) and (5) we have

$$\hat{\mathbf{Z}}_{t+1} = \Lambda\mathbf{Z}_t + \Theta\hat{\mathbf{Z}}_t \quad \text{and} \quad \hat{\mathbf{Z}}_{t+1} - \hat{\mathbf{Z}}_t = \Lambda\mathbf{a}_t. \tag{6}$$

If a new observation vector $\mathbf{Z}_t$ becomes available, then the forecast can be updated using the recursive preceding equations (6). Unless Θ is very close to the identity matrix, the coefficient matrices in (5) converge rapidly to zero and in practice an adequate approximation to the vector EWMA is obtained by suitably truncating the series.

Moreover, it is possible to deduce from (4) and (5) that the 1st difference of the shift vector $\mathbf{Z}_t$ is the 1st order vector of moving average process

$$\mathbf{Z}_t - \mathbf{Z}_{t-1} = \mathbf{a}_t - \Theta\mathbf{a}_{t-1}. \tag{7}$$

From this model the process (4) may be referred to as a vector integrated moving average process.

If we go backward in time and the process is not initialized, then summing in equation (7) gives

$$\mathbf{Z}_t = \mathbf{a}_t + \Lambda\sum_{i=1}^{t-1}\mathbf{a}_i, \tag{8}$$

with Λ is a diagonal matrix such that the elements take the values $0 < \lambda_j \leq 1$. A multivariate quality process is not initialized means an adjustment or a setting of the production process is perfect generating a forecast for the shift vector $\hat{\mathbf{Z}}_1 = 0.$ The vector process (8) can be thought as an interpolation between the white noise disturbance vector (3) obtained when Λ tends to a zero matrix and a random walk vector model defined by

$$\mathbf{Z}_t = \sum_{i=1}^{t} \mathbf{a}_i, \tag{9}$$

obtained when Λ is an identity matrix.

3. Linear Multivariate Control Procedure

In many cases, the quality process depends on several variables that may wander away from the reference vector. However, maintaining these variables at a desired level may induce high production costs that make it unbearable, i.e., there is no incentive for the producer to continue processing. So, it is important as proposed by Taguchi [9] to consider production costs for inspection, adjustment and shift with respect to the reference.

In what follows and to simplify the notations, the ranking index of the characteristics will be noted in exponent.

The procedure begins by forecasting the vector shift in quality that results from the production process. As shown, the shift vector $\mathbf{Z}_t$ is dependent on the vector of white noise $\mathbf{a}_t$, disturbances generator, the specifications of $\mathbf{Z}_t$ will be used to calculate, $C(d, m)$, the quality control cost function. Minimizing the latter function determines the quality control parameters. Hence

- variation tolerance limit for each characteristic $X_t^j(w)$ is such that

$$|Z_t^j = X_t^j(w) - \mu^j| \leq d_j,$$

- and the intervention time expressed in terms of produced units m, at which the quality process is adjusted.

According to the shift vector, $\mathbf{Z}_t$, distribution the production process is adjusted to maintain the quality level as near as possible to a desired and a fixed reference.

The nature of the function considered and the limiting positions are decided on the basis of relative costs and not on the basis of statistical test procedures significance, hence the economical qualification.

3.1. Hypothesis

The economical procedure for detecting a shift in a multivariate quality process is based on a vector IMA process defined in (8). In fact, each manufactured item is inspected and the production process needs to be adjusted only when the quality vector $\mathbf{X}_t$ wanders away from the reference vector μ. As specified by (1), centering gives a process defined by equation (3).

The monitoring of the process helps in determining when to make an adjustment. Each adjustment is associated with a fixed cost noted C_A and each inspection is associated with a fixed cost noted C_I. The shift of the *j*th characteristic with respect to its reference μ^j generates a proportional cost which is measured by $\delta(Z_t^j)^2$, where δ is a proportionality factor.

If the monitoring process detects any exceeding value of the acceptable limit for one characteristic *j* such that $|Z_t^j = X_t^j(w) - \mu^j| \geq d_j$, a decision will be made to adjust the production process, hence the proportional cost is calculated on the basis of a shift detected in one characteristic. The adjacent hypothesis is that if an item does not satisfy one customer's requirement, then this item is considered defective, inducing a loss to the manufacturer proportional to the observed shift.

