Far East Journal of Mathematical Sciences (FJMS)
Volume 28, Issue 2, 2008, Pages 417-430
Published online: March 11, 2008
This paper is available online at http://www.pphmj.com

DIMENSIONS OF SUBSPACES OF THE POLYNOMIAL ALGEBRA $\mathbf{F}_2[x_1, \ldots, x_n]$ GENERATED BY SPIKES

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Abstract

Let $n \geq 1$ and $d \geq 0$ be integers. Let $\mathbf{P}(n) = \mathbf{F}_2[x_1, \ldots, x_n]$ be the polynomial algebra in n-variables x_i over the field $\mathbf{F}_2$ of two elements and let $\mathcal{A}$ be the mod-2 Steenrod algebra. In this paper we give a formula for computing $B(n, d)$, the number of degree d monomials of the form $x_1^{2^{\lambda_1}-1} x_2^{2^{\lambda_2}-1} \cdots x_n^{2^{\lambda_n}-1}$ (called spikes) in $\mathbf{P}(n)$. Motivation for this question comes from algebraic topology where $\mathbf{P}(n)$ is identified with the mod-2 cohomology group of the n-fold product of $\mathbf{R}P^{\infty}$ with itself and thereby receives its module structure over $\mathcal{A}$. The value of $B(n, d)$ is of interest in the problem of determining a basis for the quotient vector space $\mathbf{P}(n)/\mathcal{A}^{+}\mathbf{P}(n)$ of the polynomial algebra by the image of the positive part $\mathcal{A}^{+}$ of the Steenrod algebra [1, 11, 13].

1. Introduction

For $n \geq 1$, let $\mathbf{P}(n)$ be the mod-2 cohomology group of the n-fold product of $\mathbf{R}P^{\infty}$ with itself. Then $\mathbf{P}(n)$ is the polynomial algebra $\mathbf{P}(n) = \mathbf{F}_2[x_1, \ldots, x_n]$ in n variables x_i over the field $\mathbf{F}_2$ of two elements.

2000 Mathematics Subject Classification: 05E99, 55S10.

Keywords and phrases: dimension, spike, polynomial algebra.

Received September 28, 2007

The mod-2 Steenrod algebra $\mathcal{A}$ acts on $\mathbf{P}(n)$ according to well-known rules. A polynomial u is said to be *hit* if it belongs to the set

$$\mathcal{A}^{+}\mathbf{P}(n) = \left\{ \sum_{i>0} Sq^{i}u_{i} \mid u_{i} \in \mathbf{P}(n), \quad Sq^{i} \in \mathcal{A} \right\}.$$

The papers [3, 7, 8, 9, 11, 12] are concerned with the problem of determining the set $\mathcal{A}^{+}\mathbf{P}(n)$. A closely related problem is that of determining the vector space dimension of the quotient $\mathbf{C}(n) = \mathbf{P}(n)/\mathcal{A}^{+}\mathbf{P}(n)$. Motivation for these problems stems from a conjecture by Peterson [5], proved in [12] and various other sources [6, 10].

To put our work into context we recall that $\mathbf{P}(n)$ has natural grading by degree d, that is, $\mathbf{P}(n) = \oplus_{d \geq 0}\mathbf{P}_{d}(n)$, where $\mathbf{P}_{d}(n)$ denotes the homogeneous polynomials of degree d. The quotient vector space $\mathbf{C}(n)$ inherits similar grading by d.

If $d \geq 0$ can be expressed in the form $d = \sum_{i=1}^{n}(2^{\lambda_i} - 1)$, $\lambda_i \geq 0$, then $\mathbf{P}_{d}(n)$ contains monomials $x_1^{2^{\lambda_1}-1} \cdots x_n^{2^{\lambda_n}-1}$ called *spikes*. It is known that a spike can never appear as a term in a hit polynomial. Thus spikes must be included in any generating set for $\mathbf{C}_{d}(n)$ so that if $\mathbf{P}_{d}^{*}(n)$ is the subspace of $\mathbf{P}_{d}(n)$ generated by spikes and $\dim(\mathbf{P}_{d}^{*}(n))$ is the vector space dimension of this subspace then $B(n, d) = \dim(\mathbf{P}_{d}^{*}(n))$ is a lower bound for the vector space dimension of $\mathbf{C}_{d}(n)$. Apart from a few special cases there is no simple formula for computing $B(n, d)$. Our main objective in this paper is to derive an effective recursive method for computing $B(n, d)$ that finds application in certain choices of d. This however is deferred until Section 3.

