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A CHARACTERIZATION FOR BESOV SPACE AND BLOCH SPACE OF INVARIANT HARMONIC FUNCTION IN THE UNIT BALL

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Abstract

On the unit ball $\mathbf{B}$ of $\mathbb{C}^n$, we obtain a norm equivalence of $L^p(\mathbf{B})$ space for an invariant harmonic function. Using the quantity for the $L^p(\mathbf{B})$ norm, we give characterizations of the invariant harmonic Besov space and Bloch space, which are extending the known results for the holomorphic case in the unit disk or the unit ball.

1. Introduction and Some Results

Let $\mathbf{B} = \{z \in \mathbb{C}^n : |z| < 1\}$ denote the open unit ball and $\mathbf{S}$ denote the boundary of $\mathbf{B}$ in $\mathbb{C}^n$. For each $a \in \mathbf{B}$, the Möbius transformation $\varphi_a : \mathbf{B} \to \mathbf{B}$ is defined by

$$\varphi_a(z) = \frac{a - P_a z - \sqrt{1 - |a|^2} Q_a z}{1 - \langle z, a \rangle}, \quad z \in \mathbf{B},$$

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where P_a is the orthogonal projection from $\mathbb{C}^n$ onto the subspace generated by a and Q_a is the orthogonal complement of P_a, i.e., $P_a z = \frac{\langle a, z\rangle}{|a|^2} a$, $Q_a z = z - P_a z$. Let $\mathcal{M}$ denote the group of all biholomorphic automorphisms of $\mathbf{B}$. Then any element of $\mathcal{M}$ has a unique representation by the rotations around origin of φ_a. For $a \in \mathbf{B}$ and $z \in \overline{\mathbf{B}}$, the determinant $J_{\mathbb{R}}\varphi_a(z)$ of the real Jacobian matrix of φ_a satisfies

$$J_{\mathbb{R}}\varphi_a(z) = |J_{\mathbb{C}}\varphi_a(z)|^2 = \left(\frac{1-|a|^2}{|1-\langle z, a\rangle|^2}\right)^{n+1} = \left(\frac{1-|\varphi_a(z)|^2}{1-|z|^2}\right)^{n+1} \tag{1.1}$$

and

$$d\nu(\varphi_a(z)) = J_{\mathbb{R}}\varphi_a(z)d\nu(z). \tag{1.2}$$

For $f \in C^2(\mathbf{B})$ the invariant Laplacian $\widetilde{\Delta}$ on $\mathbf{B}$ is given by

$$\widetilde{\Delta}f(z) = \Delta(f \circ \varphi_z)(0),$$

where $\Delta = 4\sum_{j=1}^{n}\frac{\partial^2}{\partial z_j \partial \bar{z}_j}$ is the usual Laplacian. An invariant harmonic or simply $\mathcal{M}$-harmonic function is a function in $C^2(\mathbf{B})$ which is annihilated by $\widetilde{\Delta}$ in $\mathbf{B}$. For a C^1 function f the invariant gradient $\widetilde{\nabla}$ is the vector field on $\mathbf{B}$ defined by

$$\widetilde{\nabla}f(z) = \nabla(f \circ \varphi_z)(0),$$

where $\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_n}\right)$, $z_j = x_j + iy_j$, $j = 1, 2, \ldots, n$. The Laplacian $\widetilde{\Delta}$ and the gradient $\widetilde{\nabla}$ are both invariant under the automorphisms of $\mathbf{B}$, i.e., $\mathcal{M}$-invariant in the sense that for $z \in \mathbf{B}$

$$\widetilde{\Delta}(f \circ \varphi_z) = (\widetilde{\Delta}f) \circ \varphi_z, \quad \widetilde{\nabla}(f \circ \varphi_z) = (\widetilde{\nabla}f) \circ \varphi_z.$$

A straightforward computation (see, Lemma 3.3 in [3]) shows that if

$\widetilde{\Delta} f = 0$ and $1 < p < \infty$, then

$$\widetilde{\Delta} | f |^p = c_p | f |^{p-2} | \widetilde{\nabla} f |^2, \tag{1.3}$$

where the constant c_p is dependent on p.