3.2. Observation process

The quality is observed directly, thus there are no measurement errors and the values of $\mathbf{Z}_t$ reflect real and precise shifts in the quality level. The observations are taken m units apart so the observed sample is $\mathbf{Z}_{im}$, where $i = 1, 2, \ldots, t$ or equivalently $\mathbf{Z}_{1m}, \mathbf{Z}_{2m}, \ldots, \mathbf{Z}_{tm}$. Having observed these vectors, we can use adequate statistical methods to obtain an appropriate forecasted valued vector of the future observation vector $\mathbf{Z}_{(t+1)m}$. This vector is denoted by $\hat{\mathbf{Z}}_{tm}$ whose components are $\mathbf{Z}_{1m}$, $\mathbf{Z}_{2m}$, ..., $\mathbf{Z}_{tm}$. The forecasted vector of the future observation $\mathbf{Z}_{(t+1)m}$ depends on the chosen model. Inspired of proceeding models that have been developed, the vector $\mathbf{Z}_t$ is a vector of integrated moving average process of order one, denoted by VIMA_1. For a VIMA_1 process, the best predicted vector $\hat{\mathbf{Z}}_{tm}$ of $\mathbf{Z}_{(t+1)m}$ that minimizes the mean squared error is an exponentially weighted moving average vector defined by equation (5) of past observation vectors $\mathbf{Z}_{1m}, \mathbf{Z}_{2m}, \ldots, \mathbf{Z}_{tm}$. Thus, an adjustment will be carried out as soon as the absolute value of the *j*th component of the forecasted vector $\hat{\mathbf{Z}}_{tm}$ exceeds its quality limit, say $\pm d_j$, for that specified characteristic. In other words, the production process is regulated at the *T*th observation, where

$$T = \min\{t \geq 1, \mathbf{Z}_t = (Z_t^1, Z_t^2, \ldots, Z_t^j, \ldots, Z_t^p) \text{ such that } |\hat{Z}_t^j| \geq d_j\}, \quad (10)$$

defining a random variable. The equation (10) implies that if $T = t$, then the *j*th characteristic exceeds the acceptable limit so the quality process is out of statistical control for the 1st time at the *t*th inspection, and the process accomplishes a cycle of length Tm.

It remains, not only, how to choose the control limit d_j, but also how many times the production system is inspected, i.e., how to fix out the sampling interval *m*. In order to find out the optimum values of the parameters d_j and *m*, a loss function is considered.

3.3. Loss function

Inspired from the loss function proposed in Box and Jenkins [1], the economical procedure formulated by Taguchi [9] is generalized to control a multivariate quality process in the following form:

$$C(d, m) = \frac{C_I}{m} + \frac{C_A}{mE(T)} + \frac{\delta}{mE(T)} E\left(\sum_{i=1}^{Tm} (Z_i^j)^2\right). \quad (11)$$

This function is named so because it measures the controlling costs of multivariate quality and evaluates the loss due to quality variability.

Minimizing the loss function (11) derives the best values of d_j and m. But to determine the optimum values of these parameters requires an evaluation of the average adjustment interval

$$\text{AAI} = mE(T) \tag{12}$$

and the mean squared shift

$$\text{MSD} = \frac{E\left(\sum_{i=1}^{Tm} (Z_i^j)^2\right)}{\text{AAI}} \tag{13}$$

during a production cycle.

4. VIMA_1 Model of m-units Apart Observations

After each adjustment of the production process, the VIMA_1 model is defined by

$$\mathbf{Z}_t = \mathbf{Z}_0 + \mathbf{a}_t + \Lambda \sum_{i=1}^{t-1} \mathbf{a}_i, \tag{14}$$

where Λ is a matrix that presents the nonstationarity coefficients with $\Lambda = \text{diag}\{\lambda_1, \lambda_2, ..., \lambda_p\}$ such that for all j, $0 < \lambda_j \leq 1$. The disturbances $\mathbf{a}_t$ form a sequence of random walk vectors whose distribution is multivariate normal and according to (2) the mean vector is null and covariance matrix is $\mathbf{\Omega}_a$. This innovation vector $\mathbf{a}_t$ is independently distributed from $\mathbf{Z}_0$, which has an $N(\mathbf{0}, \mathbf{\Omega}_0)$ distribution. If $\mathbf{\Omega}_0 = 0$, then $\mathbf{Z}_0 = \mathbf{0}$ and the adjustment of the process is considered perfect as shown in (8) of an uninitialized VIMA_1 process.

