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ON CHARACTERIZATIONS OF SPHERICAL CURVES

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Abstract

A curve is said to be a spherical curve if it lies on a sphere. There are many characterizations of spherical curves, among those is the well-known theorem of Breuer and Gottlieb [4], which was then improved by Wong without any preconditions. It states that a C^4-curve lies on a sphere if and only if

$$k\left[A\cos\left(\int_0^s \tau(s)\,ds\right) + B\sin\left(\int_0^s \tau(s)\,ds\right)\right] = 1,$$

where A, B are arbitrary constants and k is the curvature of the curve and τ is the torsion of the curve. This result was then generalized by Alodan and Deshmukh to n-dimensional submanifolds of a Euclidean space R^{n+p} which states that an n-dimensional compact connected and oriented submanifold of R^{n+p} lies on a hypersphere if and only if $F = \langle H, \psi^{\perp}\rangle = -1$, where H is the mean curvature vector field and $\psi^{\perp}$ is the normal component of the position vector field ψ of the submanifold in R^{n+p}. In this paper we prove the equivalence of Wong's result with Alodan and Deshmukh's result in two ways.

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1. Introduction

Characterizing spherical curves is an interesting goal in geometry. There are many characterizations for a curve in R^3 to lie on a sphere. An important one is the well-known theorem of Breuer and Gottlieb [4], which was then improved by Wong [12] without any preconditions, states that a C^4-curve lies on a sphere if and only if

$$k\left[A\cos\left(\int_0^s \tau(s)ds\right) + B\sin\left(\int_0^s \tau(s)ds\right)\right] = 1,$$

where A, B are arbitrary constants and k is the curvature of the curve and τ is the torsion of the curve. Alodan and Deshmukh gave recently a generalization of this result to higher dimensions which is, given an n-dimensional submanifold M of a Euclidean space R^{n+p} with immersion $\psi : M \to R^{n+p}$, $\psi(M) \subset S^{n+p-1}(c)$ for some c if and only if the smooth function $F = \langle H, \psi^{\perp}\rangle = -1$, where H is the mean curvature vector field and $\psi = \psi^T + \psi^{\perp}, \psi^T, \psi^{\perp}$ being the tangential and normal components of ψ restricted to M, and $\langle\, ,\rangle$ is the Euclidean metric on R^{n+p} (Theorem) [1].

We prove that Wong's result is equivalent to Alodan and Deshmukh's result (cf. Theorem 4.1). We also give examples of spherical and non-spherical submanifolds where the condition is met or is violated.

2. Preliminaries

We denote by $\langle\, ,\rangle$ and $\overline{\nabla}$ the Euclidean metric and the Euclidean connection on R^{n+p}. Let g and ∇ be the induced metric and the Riemannian connection on the submanifold M. Then we have the following equations for the submanifold M

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \overline{\nabla}_X N = -A_N X + \nabla^{\perp}_X N, \tag{2.1}$$

where $X, Y \in \mathfrak{X}(M)$, $N \in \Gamma(v)$, where $\mathfrak{X}(M)$ is the Lie-algebra of smooth vector fields, $\Gamma(v)$ is the space of smooth sections of the normal bundle v of M, h is the second fundamental form, A_N is the Weingarten map with respect to the normal $N \in \Gamma(v)$ which is related to the second fundamental form h by $g(A_N X, Y) = \langle h(X, Y), N \rangle$, $X, Y \in \mathfrak{X}(M)$ and $\nabla^{\perp}$ is the connection in the normal bundle v. We also have the following equations of Gauss and Codazzi for the submanifold M

$$R(X, Y, Z, W) = g(h(Y, Z), h(X, W)) - g(h(X, Z), h(Y, W)), \tag{2.2}$$

$$(Dh)(X, Y, Z) = (Dh)(Y, Z, X) = (Dh)(Z, X, Y), \tag{2.3}$$

where R is the curvature tensor field of the submanifold M and $(Dh)(X, Y, Z) = \nabla^{\perp}_X h(Y, Z) - h(\nabla_X, Y, Z) - h(Y, \nabla_X Z)$, $X, Y \in \chi(M)$.