Let $f \in C^1(\mathbf{B})$ and $\xi \in \mathbb{C}^n$. The maximal derivative of f with respect to the Bergman metric β on $\mathbf{B}$ is defined by

$$\hat{Q} f(z) = \sup_{|\xi|=1} \frac{| df(z) \cdot \xi |}{\beta(z, \xi)}, \quad z \in \mathbf{B},$$

where

$$df(z) \cdot \xi = \sum_{i=1}^{n} \left[\frac{\partial f}{\partial z_i}(z) \xi_i + \frac{\partial f}{\partial \overline{z}_i}(z) \overline{\xi}_i \right]$$

$$= \partial f(z) \cdot \xi + \overline{\partial} f(z) \cdot \overline{\xi}.$$

The following identities are easily verified. For C^1-function f and $\varphi \in \mathcal{M}$ in $\mathbf{B}$,

$$\hat{Q}(f \circ \varphi) = (\hat{Q} f) \circ \varphi, \quad \frac{1}{2} \sqrt{\widetilde{\Delta} | f |^2} \leq \hat{Q} f = 2 | \widetilde{\nabla} f | \leq \sqrt{\widetilde{\Delta} | f |^2}. \tag{1.4}$$

For $0 < p < \infty$, the Bergman space $L_a^p(\mathbf{B})$, the Hardy space $H^p(\mathbf{B})$ and the Besov space $B_p(\mathbf{B})$ for the holomorphic function on the unit ball of $\mathbb{C}^n$ are defined respectively as

$$L_a^p(\mathbf{B}) = \left\{ f \in H(\mathbf{B}) : \| f \|_{L_a^p}^p = \int_{\mathbf{B}} | f(z) |^p d\nu(z) < \infty \right\},$$

$$H^p(\mathbf{B}) = \left\{ f \in H(\mathbf{B}) : \| f \|_p^p = \sup_{0<r<1} \int_S | f(r\zeta) |^p d\sigma(\zeta) < \infty \right\}$$

and

$$B_p(\mathbf{B}) = \left\{ f \in H(\mathbf{B}) : \| f \|_{B_p}^p = \int_{\mathbf{B}} (\hat{Q} f)^p(z) d\lambda(z) < \infty \right\},$$

where we denote $d\nu$ the normalized Lebesgue measure on $\mathbf{B}$, $d\sigma$ the normalized Lebesgue measure on $\mathbf{S}$, $d\lambda(z) = \dfrac{1}{(1-|z|^2)^{n+1}} d\nu(z)$ the $\mathcal{M}$-invariant measure on $\mathbf{B}$. We use the notations $\mathcal{M}L^p(\mathbf{B})$, $\mathcal{M}H^p(\mathbf{B})$ and $\mathcal{M}B_p(\mathbf{B})$ in cases of invariant harmonic functions.

The Bloch space $\mathcal{M}B(\mathbf{B})$ consists of all invariant harmonic functions on $\mathbf{B}$ such that

$$\| f \|_{\mathcal{M}B(\mathbf{B})} := | f(0) | + \sup_{z\in\mathbf{B}} \hat{Q}f(z) < \infty.$$

In other words, $\mathcal{M}B_p(\mathbf{B})$ consists of those $\mathcal{M}$-harmonic function f for which $\hat{Q}f$ is p-integrable with respect to the $\mathcal{M}$-invariant measure $d\lambda(z)$ and $\mathcal{M}B(\mathbf{B})$ is a Besov p-space with $p = \infty$. The inclusion relations between two spaces are, for $1 < p \le q \le \infty$,

$$\mathcal{M}B_p \subseteq \mathcal{M}B_q \subseteq \mathcal{M}B_\infty = \mathcal{M}B.$$