We note that the dimension of $\mathbf{C}_{d}(n)$ is completely known if $n = 1, 2, 3$ (the case $n = 3$ being determined by Kameko [2]). Apart from a few cases where equality holds (for example $n = 1$, $d = 1$ or $n \geq 2$ and $d = 2$) $B(n, d)$ is strictly lower than the dimension of $\mathbf{C}_{d}(n)$. Thus further work is required to improve the lower bounds for the dimension of $\mathbf{C}_d(n)$ that may be obtained with the aid of the results of this paper.

It may well happen that $\mathbf{P}_d(n)$ contains no spikes. We quote some results about hit polynomials that in particular show that $\mathbf{C}_d(n) = 0$ if $\mathbf{P}_d(n)$ contains no spikes.

The results [12, 11, 3, 7, 9] all identify classes of hit monomials and are special cases of the following result of Silverman, which was conjectured in [9].

Let $b = x_1^{e_1} \cdots x_n^{e_n}$ be a monomial of degree d. Write $e_i = \sum_{j\geq 0} \alpha_j(e_i)2^j$ for the binary expansion of each exponent e_i and for each $j \geq 0$ let $w_j(b) = \sum_{i=1}^{n} \alpha_j(e_i)$. If $l \geq 0$ is an integer, define $d_l(b)$ to be the integer $d_l(b) = \sum_{j\geq l} w_j(b)2^{j-l}$. Then $d_0(b) = d$ and $d_0(b) \geq d_1(b) \geq d_2(b) \geq \cdots$.

Let $\mathcal{S}$ be the set of all ordered sequences $s = (\lambda_1, \lambda_2, \lambda_3, \ldots)$ of non-negative integers almost all of which are 0. The dimension of a sequence $|s| = \sum_{i=1}^{\infty} (2^{\lambda_i} - 1)$. If $|s| = d$, then s is called a *representation* of d [9].

Given $s \in \mathcal{S}$ and $j \geq 0$ let $w_j(s) = \sum_{i\geq 1} \alpha_j(2^{\lambda_i} - 1)$. For $l \geq 0$ define $d_l(s)$ to be the integer $d_l(s) = \sum_{j\geq l} w_j(s)2^{j-l}$. Then each d has a unique *minimal representation* $\tilde{s}(d)$ such that $d_l(\tilde{s}(d)) \geq d_l(s)$ for all $s \in \mathcal{S}$ of degree d and all $l \geq 0$ [9].

Finally define $d_l(d) = d_l(\tilde{s}(d))$ for $l \geq 0$. Then:

Theorem 1.1 [8]**.** *Let* $b \in \mathbf{P}(n)$ *be a monomial of degree* d*. If* $d_l(b) > d_l(d)$ *for some* $l \geq 1$*, then* b *is hit.*

The main result of Wood [12] is the case $l = 1$ of Theorem 1.1. The result of Wood, in particular, identifies degrees d for which all elements in $\mathbf{P}_d(n)$ are hit, that is, degrees d for which $\mathbf{C}_d(n) = 0$. Let $\alpha(m)$ denote the number of digits 1 in the binary expansion of m. If $b = x_1^{e_1} \cdots x_n^{e_n} \in \mathbf{P}_d(n)$ and $\alpha(n+d) > n$, then $d_1(b) > d_1(d)$. Hence b is hit.

It is known that $\alpha(n+d) \leq n$ if and only if there exists at least one representation $s = (\lambda_1, \lambda_2, \lambda_3, \ldots)$ of d, with $\lambda_i = 0$ for all $i > n$ [11]. We identify such a sequence s with $(\lambda_1, \ldots, \lambda_n)$ and following Singer we refer to each such a sequence as a *representation of d as n-sharp*.

It follows that $\alpha(n+d) \leq n$ if and only if $\mathbf{P}_d(n)$ contains spikes or, equivalently, if and only if the subset

$$S(n, d) = \left\{(\lambda_1, \ldots, \lambda_n) \mid d = \sum_{i=1}^{n} (2^{\lambda_i} - 1), \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0\right\}$$

of $\mathcal{S}$ is nonempty. Then $B(n, d)$ may equivalently be defined by

$$B(n, d) = \sum_{(\lambda_1, \ldots, \lambda_n) \in S(n, d)} |(\lambda_1, \ldots, \lambda_n)|,$$

where $|(\lambda_1, \ldots, \lambda_n)|$ is the number of distinct permutations of the sequence $(\lambda_1, \ldots, \lambda_n) \in S(n, d)$.

