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ON SUMS OF TWO OR THREE PENTAGONAL NUMBERS

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Abstract

Let $P_j(n)$ denote the number of representations of n as a sum of j pentagonal numbers. We obtain formulas for $P_j(n)$ when $j = 2$ and $j = 3$.

1. Introduction

The pentagonal numbers are defined for $n \in Z$ by $\omega(n) = n(3n-1)/2$. They occur frequently in the theory of partitions, notably in Euler's recurrence for the partition function $p(n)$ for $n \geq 1$, which may be written as

$$\sum_{k=-\infty}^{\infty} (-1)^k p(n - \omega(k)) = 0.$$

The first few pentagonal numbers (corresponding to $n = 0, \pm1, \pm2, \pm3, \pm4$) are: 0, 1, 2, 5, 7, 12, 15, 22, 26. If the integers $j \geq 1$ and $n \geq 0$, let $P_j(n)$ denote the number of representations of n as a sum of j pentagonal numbers. Representations that differ only in the order of terms are considered distinct, that is, we are counting *compositions* of n whose

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summands are pentagonal numbers. In this note, we obtain formulas for $P_j(n)$, where $j = 2, 3$.

2. Preliminaries

Let the integers $n \geq 0$, $k \geq 0$, $j \geq 1$, $m \geq 2$, $0 \leq i \leq m - 1$.

Definitions. $d_{i,m}(n)$ is the number of positive integers d such that $d \mid n$ and $d \equiv i \pmod{m}$

$r_j(\alpha)$ is the number of representations of α as a sum of j squares of integers if α is a non-negative integer (and is zero otherwise)

$s_j(n)$ is the number of representations of n as a sum of j squares of integers that are coprime to 6.

$t_k = k(k+1)/2$ (the kth triangular number)

$t_j(n)$ is the number of representations of n as a sum of j triangular numbers.

Proposition 1. $\frac{1}{4} r_2(n) = d_{1,4}(n) - d_{3,4}(n)$.

Proposition 2. *Let*

$$n = 2^a \prod_{i=1}^{r} p_i^{e_i} \prod_{j=1}^{s} q_j^{f_j},$$

where $a \geq 0$, *all primes* $p_i \equiv 1 \pmod{4}$, *all primes* $q_j \equiv 3 \pmod{4}$. *Then*

$$\frac{1}{4} r_2(n) = \begin{cases} \prod_{i=1}^{r} (e_i + 1) & \text{if all } f_j \equiv 0 \pmod{2} \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 3. $t_2(n) = d_{1,4}(4n+1) - d_{3,4}(4n+1)$.

Proposition 4. *Let* a, b *be positive integers such that* $(a, b) = 1$. *Let* $n \geq 0$. *Then the sequence* $\{an + b\}$ *contains infinitely many primes.*

Remarks. Proposition 1 is attributed to Jacobi. (See [5, p. 15, Theorem 2].) Proposition 2 is equivalent to Proposition 1. (See [6, p. 166, Theorem 3.22], where a slightly different notation is used.) Proposition 3 appears in both [2] and [7]. Proposition 4 is Dirichlet's celebrated theorem on primes in arithmetic progressions.

3. The Main Results

We begin with a lemma that links representations of non-negative integers as sums of pentagonal numbers to representations of non-negative integers as sums of squares.

Lemma 1. *Let* $k \geq 1,\ n \geq 0.$ *Then* $P_k(n) = s_k(24n + k).$

Proof.

$$n = \sum_{i=1}^{k} \frac{x_i(3x_i \pm 1)}{2} \leftrightarrow 24n = \sum_{i=1}^{k} (36x_i^2 \pm 24x_i) \leftrightarrow 24n + k = \sum_{i=1}^{k} (6x_i \pm 1)^2.$$

Our next result is a formula for $P_2(n)$.

Theorem 1.

$$P_2(n) = \frac{1}{4} r_2(12n + 1) = d_{1,4}(12n + 1) - d_{3,4}(12n + 1).$$

Proof. Lemma 1 implies $P_2(n) = s_2(24n + 2).$ Now

$$24n + 2 = x^2 + y^2 \to x^2 + y^2 \equiv 2 \pmod 4 \to (x, 2) = (y, 2) = 1.$$

Also

$$24n + 2 = x^2 + y^2 \to x^2 + y^2 \equiv 2 \pmod 3 \to (x, 3) = (y, 3) = 1.$$

Therefore $(x, 6) = (y, 6) = 1,$ so $s_2(24n + 2) = \frac{1}{4} r_2(24n + 2).$ But Proposition 1 implies $r_2(24n + 2) = r_2(12n + 1),$ so we are done. □

The next theorem states a property of $\frac{1}{4} r_2(n)$ that seems not to have been previously noticed, although it follows from Proposition 2. We offer a proof that is based instead on Proposition 1.