Since $\hat{Q}f$ for f $\mathcal{M}$-harmonic is $\mathcal{M}$-invariant, the $\mathcal{M}$-harmonic Besov and Bloch spaces are $\mathcal{M}$-invariant Banach spaces in the sense that $\| f \circ \varphi \| = \| f \|$, for $\varphi \in \mathcal{M}$. Considering the inclusion relations for Besov space and Bloch space, one may characterize those spaces by the supremum over the unit ball. We confirm the fact by our results for some characterizations of Besov space and Bloch space.

A. Zygmund proved that a holomorphic function f on the unit disc $\mathbf{D}$ is in $H^2(\mathbf{D})$, if and only if

$$\int_{\mathbf{D}}\int_{\mathbf{D}} | f'(z) |^2 (1 - | z |) dxdy < \infty \quad (z = x + iy).$$

This result was extended to all p, $0 < p < \infty$, by Yamashita [10] and Stoll [7] in the unit disc $\mathbf{D}$ and the unit ball $\mathbf{B}$, respectively. In [5], the result for Bergman spaces $L_a^p(\mathbf{B})$ similar to that of Yamashita's was given on the unit ball.

In this paper we extend the result for the holomorphic $L^p(\mathbf{B})$ space of [5] to the invariant harmonic version. Using the quantity for the $L^p(\mathbf{B})$ norm, we give characterizations of the invariant harmonic Besov space and Bloch space, which are extending the known results for the holomorphic case in the unit disk or the unit ball (see, [2], [4], [5], [7], [8], [9], [11]).

To obtain an analogous result, Theorem 1.2, to Bergman space for holomorphic functions in [5], we need the following lemma (see, Lemma 3.5 in [3] for another proof).

Lemma 1.1. *If* $1 < p < \infty$ *and* f *is* (*complex-valued*) $\mathcal{M}$*-harmonic on* $\mathbf{B}$, *then for* $0 < r < 1$,

$$\int_{\mathbf{S}} |f(r\zeta)|^p d\sigma(\zeta) - |f(0)|^p \approx \int_{r\mathbf{B}} G(r, z)\widetilde{\Delta}|f(z)|^p d\lambda(z),$$

where

$$G(r, z) = \frac{1}{2n}\int_{|z|}^{r} \frac{(1-\rho^2)^{n-1}}{\rho^{2n-1}} d\rho, \quad z \in r\mathbf{B} = \{z \in \mathbb{C}^n : |z| < r\}.$$

Theorem 1.2. *For* $1 < p < \infty$, $f \in \mathcal{M}L^p(\mathbf{B})$ *if and only if*

$$\int_{\mathbf{B}} |\widetilde{\nabla} f(z)|^2 |f(z)|^{p-2}(1-|z|^2)^{n+1} d\lambda(z) < \infty.$$

The integral is equivalent to $\| f \|^p_{\mathcal{M}L^p(\mathbf{B})}$.

Using Lemma 1.1 and Theorem 1.2, we give a new characterization of the invariant harmonic Besov space.

Theorem 1.3. *For* $1 < p < \infty$, $f \in \mathcal{M}B_p(\mathbf{B})$ *if and only if*

$$\int_{\mathbf{B}}\int_{\mathbf{B}} (\hat{Q}f)^2(z)\frac{|f(z) - f(w)|^{p-2}}{|1 - \langle z, w\rangle|^{2(n+1)}} d\nu(z) d\nu(w) < \infty. \tag{1.5}$$

The quantity in (1.5) *is equivalent to* $\| f \|^p_{\mathcal{M}B_p(\mathbf{B})}$.