The Ricci tensor Ric of the submanifold is given by

$$Ric(X, Y) = ng(h(X, Y), H) - \sum_i g(h(X, e_i), h(Y, e_i)), \tag{2.4}$$

where $\{e_1, \ldots, e_n\}$ is a local orthonormal frame on M and $H = \sum_i h(e_i, e_i)$, is the mean curvature vector field. The Ricci operator Q is a symmetric operator defined by $Ric(X, Y) = g(Q(X), Y)$, $X, Y \in \mathfrak{X}(M)$.

The scalar curvature S of the submanifold M is given by

$$S = n^2 \| H \|^2 - \| h \|^2, \tag{2.5}$$

where $\| h \|^2$ is the square of the length of the second fundamental form defined by

$$\| h \|^2 = \sum_{ij} \| h(e_i, e_j) \|^2.$$

If we express $\psi = \psi^T + \psi^{\perp}$, where $\psi^T \in \mathfrak{X}(M)$ is the tangential component and $\psi^{\perp} \in \Gamma(v)$ is the normal component of ψ. We denote by $A = A_{\psi^{\perp}}$ the Weingarten map with respect to the normal vector field $\psi^{\perp} \in \Gamma(v)$, then using (2.1), we have

$$\nabla_X \psi^T = X + AX, \quad \nabla_X^{\perp} \psi^{\perp} = -h(X, \psi^T), \quad X \in \mathfrak{X}(M). \tag{2.6}$$

Define a smooth function $F : M \to R$ on the submanifold M by $F = \langle H, \psi^{\perp} \rangle$, then we have the following lemmas for an n-dimensional compact submanifold $\psi : M \to R^{n+p}$.

Lemma 2.1 [1]. *Let M be an n-dimensional compact submanifold of the Euclidean space* R^{n+p}. *Then*

$$\int_M (1 + F) dv = 0.$$

Lemma 2.2 [1]. *Let M be an n-dimensional submanifold of* R^{n+p}. *Then the tensor field A satisfies*

(i) $trA = nF$,

(ii) $(\nabla A)(X, Y) - (\nabla A)(Y, X) = R(X, Y)\psi^T$,

(iii) $\sum (\nabla A)(e_i, e_i) = n\nabla F + Q(\psi^T)$,

where $(\nabla A)(X, Y) = \nabla_X AY - A\nabla_X Y$, $X, Y \in \chi(M)$ *and* $\{e_1, \ldots, e_n\}$ *is a local orthonormal frame on M.*

Lemma 2.3 [1]. *Let M be an n-dimensional submanifold of* R^{n+p}. *Then the tensor field A satisfies*

$$\int_M \{Ric(\psi^T, \psi^T) + \| A \|^2 - n^2 F^2 + n(n-1)\} dV = 0.$$

3. Spherical Submanifolds

The following well-known theorem characterizes spherical curves

Theorem 3.1 (Breuer and Gottlieb). *A curve* $\alpha(s)$ *in* R^3 *is a spherical curve if and only if* $\rho(s)$ *and* $\tau(s)$ *satisfy the explicit relation*:

$$\kappa\left[A\cos\left(\int \tau(s)ds\right) + B\sin\left(\int \tau(s)ds\right)\right] = 1,$$

where A and B are arbitrary constants.

This theorem was later improved by Wong without any preconditions on κ and τ as follows:

Theorem 3.2 (Wong). *A* C^4 *curve* $\alpha(s)$, $s \in [0, L]$ *parametrized by its arc length is a spherical curve if and only if*

$$\kappa\left[A\cos\left(\int_0^s \tau(s)ds\right) + B\sin\left(\int_0^s \tau(s)ds\right)\right] = 1,$$

where A and B are arbitrary constants. A curve satisfying the above condition lies on a sphere of radius $(A^2 + B^2)^{\frac{1}{2}}$.