2. Preliminary Results

We first note that there is a simple recursive method for computing $B(n, d)$. For all $d \geq 0$

$$B(1, d) = \begin{cases} 1 & \text{if } d = 2^j - 1 \text{ for some } j \in \mathbf{N} \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

and for $n \geq 2$ and $d \geq 0$

$$B(n, d) = \sum_{2^j - 1 \leq d} B(n-1, d - (2^j - 1)). \tag{2}$$

Equation (1) follows from the definition of a spike and (2) follows from the fact that every spike m of degree d factors uniquely as $m = u \cdot x_n^{2^j - 1}$ where u is a spike in $\mathbf{P}_{d-(2^j-1)}(n-1)$ and $j \geq 0$.

We note also that there are simple formulae for computing $B(n, d)$ in the cases $\alpha(n+d) = n$ and $\alpha(n+d) = n-1$. If $\alpha(n+d) = n$, then $B(n, d) = n!$. On the other hand if $\alpha(n+d) = n-1$ so that $n \geq 2$ we write $d + n = \sum_{i=1}^{n-1} 2^{\sum_{s=1}^{i} i_s}$, $i_1 \geq 0$ and $i_s \geq 1$, $2 \leq s \leq n-1$, and let $R(d) = \{i_s \in \{i_2, \ldots, i_{n-1}\} \mid i_s = 1\}$. Then

$$B(n, d) = \begin{cases} k\left(\frac{n!}{3!}\right) + (n-2-k)\left(\frac{n!}{2!}\right) & \text{if } i_1 = 0 \\ k\left(\frac{n!}{3!}\right) + (n-1-k)\left(\frac{n!}{2!}\right) & \text{if } i_1 > 0, \end{cases} \tag{3}$$

where $k = |R(d)|$.

Our procedure is based on the following results.

Let m, $1 \leq m \leq n$, be an integer. Then $\alpha(n+d) = m$ if and only if there exist integers i_s, $1 \leq s \leq m$, with $i_1 \geq 0$ and $i_s \geq 1$, $2 \leq s \leq m$, such that $d + n = \sum_{k=1}^{m} 2^{\sum_{s=1}^{k} i_s}$. If $i_1 \geq n - m$ or $\alpha(n+d) > n$, then:

Theorem 2.1 [2]. $\mathbf{C}_d(n) \cong \mathbf{C}_{2d+n}(n)$.

We do not give a detailed proof but all the same note that if we let $f : \mathbf{P}(n) \to \mathbf{P}(n)$ be the linear function given on monomials by

$$f(x_1^{e_1} \cdots x_n^{e_n}) = x_1^{2e_1+1} \cdots x_n^{2e_n+1},$$

then f passes to an isomorphism of quotients. In particular f is an injective mapping of sequences: $S(n, d) \to S(n, 2d+n)$

$$(\lambda_1, \ldots, \lambda_n) \mapsto (\lambda_1 + 1, \ldots, \lambda_n + 1). \tag{4}$$

The mapping (4) is bijective if $i_1 \geq n - m$. Further

$$B(n, d) = B(n, 2d+n) \text{ if } i_1 \geq n - m. \tag{5}$$

The next result generalizes (5). Write

$$d = \sum_{k=1}^{m} 2^{\sum_{s=1}^{k} i_s} - n, \text{ where } 1 \leq m \leq n. \tag{6}$$

Theorem 2.2. *Suppose that* $i_1 \geq n - m$ *or that there is an integer* t, $2 \leq t \leq m$, *such that* $i_t \geq n + 1 - m$. *Let*

$$d' = \sum_{k=1}^{t-1} 2^{\sum_{s=1}^{k} i_s} + \sum_{k=t}^{m} 2^{\left(\left(\sum_{s=1}^{k} i_s\right)+1\right)} - n. \tag{7}$$

Then $B(n, d) = B(n, d')$.