4.1. A characteristic Z_t^j model

Knowing that the components of the disturbance $\mathbf{a}_t$ are independent, each element j of the vector process (8) forms an IMA process of order one, characterized with equation

$$Z_t^j = Z_0^j + a_t^j + \lambda_j \sum_{i=1}^{t-1} a_i^j. \tag{15}$$

When each produced item is inspected, the process is said to be *uninitialized*, then $Z_0^j = 0$. In this case, the process IMA_1 defined in equation (15) is identical to the jth equation of process (14). This process may, in analogous manner, be specified by the subsequent properties

$$\begin{cases} V(Z_1^j) = V(a_1^j) = \sigma_j^2, \\ V(\nabla Z_t^j) = [2(1-\lambda_j) + \lambda_j^2]\sigma_j^2, \\ \text{Cov}(\nabla Z_t^j, \nabla Z_{(t-1)}^j) = -(1-\lambda_j)\sigma_j^2, \\ \text{Cov}(\nabla Z_t^j, \nabla Z_{(t-k)}^j) = 0, \quad \forall k \geq 2, \end{cases}$$

with $V(\cdot)$ and $\text{Cov}(\cdot)$ indicate, respectively, the variance and covariance operators. We may note that there is no reason, a priori, to differently weigh the components of the disturbance, so $\forall j$, $\lambda_j = \lambda$ and as a result,

$$\left. \begin{array}{l} V(Z_1^j) = V(a_1^j) = \sigma_j^2, \\ V(\nabla Z_t^j) = [2(1-\lambda) + \lambda^2]\sigma_j^2, \\ \text{Cov}(\nabla Z_t^j, \nabla Z_{(t-1)}^j) = -(1-\lambda)\sigma_j^2, \\ \text{Cov}(\nabla Z_t^j, \nabla Z_{(t-k)}^j) = 0, \; \forall k \geq 2. \end{array} \right\} \tag{16}$$

Equation (16) above may be used to define an IMA_1 process with nonstationarity parameter λ and variance $\sigma_a^2(j) = \sigma_j^2$.

In the considered monitoring system, observations of the jth quality component are taken m units apart, specifically we have $Z_{1m}^j, Z_{2m}^j, \ldots,$ $Z_{tm}^j, \ldots$. In this case, we assume that the initial variable Z_0^j is not null, has a small variance, such that $Z_0^j \sim N(0, \sigma_0^2)$. Consider parameters

(λ_m, σ_m^2), defined by

$$\left.\begin{aligned} \lambda_m &= \frac{m\lambda^2}{2(1-\lambda)}\left[\left(1+\frac{4(1-\lambda)}{m\lambda^2}\right)^{1/2}-1\right], \\ \sigma_m^2 &= \frac{m\lambda^2\sigma_j^2}{\lambda_m^2}. \end{aligned}\right\} \tag{17}$$

Moreover, the variance of Z_0^j is considered small which has the following form:

$$\sigma_0^2 = (1-\lambda)(\lambda_m - \lambda)\sigma_j^2. \tag{18}$$

Consider the observing produced unit interval above, inspecting each mth item according to the jth characteristic gives the sequence Z_{1m}^j, $Z_{2m}^j, \ldots, Z_{tm}^j, \ldots$. Analogously to the work of Srivastava [7], this sequence is specified by

$$\left.\begin{aligned} &V(Z_m^j) = \sigma_m^2, \\ &V(\nabla Z_{tm}^j) = [2(1-\lambda_m) + \lambda_m^2]\sigma_m^2, \\ &\mathrm{Cov}(\nabla Z_{tm}^j, \nabla Z_{(t-1)m}^j) = -(1-\lambda_m)\sigma_m^2, \\ &\mathrm{Cov}(\nabla Z_{tm}^j, \nabla Z_{(t-k)m}^j) = 0, \ \ \forall k \geq 2. \end{aligned}\right\} \tag{19}$$

Hence the independence between disturbance components $(a_t^1, a_t^2, \ldots, a_t^j, \ldots, a_t^p)'$ and the observation interval m stipulate that on the basis of the alternative definition (16), the sequence $Z_{1m}^j, Z_{2m}^j, \ldots, Z_{tm}^j, \ldots$ forms an IMA_1 process with nonstationarity parameter λ_m and variance σ_m^2 as defined in equation (17).