Let $1 \leq p < \infty$. For a function f invariant harmonic on $\mathbf{B}$, the Bloch norm $\| f \|^p_{\mathcal{M}B(\mathbf{B})}$ is equivalent to

$$\sup_{z \in \mathbf{B}} \| f \circ \varphi_z - f(z) \|^p_{\mathcal{M}L^p} \quad \text{(see [2])} \tag{1.6}$$

and

$$\sup_{a \in \mathbf{B}} \int_{\mathbf{B}} (\hat{Q}f)^p(z) \left(\frac{1 - |a|^2}{|1 - \langle z, a \rangle|^2} \right)^{n+1} d\nu(z) \quad \text{(see [1])}. \tag{1.7}$$

By (1.1), we can restate (1.7) as

$$\sup_{a \in \mathbf{B}} \int_{\mathbf{B}} (\hat{Q}f)^p(z) (1 - |\varphi_a(z)|^2)^{n+1} d\lambda(z). \tag{1.8}$$

Using above equivalent quantities for the $\mathcal{M}$-harmonic Bloch space, we obtain characterizations for the Bloch space in a parallel line with Theorem 1.3.

Theorem 1.4. *For* $1 < p < \infty$, *we have*

$$\| f \|^p_{\mathcal{M}B(\mathbf{B})} \approx \sup_{a \in \mathbf{B}} \int_{\mathbf{B}} \int_{\mathbf{B}} (\hat{Q}f)^2(z) \frac{|f(z) - f(w)|^{p-2}}{|1 - \langle z, w \rangle|^{2(n+1)}} (1 - |\varphi_a(w)|^2)^{n+1} d\nu(z) d\nu(w).$$

Applying Theorem 1.2 to the Bloch norm in (1.6), we directly obtain another characterization of $\mathcal{M}B$ space.

Theorem 1.5. *For* $1 < p < \infty$, *we obtain*

$$\| f \|^p_{\mathcal{M}B(\mathbf{B})} \approx \sup_{a \in \mathbf{B}} \int_{\mathbf{B}} |\widetilde{\nabla} f(z)|^2 |f(z) - f(a)|^{p-2} (1 - |\varphi_a(z)|^2)^{n+1} d\lambda(z).$$

2. Proofs of the Results

We use the notation $A \lesssim B$ for the two expressions A and B which means that there is a constant C, independent of the quantities under consideration, such that $A \leq C \cdot B$. When $A \lesssim B$ and $B \lesssim A$, we use the notation $A \approx B$.

Proof of Lemma 1.1. By Green's formula for the invariant Laplacian $\widetilde{\Delta}$, for $0 < \rho < 1$,

$$\frac{\pi\rho^{2n-2}}{(n-1)!}(1-\rho^2)^{-n+1}\int_{\mathbf{S}} \mathcal{R}(|f|^p)(\rho\zeta)d\sigma(\zeta) = \int_{\rho\mathbf{B}} \widetilde{\Delta}|f(z)|^p d\lambda(z),$$

where $\mathcal{R}(f)$ is the radial derivative of f. Since

$$\frac{d}{d\rho}|f(\rho\zeta)|^p = \frac{1}{\rho}\mathcal{R}(|f|^p)(\rho\zeta),$$

we have

$$\frac{d}{d\rho}\int_{\mathbf{S}}|f(\rho\zeta)|^p d\sigma(\zeta) \approx \frac{(1-\rho^2)^{n-1}}{\rho^{2n-1}}\int_{\rho\mathbf{B}}\widetilde{\Delta}|f(z)|^p d\lambda(z).$$

In integrating both side over $r \in [0, \rho]$, we obtain

$$\int_{\mathbf{S}}|f(r\zeta)|^p d\sigma(\zeta) - |f(0)|^p \approx \int_0^r \frac{(1-\rho^2)^{n-1}}{\rho^{2n-1}}d\rho\int_{\rho\mathbf{B}}\widetilde{\Delta}|f(z)|^p d\lambda(z). \tag{2.1}$$

By the change of integration order in (2.1), the proof is completed. □

Let

$$\chi_{|z|}(t) = \begin{cases} 1, & |z| < t \\ 0, & \text{otherwise.} \end{cases}$$

The proof of Theorem 1.2 goes along the similar process as that of [5] (see Theorem 1 in [5]).