The result of the theorem may easily be justified by noting that the formula

$$2^{p+1} = 2^p + 2^p \quad \text{for } p \geq 0, \tag{8}$$

when applied to (6) and (7), determines a bijection from $S(n, d)$ to $S(n, d')$. Further that if $(\lambda_1', \ldots, \lambda_n') \in S(n, d')$ is the image of $(\lambda_1, \ldots, \lambda_n) \in S(n, d)$ under the bijection, then $|(\lambda_1, \ldots, \lambda_n)| = |(\lambda_1', \ldots, \lambda_n')|$.

3. Formula for Computing $B(n, d)$

Let n be fixed and suppose that d is an integer for which $\alpha(n + d) = m$, where m, $1 \leq m \leq n$, is fixed. To compute $B(n, d)$, we see, by the result of Theorem 2.2, that it is sufficient to consider all degrees d for which $i_1 \leq n - m$ and $i_s \leq n + 1 - m$ for all s, $2 \leq s \leq m$. All such degrees d lie in the range $0 \leq d \leq d(m)$, where $d(m)$ is defined by $d(m) = \sum_{k=0}^{m-1} 2^{n-m+(n+1-m)k} - n$.

Our next result is elementary and is given without proof.

Lemma 3.1. *Suppose that* $\alpha(n + d) = m$, *where* $1 \leq m \leq n$. *Then* $d(m) \leq d\left(\frac{n+2}{2}\right)$ *if n is even and* $d(m) \leq d\left(\frac{n+1}{2}\right)$ *if n is odd.*

From the result of the lemma we see that if n is fixed and we require to compute $B(n, d)$, then we need only consider degrees d in the range

$$0 \leq d \leq d\left(\frac{n+2}{2}\right) = \sum_{k=0}^{\frac{n}{2}} 2^{\left(\frac{(k+1)n-2}{2}\right)} - n \tag{9}$$

when n is even and $0 \leq d \leq d\left(\frac{n+1}{2}\right) = \sum_{k=0}^{\frac{n-1}{2}} 2^{\left(\frac{(k+1)n+k-1}{2}\right)} - n$ when n is odd.

We now give a formula for computing $B\left(n, d\left(\frac{n+2}{2}\right)\right)$ and $B\left(n, d\left(\frac{n+1}{2}\right)\right)$. The procedure is not limited to these cases. In particular it applies to $B(n, d(m))$ for all m, $1 \leq m < n$, as we shall illustrate.

For any positive integer l, let $T(l)$ denote the set of all j-tuples of positive integers $(l_1, l_2, ..., l_j)$ such that $l_1 + l_2 + \cdots + l_j = l$. Then an element of $T(l)$ may be represented in the form $[1^{\alpha_1} 2^{\alpha_2} \cdots l^{\alpha_l}]$ where for each i, $1 \leq i \leq l$, α_i is the number of parts of size i. Thus, making trivial abbreviations, we have: $[1^3 3] = (1, 1, 1, 3) \in T(6)$.

Let $[1^{\alpha_1} 2^{\alpha_2} \cdots l^{\alpha_l}] \in T(l)$ and let k be a positive integer. We shall use a more compact notation $F_{\alpha_1}^{\alpha_l}(k)$ for

$$\binom{k}{\sum_{i=1}^{l} \alpha_i} \binom{\sum_{i=1}^{l} \alpha_i}{\alpha_1, \ldots, \alpha_l}.$$

Theorem 3.2. *Let* $n \geq 3$ *be an integer. If* n *is even, then* $B\left(n, d\left(\frac{n+2}{2}\right)\right)$ *is given by*

$$\sum \frac{n!}{\prod_{i=1}^{\frac{n-2}{2}} ((i+1)!)^{\alpha_i}} F_{\alpha_1}^{\alpha_{\left(\frac{n-2}{2}\right)}} \left(\frac{n+2}{2}\right) \prod_{i=1}^{\frac{n-2}{2}} [B(i+1, 2^i - (i+1))]^{\alpha_i} \tag{10}$$

and if n is odd, then $B\left(n, d\left(\frac{n+1}{2}\right)\right)$ *is given by*

$$\sum \frac{n!}{\prod_{i=1}^{\frac{n-1}{2}} ((i+1)!)^{\alpha_i}} F_{\alpha_1}^{\alpha_{\left(\frac{n-1}{2}\right)}} \left(\frac{n+1}{2}\right) \prod_{i=1}^{\frac{n-1}{2}} [B(i+1, 2^i - (i+1))]^{\alpha_i}, \tag{11}$$

where, in both cases, the sum is taken over all elements

$$\left[1^{\alpha_1} \cdots \left(\frac{n-2}{2}\right)^{\alpha\left(\frac{n-2}{2}\right)}\right] \in T\left(\frac{n-2}{2}\right).$$