In other words, it is possible to write

$$Z_{tm}^j = u_t^j + \lambda_m \sum_{i=1}^{t-1} u_i^j \quad \text{such that } Z_m^j = u_1^j,$$

with $\{u_i^j\}$ is iid $N(0, \sigma_m^2)$ sequence.

4.2. Linearized loss function

It is shown by Srivastava [7] that the predicted value of $Z^j_{(t+1)m}$ is given by

$$\hat{Z}^j_{tm} = \lambda_m [Z^j_{tm} + \theta_m Z^j_{(t-1)m} + \cdots + \theta_m^{t-1} Z^j_m],$$

$$\hat{Z}^j_{tm} = \lambda_m Z^j_{tm} + \theta_m \hat{Z}^j_{(t-1)m}, \tag{20}$$

where $\theta_m = 1 - \lambda_m$, defines an exponentially weighted moving average process of observations $Z^j_{1m}, Z^j_{2m}, \ldots, Z^j_{tm}, \ldots$. So $\hat{Z}^j_{tm}$ is a random walk with mean zero and variance $\lambda_m^2 \sigma_m^2 = m\lambda^2\sigma_j^2$. Standardizing this changes the variable defined in (10) to

$$\tau = \min\left\{ t \geq 1 \text{ telle que } \left| R_t = \frac{\hat{Z}^j_{tm}}{\lambda\sigma_j\sqrt{m}} \right| \geq \nu_j \right\},$$

where $\nu_j = \dfrac{d_j}{\lambda\sigma_j\sqrt{m}} = \left(\dfrac{h_j}{m}\right)^{1/2}$ with $h_j = \left(\dfrac{d_j}{\lambda\sigma_j}\right)^2$ denote the control limits crossing interval in terms of produced units.

In order to obtain linear form of the cost function (11), the mean squared shift of (13) may be simplified using the result of Srivastava and Wu [8], as follows:

$$E\left(\sum_{i=1}^{\tau m} (Z_i^j)^2\right) = \lambda^2\sigma_j^2\left[\frac{m^2 E(\tau^2)}{6} + \frac{mE(\tau)}{2}\right] + m\sigma_j^2(1-\lambda)(1+\lambda_m)E(\tau),$$

where the approximations of $E(\tau)$ and $E(\tau^2)$ are evaluated in Srivastava and Wu [8] by

$$E(\tau) = \begin{cases} \nu^2 + 2\rho_1\nu + \rho_2, & \nu \geq 1, \\ 1 + \varepsilon_1\nu + \varepsilon_2\nu^2 + \varepsilon_3\nu^3, & \nu < 1, \end{cases}$$

$$E(\tau^2) = \begin{cases} \nu^4 + 4\rho_1\nu^3 + 6\rho_2\nu^2 + 4\rho_3\nu + \rho_4, & \nu \geq 1, \\ 3(1 + \varepsilon_1\nu + \varepsilon_2\nu^2 + \varepsilon_4\nu^3), & \nu < 1, \end{cases}$$

with $\rho_1 = 0.583, \rho_2 = 0.590, \rho_3 = 0.796, \rho_4 = 1.320, \varepsilon_1 = 0.798, \varepsilon_2 = 0.637,$ $\varepsilon_3 = 0.375$ and $\varepsilon_4 = 0.907$. Thus, including the simplified form of the squared shift alters the cost function of equation (11). According to Box and Jenkins [1] specifications, two types of cost function are obtained.

4.2.1. 1st case: machine tool process where $h_j \geq m$

If the control limits crossing interval exceeds the inspection interval in terms of manufactured units, the loss function is linearized using the first approximations of the moments, $E(\tau)$ and $E(\tau^2)$, then

$$C(d, m) \approx \frac{C_I}{m} + \frac{C_A}{h_j + 2\rho_1 (mh_j)^{1/2} + \rho_2 m}$$

$$+ \delta\lambda^2\sigma_j^2 \left[\frac{(1-\lambda)(1+\lambda_m)}{\lambda^2} + \frac{1}{6}(h_j + 2\rho_1 (mh_j)^{\frac{1}{2}} + \rho_2 m) + \frac{m}{6} + \frac{1}{2} \right]. \quad (21)$$

