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SOME RESULTS ASSOCIATED WITH THE ANALYTIC PART OF HARMONIC UNIVALENT FUNCTIONS

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Abstract

In this paper we introduce a new class $\mathcal{MA}_{\mathcal{H}}(\alpha, \beta, \gamma)$ of harmonic univalent functions defined in the open unit disk. We derive the fundamental geometric properties like the coefficient bound inequality, distortion theorems and radii of starlikeness, close-to-convexity and convexity for the analytic part of the harmonic function belonging to the class $\mathcal{MA}_{\mathcal{H}}(\alpha, \beta, \gamma)$. Applications of fractional calculus operators are exhibited in establishing the distortion theorems.

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1. Introduction

A continuous function $f = u + iv$ is a complex-valued harmonic function in a domain $D \subset \mathbb{C}$ if both u and v are real harmonic in D. In any simply-connected domain the function f can be represented by

$$f = h + \overline{g}, \tag{1.1}$$

where h and g are analytic in D. The functions h and g are, respectively, called the analytic and co-analytic parts of the function f. It is observed (see [3], [7]) that f is locally univalent and sense preserving in D if and only if

$$|g'(z)| < |h'(z)| \quad (z \in D).$$

We denote by $\mathcal{H}$ the class of functions f of the form (1.1) that are harmonic univalent and sense-preserving in the unit disk $\mathcal{U} = \{z : |z| < 1\}$ for which

$$f(0) = f_z(0) - 1 = 0,$$

and we may write then

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} b_n z^n, \quad |b_1| < 1. \tag{1.2}$$

Jahangiri [3] defined a class $\mathcal{MA}_{\mathcal{H}}(\alpha)$ consisting of harmonic starlike functions $f = h + \overline{g}$ which are of order α $(0 \le \alpha < 1)$, where

$$h(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, \quad g(z) = \sum_{n=1}^{\infty} |b_n| z^n \tag{1.3}$$

and which satisfy the condition that

$$\frac{\partial}{\partial \theta}(\arg f(re^{i\theta}) \ge \alpha, \quad 0 \le \alpha < 1, \quad |z| = r < 1. \tag{1.4}$$

It was proved in [3] that if $f = h + \overline{g}$ where h, g are given by (1.1) and if

$$\sum_{n=1}^{\infty} \left(\frac{n - \alpha}{1 - \alpha} |a_n| + \frac{n + \alpha}{1 - \alpha} |b_n| \right) \le 2 \quad (0 \le \alpha < 1,\ a_1 = 1), \tag{1.5}$$

then f is harmonic, univalent and starlike of order α in $\mathcal{U}$. The condition in (1.5) is shown to be necessary also if h and g are of the form (1.3). Avci and Zlotkiewicz in [1] showed that if $\sum_{n=2}^{\infty} n(|a_n| + |b_n|) \leq 1$, then $f \in \mathcal{MA}_{\mathcal{H}}(0)$. Silverman in [7] proved that the last condition is also necessary if $f = h + \overline{g}$ has negative coefficients.

We introduce here a new class $\mathcal{MA}_{\mathcal{H}}(\alpha, \beta, \gamma)$ of harmonic univalent functions defined as follows:

A function $f(= h + \overline{g}) \in \mathcal{MA}_{\mathcal{H}}(\alpha)$ is said to be in the class $\mathcal{MA}_{\mathcal{H}}(\alpha, \beta, \gamma)$ if the analytic functions h and g satisfy the condition

$$\Re\{1 + \alpha z^2 h''(z) - \beta z h'(z) + \beta g(z)\} > 1 - |\gamma|, \tag{1.6}$$

where $0 \leq \beta \leq \alpha$, $\alpha \geq 0$, $\gamma \in \mathbb{C}$, $z \in \mathcal{U}$.

In this paper we obtain sufficient conditions for a function f to be in the class $\mathcal{MA}_{\mathcal{H}}(\alpha, \beta, \gamma)$ and derive distortion theorems by using fractional calculus operators. We also obtain the radii of starlikeness, close-to-convexity and convexity of functions belonging to the above class.