Proof. Let $n \geq 4$ be an even integer. We first show that there is a partition of $S\left(n, d\left(\frac{n+2}{2}\right)\right)$ indexed by $T\left(\frac{n-2}{2}\right)$. An element $(\lambda_1, \ldots, \lambda_n) \in S\left(n, d\left(\frac{n+2}{2}\right)\right)$ may be obtained from the representation (9) of $d\left(\frac{n+2}{2}\right)$ by splitting powers of 2 in the sum

$$\sum_{k=0}^{\frac{n}{2}} 2^{\left(\frac{(k+1)n-2}{2}\right)} \tag{12}$$

by means of formula (8). In other words, an additional $\frac{n-2}{2}$ powers of 2 have to be obtained from (12) by means of (8) resulting in a representation of $d\left(\frac{n+2}{2}\right)$ as n-sharp. A term, $2^{\left(\frac{(k+1)n-2}{2}\right)}$, in (12) may be split in one of $\frac{n-2}{2}$ admissible ways. Put $p = p_k = 2^{\left(\frac{(k+1)n-2}{2}\right)}$, $0 \leq k \leq \frac{n}{2}$, and let i, $1 \leq i \leq \frac{n-2}{2}$, be an integer. Define an order i splitting of 2^p as an expansion of this power into a sum of $i+1$ terms of powers of 2. This may be achieved in several but a finite number of ways. For instance $2^p = 2^{p-1} + 2^{p-2} + 2^{p-3} + \cdots + 2^{p-(i+1)} + 2^{p-i} + 2^{p-i}$ is an example of an order i splitting of 2^p. In general an order i splitting of 2^p yields a representation of $2^p - (i+1)$ as $(i+1)$-sharp. There, therefore, exists a bijective correspondence between the set of all order i splittings of 2^p and the set $S(i+1, 2^p - (i+1))$ of all representations of $2^p - (i+1)$ as $(i+1)$-sharp.

Let $(l_1, l_2, ..., l_j)$ be a j-tuple of integers with $1 \leq l_i \leq \frac{n-2}{2}$ for all i. We can think of each l_i as representing an element in a class or set of order l_i splittings of a term in (12) (one term for each l_i). Suppose that

$$l_1 + l_2 + \cdots + l_j = l. \tag{13}$$

With the given interpretation of l_i, we see that corresponding to the expression (13) is a class of representations of $d\left(\frac{n+2}{2}\right)$ involving $\frac{n+2}{2} + l$ powers of 2. If $l = \frac{n-2}{2}$, then (13) determines a class of sequences $(\lambda_1, ..., \lambda_n) \in S\left(n, d\left(\frac{n+2}{2}\right)\right)$. Clearly distinct such j-tuples determine distinct classes of sequences in $S\left(n, d\left(\frac{n+2}{2}\right)\right)$. This is the case since splitting 2^{p_k} in any of the $\frac{n-2}{2}$ ways above we obtain terms 2^q with $2^q > 2^{p_{k-1}}$ for all q and all k. Furthermore since every element of $S\left(n, d\left(\frac{n+2}{2}\right)\right)$ may be obtained by applying the formula (8) to (12) it follows that the j-tuples $(l_1, l_2, ..., l_j) \in T\left(\frac{n-2}{2}\right)$ index a partition of $S\left(n, d\left(\frac{n+2}{2}\right)\right)$.

Each element $\left[1^{\alpha_1} \cdots \left(\frac{n-2}{2}\right)^{\alpha\left(\frac{n-2}{2}\right)}\right] \in T\left(\frac{n-2}{2}\right)$ therefore determines a unique subset $S\left(n, d\left(\frac{n+2}{2}\right), \left[1^{\alpha_1} \cdots \left(\frac{n-2}{2}\right)^{\alpha\left(\frac{n-2}{2}\right)}\right]\right)$ of $S\left(n, d\left(\frac{n+2}{2}\right)\right)$. The sum $\sum_{i=1}^{\frac{n-2}{2}} \alpha_i$ is the number of terms in the expression (12) split to generate elements $(\lambda_1, ..., \lambda_n) \in S\left(n, d\left(\frac{n+2}{2}\right), \left[1^{\alpha_1} \cdots \left(\frac{n-2}{2}\right)^{\alpha\left(\frac{n-2}{2}\right)}\right]\right)$. Since there are $\frac{n+2}{2}$ terms in the expression (12) $S\left(n, d\left(\frac{n+2}{2}\right),\right.$