This result was then generalized by Alodan and Deshmukh for n-dimensional submanifolds of R^{n+p} to lie on a hypersphere

Theorem 3.3 (Alodan and Deshmukh). *Let* $\Psi : M \to R^{n+p}$ *be an n-dimensional connected compact oriented submanifold. Then* $\Psi(M) \subset S^{n+p-1}(c)$, *for some constant* $c > 0$, *if and only if the function* $F = \langle H, \Psi^{\perp}\rangle = -1$.

4. Equivalence Theorem

In this section, we prove that the result of Wong (Theorem 3.2) is equivalent to the result of Alodan and Deshmukh (Theorem 3.3).

Theorem 4.1. *For a unit speed curve* $\Psi : [a, b] \to R^3$, $F = \langle H, \Psi^{\perp} \rangle = -1$ *holds if and only if*

$$k\left[A \cos\left(\int_0^s \tau(s) ds\right) + B \sin\left(\int_0^s \tau(s) ds\right)\right] = 1,$$

where A, B *are arbitrary constants.*

Proof. It is known that

$$k\left[A \cos\left(\int_0^s \tau(s) ds\right) + B \sin\left(\int_0^s \tau(s) ds\right)\right] = 1$$

holds if and only if there exists a differentiable function f such that $f\tau = \rho'$ and $f' + \tau\rho = 0$. Thus it is sufficient to prove that $F = -1$ if and only if there exists a differential function f such that $f\tau = \rho'$ and $f' + \tau\rho = 0$.

Suppose that $F = -1$. Then Ψ is a spherical curve. Then in this case $\| H \| = -1$ and $\mathbf{n}$ along Ψ is given by $\mathbf{n}(s) = \Psi(s)$. Thus

$$\langle \Psi, \mathbf{n} \rangle = 1 \tag{4.1}$$

which gives after differentiating with respect to s

$$\langle \mathbf{T}, \Psi \rangle = 0. \tag{4.2}$$

Differentiating equation (4.2) with respect to s and using the Frenet-Serret theorem we have

$$-\langle \mathbf{N}, \Psi \rangle = \rho.$$

Now

$$\begin{aligned} \rho' &= k\langle \mathbf{T}, \Psi \rangle - \tau\langle \mathbf{B}, \Psi \rangle \\ &= -\tau\langle \mathbf{B}, \Psi \rangle. \end{aligned}$$

Take $f = -\langle \mathbf{B}, \Psi \rangle$ which is a differentiable function and

$$f' = -\tau\rho.$$

Thus we get $f' + \tau\rho = 0$ and $f\tau = \rho'$.

Conversely, suppose that there exists a differentiable function f which satisfies $f' + \tau\rho = 0$ and $f\tau = \rho'$.

Now as

$$\rho = -\langle \mathbf{N}, \Psi \rangle$$

we get $\frac{d}{ds}\langle \mathbf{T}, \Psi \rangle = 0$ and hence $\langle \mathbf{T}, \Psi \rangle =$ constant. But as Ψ is spherical, then the constant must be zero. Now as $\Psi = \mathbf{n}$, $\mathbf{T} = \frac{d\mathbf{n}}{ds}$, we arrive at

$$\langle \mathbf{T}, \Psi \rangle = \frac{d}{ds}\langle \mathbf{n}, \Psi \rangle - \langle \mathbf{n}, \mathbf{T} \rangle = 0.$$

Thus $\frac{d}{ds}\langle \mathbf{n}, \Psi \rangle = 0$ and hence $\langle \mathbf{n}, \Psi \rangle =$ constant. But as $\mathbf{n} = \Psi$

$$\langle \mathbf{n}, \Psi \rangle = 1$$

$$F = -1.$$

This proves the theorem.