2. Coefficient Bound Inequalities

We begin by proving the following:

Theorem 1. *Let* $f = h + \overline{g}$ *(h and g being given by* (1.3)*). If* $f \in \mathcal{MA}_{\mathcal{H}}(\alpha, \beta, \gamma)$, *then*

$$\sum_{n=2}^{\infty}\left[n(\alpha(n-1) - \beta)|a_n| - \beta\frac{1 - 3\alpha}{n + \alpha}\right] \leq |\gamma|, \tag{2.1}$$

where $a_1 = b_1 = 1$, $0 < \alpha \leq 1/3$, $0 \leq \beta < \alpha$ *and* $\gamma \in \mathbb{C}$. *The result is sharp.*

Proof. Assume that $f \in \mathcal{MA}_{\mathcal{H}}(\alpha, \beta, \gamma)$. Using (1.6), we get

$$\Re\left\{1 - \sum_{n=2}^{\infty} \alpha n(n-1)|a_n| z^n + \sum_{n=2}^{\infty} \beta n|a_n| z^n + \sum_{n=2}^{\infty} \beta|b_n| z^n\right\} > 1 - |\gamma|.$$

Choosing z real and letting $z \to 1^{-}$, we obtain

$$1 - \left[\sum_{n=2}^{\infty} n(\alpha(n-1) - \beta) | a_n | - \sum_{n=2}^{\infty} \beta | b_n | \right] \geq 1 - | \gamma |$$

which implies

$$\sum_{n=2}^{\infty} n(\alpha(n-1) - \beta) | a_n | - \beta | b_n | \leq | \gamma |. \tag{2.2}$$

Since $f \in \mathcal{MA}_{\mathcal{H}}(\alpha)$, therefore, it follows that

$$\sum_{n=1}^{\infty} \frac{n+\alpha}{1-\alpha} | b_n | \leq 2$$

and from the above inequality we infer that

$$| b_n | \leq \left(\frac{1-3\alpha}{n+\alpha} \right) \quad (n \geq 2),$$

and consequently

$$\beta | b_n | \leq \beta \frac{1-3\alpha}{n+\alpha}. \tag{2.3}$$

The assertion (2.1) of Theorem 1 now follows upon combining (2.2) and (2.3).

Corollary 1. *If the function* $f = h + \overline{g}$, *where* h *and* g *are given by* (1.3), *belongs to the class* $\mathcal{MA}_{\mathcal{H}}(\alpha, \beta, \gamma)$, *then*

$$| a_n | \leq \frac{(n+\alpha) | \gamma | + \beta(1-3\alpha)}{n(\alpha(n-1) - \beta)(n+\alpha)} \quad (n \geq 2,\ 0 < \alpha \leq 1/3,\ 0 \leq \beta < \alpha,\ \gamma \in \mathbb{C}). \tag{2.4}$$

The result is sharp for the functions $h(z)$ *and* $g(z)$, *respectively, given by*

$$h(z) = z - \frac{(n+\alpha) | \gamma | + \beta(1-3\alpha)}{n(\alpha(n-1) - \beta)(n+\alpha)} z^n \quad (n \geq 2), \tag{2.5}$$

and

$$g(z) = z + \frac{\beta(1-3\alpha)}{n+\alpha} z^n \quad (n \geq 2). \tag{2.6}$$

3. Distortion Theorems Involving Fractional Calculus Operators

We require the following Riemann-Liouville type operators of fractional calculus due to Owa [4].

The fractional integral of order μ of a function $h(z)$ is defined by

$$D_z^{-\mu}\{h(z)\} = \frac{1}{\Gamma(\mu)}\int_0^z \frac{h(\xi)}{(z-\xi)^{1-\mu}}\,d\xi \quad (\mu > 0), \tag{3.1}$$

where $h(z)$ is analytic in a simply-connected region of the z-plane containing the origin and the multiplicity of $(z-\xi)^{\mu-1}$ is removed by requiring $\log(z-\xi)$ to be real when $z-\xi > 0$.

The fractional derivative of order μ of a function $h(z)$ is defined by

$$D_z^{\mu}\{h(z)\} = \frac{1}{\Gamma(1-\mu)}\frac{d}{dz}\int_0^z \frac{h(\xi)}{(z-\xi)^{\mu}}\,d\xi \quad (0 \le \mu < 1), \tag{3.2}$$

where $h(z)$ is chosen as in (3.1), and the multiplicity of $(z-\xi)^{-\mu}$ is removed in the same manner as stated above with (3.1).

Further, under the hypotheses stated with (3.2), the fractional derivative of order $j+\mu$ is defined by

$$D_z^{\mu+j}h(z) = \frac{d^j}{dz^j}D_z^{\mu}h(z), \quad 0 \le \mu < 1; \quad j \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}. \tag{3.3}$$

Making use of the operators (3.1) and (3.2), we establish the following distortion theorems.

Theorem 2. *If the function* $f = h + \overline{g}$ *(*h *and* g *being given by* (1.3)*) belongs to the class* $\mathcal{MA}_{\mathcal{H}}(\alpha, \beta, \gamma)$*, then*

$$|D_z^{-\mu}h(z)| \ge \frac{|z|^{1+\mu}}{\Gamma(2+\mu)}\left\{1 - \frac{(2+\alpha)|\gamma| + \beta(1-3\alpha)}{(\alpha-\beta)(2+\alpha)(2+\mu)}|z|\right\} \tag{3.4}$$

and

$$|D_z^{-\mu}h(z)| \le \frac{|z|^{1+\mu}}{\Gamma(2+\mu)}\left\{1 + \frac{(2+\alpha)|\gamma| + \beta(1-3\alpha)}{(\alpha-\beta)(2+\alpha)(2+\mu)}|z|\right\}, \tag{3.5}$$

for $\mu > 0$, $0 < \alpha \leq 1/3$, $0 \leq \beta < \alpha$, $\gamma \in \mathbb{C}$ *and* $z \in \mathcal{U}$. *The results are sharp*.