Theorem 2. $\frac{1}{4}r_2(n)$ *is a multiplicative function.*

Proof. Let

$$\chi_2(n) = \begin{cases} 1 & \text{if } n \equiv 1 \pmod 4 \\ -1 & \text{if } n \equiv 3 \pmod 4 \\ 0 & \text{otherwise.} \end{cases}$$

Then $\chi_2(n)$ is a Dirichlet character $(\bmod 4)$, and is therefore multiplicative. (See [1, Theorem 6.15, p. 138].) Let $u(n) = 1$ be the unit function. Invoking Proposition 1, we have

$$\frac{1}{4}r_2(n) = d_{1,4}(n) - d_{3,4}(n) = \sum_{d \mid n} \chi_2(d) = \sum_{d \mid n} \chi_2(d)u(n/d) = (\chi_2 * u)(n),$$

where $*$ denotes the Dirichlet product. Since each of $\chi_2(n)$, $u(n)$ is multiplicative, it follows that $(\chi_2 * u)(n)$ is multiplicative. (See [1, Theorem 2.14, p. 35].) □

We now present an alternate proof of Theorem 1, based on a connection between sums of pentagonal numbers and sums of triangular numbers.

Lemma 2. $P_2(n) = t_2(3n)$.

Proof.

$$n = \frac{x(3x \pm 1)}{2} + \frac{y(3y \pm 1)}{2} \leftrightarrow 3n = \frac{3x(3x \pm 1)}{2} + \frac{3y(3y \pm 1)}{2},$$

that is, n is a sum of two pentagonal numbers if and only if $3n$ is a sum of two triangular numbers that are multiples of 3. But if t_i is a triangular number, then we must have $t_i \equiv 0, 1 \pmod 3$. Therefore if $t_i + t_j \equiv 0 \pmod 3$, it follows that $t_i \equiv t_j \equiv 0 \pmod 3$. Thus there are no representations of $3n$ as a sum of two triangular numbers that are not both multiples of 3. The conclusion now follows. □

Alternate Proof of Theorem 1. This follows immediately from Lemma 2 and Proposition 3. □

The next theorem concerns solutions of the equation $P_2(n) = k$, where k is a given non-negative integer.

Theorem 3. *For every non-negative integer k, there are infinitely many n such that $P_2(n) = k$.*

Proof.

Case 1 $k = 0$**.** By Proposition 4, there are infinitely many primes, q, such that $q \equiv 7 \pmod{12}$. For each such pair q_1, q_2 of distinct primes, let $n = ((q_1 q_2) - 1)/12$. Then, by Theorem 1, $P_2(n) = \frac{1}{4} r_2(q_1 q_2) = 0$.

Case 2 $k = 1$**.** By Proposition 4, there are infinitely many primes, q, such that $q > 3$ and $q \equiv 3 \pmod 4$. For each such prime, q, let $n = (q^2 - 1)/12$. Then $P_2(n) = \frac{1}{4} r_2(q^2) = 1$.

Case 3 $k \geq 2$**.** By Proposition 4, there are infinitely many primes, q, such that $q \equiv 1 \pmod 4$. For each such prime, q, and each $k \geq 2$, let $n = (q^{k-1} - 1)/12$. Then $P_2(n) = \frac{1}{4} r_2(q^{k-1}) = k$. □

We now present a formula for sums of three pentagonal numbers.

Theorem 4.

$$P_3(n) = \frac{1}{8}\left\{r_3(24n + 3) - r_3\left(\frac{8n + 1}{3}\right)\right\}.$$

Proof. Lemma 1 implies $P_3(n) = s_3(24n + 3)$. Let $24n + 3 = x^2 + y^2 + z^2$. Clearly, x, y, z are all odd, and either all of them or none of them are multiples of 3. If $n \not\equiv 1 \pmod 3$, then $x^2 + y^2 + z^2 \equiv \pm 3 \pmod 9$, so $xyz \not\equiv 0 \pmod 3$. Therefore $s_3(24n + 3) = \frac{1}{8} r_3(24n + 3)$. (The factor $\frac{1}{8}$ occurs because $r_3(n)$ counts squares of both positive and negative integers.) If $n \equiv 1 \pmod 3$, then $x^2 + y^2 + z^2 \equiv 0 \pmod 9$, so it is possible that $x = 3r$, $y = 3s$, $z = 3t$. In this case, we have $r^2 + s^2 + t^2 = \frac{8n + 1}{3}$.

Since we do not wish to count such representations, we have $s_3(24n+3)$ $= \frac{1}{8}\left\{r_3(24n+3) - r_3\left(\frac{8n+1}{3}\right)\right\}$. The conclusion now follows, recalling the definition of $r_j(\alpha)$. □

Remarks. $r_3(n)$ may be computed in either of the following ways:

(1) Let $R_3(n)$ denote the number of primitive representations of n as a sum of three squares, that is, $n = x^2 + y^2 + z^2$, where $GCD(x, y, z) = 1$. Then

$$R_3(n) = \begin{cases} 12h(\sqrt{-n}) & \text{if } n \equiv 1, 2, 5, 6 \pmod 8 \\ 24h(\sqrt{-4n}) & \text{if } n \equiv 3 \pmod 8, \end{cases}$$

where $h(d)$ denotes the class number of an imaginary quadratic field of discriminant d, and

$$r_3(n) = \sum_{d^2 \mid n} R_3\left(\frac{n}{d^2}\right).$$

(See [4, p. 187, Theorem 7.8].)

(2) Let $q_0(n)$ denote the number of self-conjugate partitions of n (or the number of partitions of n into distinct, odd parts). Then

$$r_3(n) = \sum_{k=-\infty}^{\infty} (-1)^{\omega(k)}(1-6k)q_0(n-\omega(k)).$$

This identity is due to Ewell [3].

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