4.2.2. 2nd case: chemical process where $h_j < m$

If the control limits crossing interval does not exceed the inspection interval in terms of manufactured units, the loss function is linearized using the second approximations of the moments, $E(\tau)$ and $E(\tau^2)$, then

$$C(d, m) \approx \frac{C_I}{m} + \frac{C_A}{m}\left(1 - \varepsilon_1 \left(\frac{h_j}{m}\right)^{1/2} + (2\varepsilon_1\varepsilon_2 - \varepsilon_3)\left(\frac{h_j}{m}\right)^{2/3}\right)$$

$$+ \delta\lambda^2\sigma_j^2 \left[\frac{m}{2}\left(1 + \frac{2}{3}\varepsilon_1 \left(\frac{h_j}{m}\right)^{3/2}\right) + \frac{(1-\lambda)(1+\lambda_m)}{\lambda^2} + \frac{1}{2} \right]. \quad (22)$$

5. Optimum Values of the Control Parameters

The optimum values of the control parameters $(d_j^\star, m^\star)$ can be obtained by minimizing (11), using adequate numerical methods. But to get explicit expressions for these values, the approximation given in (21) or in (22) may generate good solutions.

Knowing that the derivative of the nonstationarity parameter defined in (17) is

$$\lambda'_m = \frac{\lambda_m}{m} - \frac{1}{m}\left(1 + \frac{4(1-\lambda)}{m\lambda^2}\right)^{\frac{1}{2}}$$

such that $\left(\frac{4(1-\lambda)}{m\lambda^2}\right) < 1$, minimizing (21) or (22) leads to the required solution.

5.1. The case of machine tool process

Using the same procedure of Hajlaoui and Limam [4], minimizing (21) gives:

- an optimal inspection interval

$$m^{\star} = \max\left\{1, \left(6\,\frac{C_I}{\delta\lambda^2\sigma_j^2}\right)^{\frac{1}{2}}\left[1 - \frac{\delta\sigma_j^2(1-\lambda)^2}{\lambda^2 C_I}\right]^{\frac{1}{2}}\right\}, \tag{23}$$

- an optimal control limit

$$d_j^{\star} = \lambda\sigma_j\left[\left(6\,\frac{C_A}{\delta\lambda^2\sigma_j^2}\right)^{\frac{1}{4}} - \rho_1 m^{\star\frac{1}{2}}\right]. \tag{24}$$

5.2. The case of chemical process

As Hajlaoui and Limam [4], minimizing (22) gives:

- an optimal inspection interval

$$m^{\star} = \max\left\{1, \left[2\left(\frac{C_I + C_A}{\delta\lambda^2\sigma_j^2} - \frac{(1-\lambda)^2}{\lambda^4}\right)\right]^{\frac{1}{2}}\right\}, \tag{25}$$

- an optimal control limit

$$d_j^{\star} = \left[\frac{C_A}{\delta m^{\star}}\right]^{\frac{1}{2}}. \tag{26}$$

6. Conclusion

Determining the optimal values of the control parameters yields considerable gains for the firm. These gains come from controlling the costs of quality control by considering the economical aspect of this control. This means to rationalize the intervention during processing items in either case to adjust or to inspect the production process. In order to control costs generated by inspection or by adjustment actions, the firm is warned out through the process (20) of a possible decrease in quality level of the *j*th characteristic. It is also told of how much to adjust the production process that it is necessary to do for the characteristic of concern in order to maintain the quality level as near as possible to the reference value.

To avoid reiteration of outputs, the production process is monitored twice:

• supervising the process according to the parameters $(d_j^\star, m^\star)$, this is done by $\hat{Z}^j_{tm^\star}$ such that

$$\hat{Z}^j_{tm^\star} = \lambda_{m^\star} Z^j_{tm^\star} + (1 - \lambda_{m^\star}) \hat{Z}^j_{(t-1)m^\star}, \tag{27}$$

• verifying the *j*th characteristic or marking operation through (10),

$$T = \min\{t \geq 1, \mathbf{Z}_t = (Z_t^1, Z_t^2, \ldots, Z_t^j, \ldots, Z_t^p) \text{ such that} : |\hat{Z}_t^j| \geq d_j\}.$$

Hence, if $\hat{Z}^j_{tm^\star}$ exceeds $\pm d_j^\star$ at the *t*th inspection, the process is adjusted by $-\hat{Z}^j_{tm^\star}$ to maintain the *j*th characteristic of the multivariate quality in the limits. But if the *j*th marked characteristic changes, the process (27) must be reviewed.

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