$\left[1^{\alpha_1} \cdots \left(\frac{n-2}{2}\right)^{\alpha\left(\frac{n-2}{2}\right)}\right]\Big)$ is the disjoint union of

$$\binom{\frac{n+2}{2}}{\sum_{i=1}^{\frac{n-2}{2}} \alpha_i}$$

subclasses, each with the same number of elements. Now let $(\lambda_1, \ldots, \lambda_n)$ be an element in one of these subclasses and let $(l_1, l_2, \ldots, l_j)$ be the j-tuple notation for $\left[1^{\alpha_1} \cdots \left(\frac{n-2}{2}\right)^{\alpha\left(\frac{n-2}{2}\right)}\right]$. Then corresponding to distinct permutations of the j-tuple $(l_1, l_2, \ldots, l_j)$ are distinct permutation representations $(\lambda'_1, \ldots, \lambda'_n)$ of $(\lambda_1, \ldots, \lambda_n)$ all of which belong to the same subclass. Thus associated with each element $(\lambda_1, \ldots, \lambda_n)$ in a subclass of $S\left(n, d\left(\frac{n+2}{2}\right), \left[1^{\alpha_1} \cdots \left(\frac{n-2}{2}\right)^{\alpha\left(\frac{n-2}{2}\right)}\right]\right)$ are

$$\binom{\sum_{i=1}^{\frac{n-2}{2}} \alpha_i}{\alpha_1, \ldots, \alpha_l}$$

permutation representatives.

Finally let $\left[1^{\alpha_1} \cdots \left(\frac{n-2}{2}\right)^{\alpha\left(\frac{n-2}{2}\right)}\right] \in T\left(\frac{n-2}{2}\right)$ and i, $1 \leq i \leq \frac{n-2}{2}$, be an integer. For a fixed i an order i splitting may be applied to more than one term in (12). The power α_i of i represents this multiplicity. To show how each element $\left[1^{\alpha_1} \cdots \left(\frac{n-2}{2}\right)^{\alpha\left(\frac{n-2}{2}\right)}\right] \in T\left(\frac{n-2}{2}\right)$ determines a term in (10) we illustrate with the case $\left[\frac{n-2}{2}\right]$. Let $(\lambda_1, \ldots, \lambda_n) \in S\left(n, d\left(\frac{n+2}{2}\right), \left[\frac{n-2}{2}\right]\right)$. Then λ_i are distinct apart possibly for the $\frac{n}{2}$ terms obtained from an order $\frac{n-2}{2}$ splitting of a power of 2, 2^p, in (12).

Let $S^k\left(n,\, d\left(\frac{n+2}{2}\right), \left[\frac{n-2}{2}\right]\right)$ be the subclass consisting of all $(\lambda_1, \ldots, \lambda_n)$ $\in S\left(n,\, d\left(\frac{n+2}{2}\right), \left[\frac{n-2}{2}\right]\right)$ obtained from order $\frac{n-2}{2}$ splittings of the term 2^{p_k}, (k fixed) in (12). Then

$$\sum_{(\lambda_1, \ldots, \lambda_n) \in S^k\left(n,\, d\left(\frac{n+2}{2}\right), \left[\frac{n-2}{2}\right]\right)} |(\lambda_1, \ldots, \lambda_n)|$$

is equal to

$$\frac{n!}{\left(\frac{n}{2}\right)!}\left(\sum_{(\sigma_1, \ldots, \sigma_{\frac{n}{2}}) \in S\left(\frac{n}{2},\, 2^{\left(\frac{n-2}{2}\right)} - \frac{n}{2}\right)} |(\sigma_1, \ldots, \sigma_{\frac{n}{2}})|\right) = \frac{n!}{\left(\frac{n}{2}\right)!} B\left(\frac{n}{2},\, 2^{\left(\frac{n-2}{2}\right)} - \frac{n}{2}\right).$$

Summing over all k we obtain the term

$$\frac{n!}{\left(\frac{n}{2}\right)!}\binom{\frac{n+2}{2}}{1} B\left(\frac{n}{2},\, 2^{\left(\frac{n-2}{2}\right)} - \frac{n}{2}\right)$$

in (10). A similar argument works for any choice of $\left[1^{\alpha_1} \cdots \left(\frac{n-2}{2}\right)^{\alpha\left(\frac{n-2}{2}\right)}\right] \in T\left(\frac{n-2}{2}\right)$ noting, by Theorem 2.2, that $B(i+1,\, 2^p - (i+1)) = B(i+1,\, 2^i - (i+1))$ for all $p \geq \frac{n-2}{2}$ and all i, $1 \leq i \leq \frac{n-2}{2}$. The case n odd may be proved similarly. This completes the proof of the theorem.