Proof of Theorem 1.2. It follows from Lemma 1.1 and (1.3) for $\mathcal{M}$-harmonic function f that for $0 < \rho < 1$ we obtain

$$\begin{aligned}\int_{\mathbf{S}}|f(\rho\zeta)|^p d\sigma(\zeta) &\approx \int_{\rho\mathbf{B}}|f(z)|^{p-2}|\widetilde{\nabla}f(z)|^2 d\lambda(z)\int_{|z|}^{\rho}\frac{(1-r^2)^{n-1}}{r^{2n-1}}dr \\ &\approx \int_{\rho\mathbf{B}}|f(z)|^{p-2}|\widetilde{\nabla}f(z)|^2 d\lambda(z)\int_0^{\rho}\frac{(1-r^2)^{n-1}}{r^{2n-1}}\chi_{|z|}(r)dr \\ &= \int_0^{\rho}\frac{(1-r^2)^{n-1}}{r^{2n-1}}dr\int_{r\mathbf{B}}|f(z)|^{p-2}|\widetilde{\nabla}f(z)|^2 d\lambda(z). \end{aligned} \tag{2.2}$$

Following the same process as in Theorem 1 of [5], we obtain the equivalent quantity for the last one of (2.2)

$$\int_0^{\rho}\frac{(1-r^2)^{n-1}}{r^{2n-1}}dr\int_{r\mathbf{B}}|f(z)|^{p-2}|\widetilde{\nabla}f(z)|^2 d\lambda(z)$$

$$\approx \rho^{-2n+1}\int_0^{\rho}(1-r^2)^{n-1}dr\int_{r\mathbf{B}}|f(z)|^{p-2}|\widetilde{\nabla}f(z)|^2 d\lambda(z). \tag{2.3}$$

By (2.2) and (2.3), we have

$$\begin{aligned}\|f\|_{\mathcal{M}L^p}^p &= \int_{\mathbf{B}}|f(z)|^p d\nu(z)\\ &= 2n\int_0^1\rho^{2n-1}d\rho\int_S|f(\rho\zeta)|^p d\sigma(\zeta)\\ &\approx 2n\int_0^1 d\rho\int_0^{\rho}(1-r^2)^{n-1}dr\int_{r\mathbf{B}}|f(z)|^{p-2}|\widetilde{\nabla}f(z)|^2 d\lambda(z).\end{aligned}$$

By interchanging the order of the integration, we get

$$\begin{aligned}\|f\|_{\mathcal{M}L^p}^p &= \int_0^1(1-r^2)^{n-1}dr\int_{r\mathbf{B}}|f(z)|^{p-2}|\widetilde{\nabla}f(z)|^2 d\lambda(z)\int_r^1 d\rho\\ &\approx \int_0^1(1-r^2)^{n}dr\int_{r\mathbf{B}}|f(z)|^{p-2}|\widetilde{\nabla}f(z)|^2 d\lambda(z)\\ &= \int_{\mathbf{B}}|f(z)|^{p-2}|\widetilde{\nabla}f(z)|^2 d\lambda(z)\int_0^1(1-r^2)^n\cdot\chi_{|z|}(r)dr\\ &= \int_{\mathbf{B}}|f(z)|^{p-2}|\widetilde{\nabla}f(z)|^2 d\lambda(z)\int_{|z|}^1(1-r^2)^n dr\\ &= \int_{\mathbf{B}}|f(z)|^{p-2}|\widetilde{\nabla}f(z)|^2(1-|z|^2)^{n+1}d\lambda(z).\end{aligned}$$