Remark. The proof we gave for Theorem 4.1 is certainly not the only possible proof. Another proof can given with more use of submanifold theory. We shall give such proof in what follows.

Suppose Ψ is a unit speed curve and $F = -1$, then Ψ is a spherical curve. Now $\Psi : [a, b] \to R^3$, can be expressed as follows:

$$\Psi = f\mathbf{T} + \lambda\mathbf{N} + \mu\mathbf{B}$$

f, λ, μ are smooth functions. Now as $f = \langle \Psi, \mathbf{T} \rangle$, then

$$\mathbf{T}(f) = \langle \mathbf{T}, \mathbf{T} \rangle + \langle \Psi, \nabla_{\mathbf{T}}\mathbf{T} + h(\mathbf{T}, \mathbf{T}) \rangle.$$

But as $\nabla_{\mathbf{T}}\mathbf{T} = 0$

$$\mathbf{T}(f) = f' = 1 + \langle \Psi^{\perp}, h(\mathbf{T}, \mathbf{T}) \rangle$$

$$= 1 + F$$

and using $F = -1$, we get $f' = 0$, and hence $f = \text{constant}$. Thus $\langle \Psi^{\top}, \mathbf{T} \rangle = 0$. This implies that $\Psi = \lambda\mathbf{N} + \mu\mathbf{B}$.

Now

$$\Psi' = \overline{\nabla}_{\mathbf{T}}\Psi = \lambda'\mathbf{N} + \lambda(-k\mathbf{T} + \tau\mathbf{B}) + \mu'\mathbf{B} + \mu(-\tau\mathbf{N}),$$

$$\mathbf{T} = -\lambda k\mathbf{T} + (\lambda' - \mu\tau)\mathbf{N} + (\lambda\tau + \mu')\mathbf{B},$$

$$0 = -(1 + \lambda k)\mathbf{T} + (\lambda' - \mu\tau)\mathbf{N} + (\lambda\tau + \mu')\mathbf{B}.$$

As $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ is a linearly independent set we get

$$-1 = \lambda k, \quad \lambda' = \mu\tau, \quad \lambda\tau = -\mu'.$$

Thus $k \neq 0$ and $\lambda = -\rho$. Hence $\rho' = -\mu\tau$ and $\mu = -\rho'\sigma$. But $\mu = \langle \Psi, \mathbf{B} \rangle$ gives

$$\mu' = -\tau\langle \Psi, \mathbf{N} \rangle = \tau\rho$$

and

$$\tau\rho = (-\rho'\sigma)'.$$

This is Darboux's condition which is equivalent to Wong's condition.

Conversely, suppose Ψ is a unit speed curve and there exists a differential function f such that $f\tau = \rho'$ and $f' + \tau\rho = 0$, i.e., Ψ is a spherical curve. We have

$$\Psi = \Psi^{\perp} = -\rho\mathbf{N} - \rho'\sigma\mathbf{B}.$$

As the curve is a 1-dimensional submanifold of R^3, the mean curvature vector is expressed as

$$H = h(\mathbf{T}, \mathbf{T}).$$

Now

$$\begin{aligned}\mathbf{T}' &= \overline{\nabla}_{\mathbf{T}}\mathbf{T} = \nabla_{\mathbf{T}}\mathbf{T} + h(\mathbf{T}, \mathbf{T})\\ &= \frac{1}{\rho}\mathbf{N}\end{aligned}$$

which gives

$$\begin{aligned}F &= \langle H, \Psi^{\perp}\rangle\\ &= \left\langle \frac{1}{\rho}\mathbf{N}, -\rho\mathbf{N} - \rho'\sigma\mathbf{B}\right\rangle\\ &= -1.\end{aligned}$$

5. Examples of Spherical and Non-spherical Submanifolds

In this section we give examples of spherical submanifolds where the condition $F = -1$ is met and examples of non-spherical submanifolds where the condition $F = -1$ is violated.