The result of Theorem 3.2 is, of course, also true for all degrees obtained from $d\left(\frac{n+2}{2}\right)$ in the manner outlined in Theorem 2.2.

A truncated version of (10) which gives exact equality when $2 \leq n \leq 8$ is

$$\sum_{i=1}^{\frac{n-4}{2}} i\binom{\frac{n+2}{2}}{i}\frac{n!}{\left(\frac{n+2-2i}{2}\right)!2^{i-1}}B\left(\frac{n+2-2i}{2},\, 2^{\frac{n-2i}{2}}-\left(\frac{n+2-2i}{2}\right)\right)+\binom{\frac{n+2}{2}}{\frac{n-2}{2}}\frac{n!}{2^{\left(\frac{n-2}{2}\right)}}.$$

If we now make use of the fact that $B(3, 1) = 3$ and $B(4, 4) = 13$ we obtain $B(4, 38) = 36$, $B(6, 2334) = 2520$ and $B(8, 559232) = 361200$.

We conclude by using the fact that $B(5, 11) = 75$, $B(6, 26) = 525$ and $B(7, 57) = 4347$ to obtain explicit formulae for $B(n, d(n-i))$ in the cases $\alpha(n+d) = n-i$, $1 \leq i \leq 6$. The respective formulae are given in the table below. We note that (10) is an expression for $B(n, d(n-i))$ when n is even and $i = \frac{n-2}{2}$ and (11) is an expression for $B(n, d(n-i))$ when n is odd and $i = \frac{n-1}{2}$. By analogy we may therefore obtain an expression for $B(n, d(m))$ for all m, $1 \leq m < n$.

$\alpha(n+d)$	$B(n, d(n-i))$
$n-1$	$(n-1)\frac{n!}{2}$
$n-2$	$(n-2)\frac{n!}{3!}(3)+\binom{n-2}{2}\frac{n!}{2^2}$
$n-3$	$(n-3)\frac{n!}{4!}(13)+2\binom{n-3}{2}\frac{n!}{3!2}(3)+\binom{n-3}{2}\frac{n!}{2^3}$
$n-4$	$\sum_{i=1}^{3} i\binom{n-4}{i}\frac{n!}{(6-i)!2^{i-1}}B(6-i,\, 2^{5-i}-(6-i))+\binom{n-4}{4}\frac{n!}{2^4}$ $+\binom{n-4}{2}\frac{n!}{(3!)^2}(3^2)$
$n-5$	$\sum_{i=1}^{4} i\binom{n-5}{i}\frac{n!}{(7-i)!2^{i-1}}B(7-i,\, 2^{6-i}-(7-i))+\binom{n-5}{5}\frac{n!}{2^5}$ $+3\binom{n-5}{3}\frac{n!}{2(3!)^2}(3^2)+2\binom{n-5}{2}\frac{n!}{3!4!}(3)(13)$

$n-6$	$\sum_{i=1}^{5} i\binom{n-6}{i}\frac{n!}{(8-i)!2^{i-1}}B(8-i,\ 2^{7-i}-(8-i))+\binom{n-6}{6}\frac{n!}{2^6}$ $+\,6\binom{n-6}{4}\frac{n!}{2^2(3!)^2}(3^2)+\binom{n-6}{3}\frac{n!}{(3!)^3}(3^3)+\binom{n-6}{2}\frac{n!}{(4!)^2}(13)^2$ $+\,2\binom{n-6}{2}\frac{n!}{(3!)(5!)}(3)(75)+6\binom{n-6}{3}\frac{n!}{(2!)(3!)(4!)}(3)(13)$

Computation of $B(n, d(m))$ is therefore dependent on the values $B(j, 2^{j-1}-j)$, for all $j \geq 3$. We do not consider this problem in this paper except to note that formula (2) is effective for small values of j.

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