This completes Theorem 1.2. □

Remark 2.1. Taking $\rho\to 1$ in (2.3), we obtain

$$\|f\|_{\mathcal{M}H^p}\approx\int_0^1(1-r^2)^{n-1}dr\int_{r\mathbf{B}}|f(z)|^{p-2}|\widetilde{\nabla}f(z)|^2 d\lambda(z)$$

$$= \int_{\mathbf{B}} |f(z)|^{p-2} |\widetilde{\nabla} f(z)|^2 d\lambda(z) \int_{|z|}^{1} (1 - r^2)^{n-1} dr$$

$$\approx \int_{\mathbf{B}} |f(z)|^{p-2} |\widetilde{\nabla} f(z)|^2 (1 - |z|^2)^n d\lambda(z).$$

The result gives another proof of Proposition 5 in [7]: Let $1 < p < \infty$. Then $\mathcal{M}H^p(\mathbf{B})$ if and only if

$$\int_{\mathbf{B}} |\widetilde{\nabla} f(z)|^2 |f(z)|^{p-2} (1 - |z|^2)^n d\lambda(z) < \infty.$$

By modifying Hardy's inequality, K. T. Hahn obtained the following theorem.

Theorem 2.2 [2]. *For* $1 < p < \infty$, $f \in \mathcal{M}B_p(\mathbf{B})$ *if and only if*

$$\int_{\mathbf{B}} \int_{\mathbf{B}} |f(z) - f(w)|^p \frac{|K(z, w)|^2}{K(z, z)K(w, w)} d\lambda(z) d\lambda(w) < \infty,$$

where

$$K(z, w) = \frac{1}{(1 - \langle z, w \rangle)^{n+1}}, \quad z, w \in \mathbf{B}$$

denotes the Bergman kernel of $\mathbf{B}$.

Using K. T. Hahn's result, we can obtain a characterization of the $\mathcal{M}$-harmonic Besov space.

Proof of Theorem 1.3. By Theorem 2.2, we want to show that

$$\int_{\mathbf{B}} \int_{\mathbf{B}} (\hat{Q} f)^2 (z) \frac{|f(z) - f(w)|^{p-2}}{|1 - \langle z, w \rangle|^{2(n+1)}} d\nu(z) d\nu(w)$$

$$\approx \int_{\mathbf{B}} \int_{\mathbf{B}} \frac{|f(z) - f(w)|^p}{|1 - \langle z, w \rangle|^{2(n+1)}} d\nu(z) d\nu(w).$$

It follows from (1.1) and (1.2) that we have the identity

$$\int_{\mathbf{B}} \int_{\mathbf{B}} (\hat{Q} f)^2 (z) |f(z) - f(w)|^{p-2} \frac{1}{|1 - \langle z, w \rangle|^{2(n+1)}} d\nu(z) d\nu(w)$$

$$= \int_{\mathbf{B}}\int_{\mathbf{B}} (\hat{Q}f)^2(z) | f(z) - f(w) |^{p-2} d\nu(\varphi_w(z)) \frac{d\nu(w)}{(1 - | w |^2)^{n+1}}.$$

By the change of variable $u = \varphi_w(z)$ and the invariance of $\hat{Q}f$ under the automorphism on **B**, the integral above is

$$\int_{\mathbf{B}}\int_{\mathbf{B}} (\hat{Q}f)^2(\varphi_w(u)) | f(\varphi_w(u)) - f(w) |^{p-2} d\nu(u) \frac{d\nu(w)}{(1 - | w |^2)^{n+1}}$$

$$= \int_{\mathbf{B}}\int_{\mathbf{B}} (\hat{Q}(f \circ \varphi_w))^2(u) | f \circ \varphi_w(u) - f(w) |^{p-2} d\nu(u) \frac{d\nu(w)}{(1 - | w |^2)^{n+1}}. \quad (2.4)$$