Example 5.1. Consider the hyperspheres $\Psi_1 : S^2(c_1) \to R^3$,

$$\Psi_1(u, v) = \left(u, v, \sqrt{\frac{1}{c_1} - u^2 - v^2}\right)$$

and $\Psi_2 : S^2(c_2) \to R^3$,

$$\Psi_2(r, s) = \left(r, s, \sqrt{\frac{1}{c_2} - r^2 - s^2}\right)$$

and let $M = S^2(c_1) \times S^2(c_2)$. Define $\Psi : M \to R^6$ by $\Psi = (\Psi_1, \Psi_2)$, such that $\langle \Psi_1, \Psi_2 \rangle = 0$.

It is easy to check

$$F = \langle H, \Psi \rangle$$

$$= -1.$$

Thus M is a spherical submanifold.

Example 5.2. Consider the unit sphere $S^n(1) \subset R^{n+1}$ with the natural embedding $i : S^n(1) \to R^{n+1}$. Let $M = S^m\left(\sqrt{\frac{m}{n-1}}\right) \times S^k\left(\sqrt{\frac{k}{n-1}}\right)$ be the product of the spheres $S^m\left(\sqrt{\frac{m}{n-1}}\right) \times S^k\left(\sqrt{\frac{k}{n-1}}\right)$, where $m + k = n - 1$. Then from Simon [11], we know that there exists an immersion $f : M \to S^n(1)$ which is a minimal hypersurface.

Now consider $\Psi = i \circ f : M \to R^{n+1}$ as an $(n-1)$-dimensional submanifold of R^{n+1}. Let N_1 be the unit normal of M in $S^n(1)$ and $N_2 = i$ the unit normal of $S^n(1)$ in R^{n+1}. Then $\{N_1, N_2\}$ is an orthonormal frame of normals for the submanifold M in R^{n+1}.

The mean curvature vector H of M in R^{n+1} is given by

$$nH = -nN_2,$$

where we used $trA_{N_1} = 0$ as M is a minimal hypersurface of $S^n(1)$. Thus

$$H = -N_2.$$

Also as

$$\Psi(p) = (i \circ f)(p)$$

$$= N_2(f(p)),$$

where $i = N_2$, treating i as a position vector field of $S^n(1)$.

Thus we have

$$F = \langle H, \Psi^{\perp} \rangle$$

$$= -1$$

which gives an example of a spherical submanifold M.

Example 5.3. Consider the surface of revolution given by

$$x(\theta, \phi) = ((2 + \cos\theta)\cos\phi, (2 + \cos\theta)\sin\phi, \sin\theta),$$

where $x : U \to R^3$, $U = \{(\theta, \phi), 0 < \theta < \pi, -\pi < \phi < \pi\}$. Then x is an immersion and

$$x^{\perp} = (2\cos\theta + 1)(\cos\theta\cos\phi, \cos\theta\sin\phi, \sin\theta),$$

we have

$$F = \langle H, x^{\perp} \rangle$$

$$= \frac{3\cos\theta + 2\cos^2\theta + 1}{2 + \cos\theta}$$

$$\neq -1.$$

This gives an example of a submanifold with $F \neq -1$.

Remark. In the previous example, we found that $F \neq -1$ as $x^{\top} \neq 0$.

Example 5.4. Consider the simple surface $x : U \to R^3$ defined by

$$x(s, \theta) = \left(s \cos \theta,\ s \sin \theta,\ \sqrt{1 + s^2}\right),$$

where $U = \{(s, \theta), s > 0, 0 < \theta < 2\pi\}$. Then clearly x is an immersion and

$$x^{\perp} = \frac{1}{\sqrt{1 + 2s^2}} \mathbf{n}$$

and

$$\begin{aligned} F &= \langle H, x^{\perp} \rangle \\ &= \left(\frac{(1 + s^2)}{(1 + 2s^2)^2} \right) \\ &\neq -1. \end{aligned}$$

This is another example with $F \neq -1$.

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