Replacing $f \circ \varphi_w(u) - f(w)$ by $F_w(u)$, (2.4) is equivalent to

$$\int_{\mathbf{B}}\int_{\mathbf{B}} (\hat{Q}F_w)^2(u) | F_w(u) |^{p-2} d\nu(u) \frac{d\nu(w)}{(1 - | w |^2)^{n+1}},$$

by applying Theorem 1.2 to F_w and by (1.4), which is equivalent to

$$\int_{\mathbf{B}}\int_{\mathbf{B}} | \tilde{\nabla} F_w(u) |^2 | F_w(u) |^{p-2} d\nu(u) \frac{d\nu(w)}{(1 - | w |^2)^{n+1}}$$

$$= \int_{\mathbf{B}}\int_{\mathbf{B}} | F_w(u) |^p d\nu(u) \frac{d\nu(w)}{(1 - | w |^2)^{n+1}}. \quad (2.5)$$

Replacing $F_w(u)$ by $f \circ \varphi_w(u) - f(w)$ in (2.5), we get

$$\int_{\mathbf{B}}\int_{\mathbf{B}} | F_w(u) |^p d\nu(u) \frac{d\nu(w)}{(1 - | w |^2)^{n+1}}$$

$$= \int_{\mathbf{B}}\int_{\mathbf{B}} | f \circ \varphi_w(u) - f(w) |^p d\nu(u) \frac{d\nu(w)}{(1 - | w |^2)^{n+1}}.$$

By changing of variable $u = \varphi_w(z)$ and (1.1), the proof is completed. □

Proof of Theorem 1.4. Following the step to obtain (2.5), we have

$$\sup_{a \in \mathbf{B}} \int_{\mathbf{B}}\int_{\mathbf{B}} \frac{| f(z) - f(w) |^{p-2}}{| 1 - \langle z, w \rangle |^{2(n+1)}} (\hat{Q}f)^2(z) (1 - | \varphi_a(w) |^2)^{n+1} d\nu(z) d\nu(w)$$

$$\approx \sup_{a \in \mathbf{B}} \int_{\mathbf{B}}\int_{\mathbf{B}} | F_w(z) |^{p-2} | \tilde{\nabla} F_w(z) |^2 (1 - | z |^2)^{n+1} (1 - | \varphi_a(w) |^2)^{n+1} d\lambda(z) d\lambda(w), \quad (2.6)$$

where $F_w := f \circ \varphi_w - f(w)$. By Theorem 1.2, (2.6) is equivalent to

$$\sup_{a \in \mathbf{B}} \int_{\mathbf{B}} \int_{\mathbf{B}} | F_w(z) |^p (1 - | \varphi_a(w) |^2)^{n+1} d\nu(z) d\lambda(w). \tag{2.7}$$

By Proposition 10.2 in [8], $\mathcal{M}$-subharmonic mean value property,

$$| \widetilde{\nabla} f(w) |^p (1 - | w |^2)^p \lesssim \int_{\mathbf{B}} | f \circ \varphi_w(z) - f(w) |^p d\nu(z).$$

So, (2.7) is greater than or equal to a constant times to

$$\sup_{a \in \mathbf{B}} \int_{\mathbf{B}} | \widetilde{\nabla} f(w) |^p (1 - | w |^2)^p (1 - | \varphi_a(w) |^2)^{n+1} d\lambda(w),$$

which is, by (1.4) and (1.1), equivalent to the left side quantity in (1.7).

Conversely, (2.7) is less than or equal to

$$\sup_{w \in \mathbf{B}} \int_{\mathbf{B}} | F_w(z) |^p d\nu(z) \cdot \sup_{a \in \mathbf{B}} \int_{\mathbf{B}} (1 - | \varphi_a(w) |^2)^{n+1} d\lambda(w),$$

which equals

$$\sup_{w \in \mathbf{B}} \int_{\mathbf{B}} | f \circ \varphi_w(z) - f(w) |^p d\nu(z) \approx \| f \|_{\mathcal{M}B}^p$$

since

$$\int_{\mathbf{B}} (1 - | \varphi_a(w) |^2)^{n+1} d\lambda(w) = 1. \qquad \square$$

3. Comparison with a Holomorphic Case

Remark 3.1. When $n = 1$ and f is holomorphic in the unit disc $\mathbf{D}$, the left side of (1.5) equals

$$\int_{\mathbf{D}} \int_{\mathbf{D}} | f'(z) |^2 (1 - | z |^2)^2 \frac{| f(z) - f(w) |^{p-2}}{| 1 - z\overline{w} |^4} dA(z) dA(w), \tag{3.1}$$

in fact, the Bergman metric $\beta(z, \zeta)$ for $\mathbf{D}$ is $\sqrt{2} | \zeta | / (1 - | z |^2)$.

Lemma 3.2. *For* $z, w \in \mathbf{D}$, $| f'(z) | (1 - | z |^2)$ *is comparable to* $| f(z) - f(w) |$.

Proof. From Theorem 3.1 in [9], we see the inequality for $0 < r < 1$

$$|f'(z)|(1-|z|^2) \lesssim \frac{1}{r} \sup_{|\varphi_w(z)|<r} |f(z)-f(w)|.$$

On the other hand it follows from the inequality

$$|f(u)-f(0)| = \left| \int_0^1 \frac{d}{dt} f(tu)dt \right|$$

$$\leq \sup_{|tu|<1} |f'(tu)|(1-|tu|^2) \int_0^1 \frac{|u|}{1-t^2|u|^2} dt$$

for each $u \in \mathbf{D}$ that

$$|f(u)-f(0)| \leq \sup_{|tu|<1} |f'(tu)|(1-|tu|^2) \cdot \frac{1}{2} \log \frac{1+|u|}{1-|u|}. \tag{3.2}$$

Thus we have, for the Möbius transformation φ_a on the unit disc $\mathbf{D}$,

$$|f(z)-f(w)| = |f \circ \varphi_w \circ \varphi_w(z) - f \circ \varphi_w(0)|$$

$$= |F(u)-F(0)| \quad (f \circ \varphi_w = F,\ \varphi_w(z) = u).$$

Applying (3.2) to F, we then obtain

$$|F(u)-F(0)| \leq \frac{1}{2} \log \frac{1+|u|}{1-|u|} \cdot \sup_{|tu|<1} |F'(tu)|(1-|tu|^2)$$

$$= \frac{1}{2} \log \frac{1+|\varphi_w(z)|}{1-|\varphi_w(z)|} \cdot \sup_{0<t<1} |(f \circ \varphi_w)'(t\varphi_w(z))|(1-t^2|\varphi_w(z)|^2).$$

By (1.1), we have

$$|f(z)-f(w)| \leq \frac{1}{2} \log \frac{1+|\varphi_w(z)|}{1-|\varphi_w(z)|} \cdot \sup_{0<t<1} |f'(\varphi_w(tu))|(1-|\varphi_w(tu)|^2)$$

$$= \frac{1}{2} \log \frac{1+|\varphi_w(z)|}{1-|\varphi_w(z)|} \cdot \sup_{0<t<1} |f'(v)|(1-|v|^2), \quad (v = \varphi_w(tu)).$$

Therefore

$$\sup_{|\varphi_w(z)|<r} |f(z) - f(w)| \leq \frac{1}{2} \log \frac{1+r}{1-r} \cdot \sup_{0<t<1} |f'(v)|(1 - |v|^2).$$

The proof is complete. □

Then we see that the following characterization of the holomorphic Besov space $B_p(\mathbf{D})$ by K. Zhu is obtained by (3.1) and Lemma 3.2.

Theorem 3.3 [11]**.** *If* $p > 1$ *and f is holomorphic in* $\mathbf{D}$, *then f is in* B_p *if and only if*

$$\int_{\mathbf{D}} \int_{\mathbf{D}} \frac{|f(z) - f(w)|^p}{|1 - z\overline{w}|^4} dA(z) dA(w) < \infty.$$

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