Proof. Applying (3.1) to (1.3), we find that

$$D_z^{-\mu} h(z) = \frac{1}{\Gamma(2+\mu)} z^{1+\mu} - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n+1+\mu)} | a_n | z^{n+\mu}$$

which yields

$$\Gamma(2+\mu) z^{-\mu} D_z^{-\mu} f(z) = z - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2+\mu)}{\Gamma(n+1+\mu)} | a_n | z^n.$$

We observe that the function $\varphi(n) = \dfrac{\Gamma(n+1)\Gamma(2+\mu)}{\Gamma(n+1+\mu)}$ $(n \geq 2)$ is a decreasing function of n, and this implies that

$$0 < \varphi(n) \leq \varphi(2) = \frac{2}{2+\mu} \quad (n \geq 2).$$

In view of Theorem 1, we conclude that

$$\left| \Gamma(2+\mu) z^{-\mu} D_z^{-\mu} h(z) \right| \geq | z | - \varphi(2) | z |^2 \sum_{n=2}^{\infty} | a_n |$$

$$\geq | z | - \frac{(2+\alpha) | \gamma | + \beta(1-3\alpha)}{(\alpha-\beta)(2+\alpha)(2+\mu)} | z |^2$$

which gives the desired assertion (3.4) of Theorem 2. Similarly, it follows that

$$\left| \Gamma(2+\mu) z^{-\mu} D_z^{-\mu} h(z) \right| \leq | z | + \varphi(2) | z |^2 \sum_{n=2}^{\infty} | a_n |$$

$$\leq | z | + \frac{(2+\alpha) | \gamma | + \beta(1-3\alpha)}{(\alpha-\beta)(2+\alpha)(2+\mu)} | z |^2.$$

This yields the second assertion (3.5) of Theorem 2. The sharpness of the inequalities (3.4) and (3.5) are achieved when $h(z)$ is given by

$$D_z^{-\mu}h(z) = \frac{z^{1+\mu}}{\Gamma(2+\mu)}\left\{1 - \frac{(2+\alpha)|\gamma| + \beta(1-3\alpha)}{2(\alpha-\beta)(2+\alpha)(2+\mu)}z\right\}.$$

Following similar steps as used in proving Theorem 2, wherein, the fractional derivative operator (3.2) is applied to (1.3), we can prove the following.

Theorem 3. *If the function* $f = h + \overline{g}$ *(h and g being given by* (1.3)) *belongs to the class* $\mathcal{MA}_{\mathcal{H}}(\alpha, \beta, \gamma)$, *then*

$$|D_z^{\mu}h(z)| \geq \frac{|z|^{1-\mu}}{\Gamma(2-\mu)}\left\{1 - \frac{(2+\alpha)|\gamma| + \beta(1-3\alpha)}{(\alpha-\beta)(2+\alpha)(2-\mu)}|z|\right\} \tag{3.6}$$

and

$$|D_z^{\mu}h(z)| \leq \frac{|z|^{1-\mu}}{\Gamma(2-\mu)}\left\{1 + \frac{(2+\alpha)|\gamma| + \beta(1-3\alpha)}{(\alpha-\beta)(2+\alpha)(2-\mu)}|z|\right\}, \tag{3.7}$$

where $0 \leq \mu < 1$, $0 < \alpha \leq 1/3$, $0 \leq \beta < \alpha$, $\gamma \in \mathbb{C}$ *and* $z \in \mathcal{U}$. *The results are sharp for the function given by*

$$D_z^{\mu}h(z) = \frac{z^{1-\mu}}{\Gamma(2-\mu)}\left\{1 - \frac{(2+\alpha)|\gamma| + \beta(1-3\alpha)}{(\alpha-\beta)(2+\alpha)(2-\mu)}z\right\}. \tag{3.8}$$

Also, by letting $\mu \to 0$ in Theorem 2, we obtain

Corollary 2. *If the function* $f = h + \overline{g}$ *(h and g being given by* (1.3)) *belongs to the class* $\mathcal{MA}_{\mathcal{H}}(\alpha, \beta, \gamma)$, *then*

$$r - \frac{(2+\alpha)|\gamma| + \beta(1-3\alpha)}{2(\alpha-\beta)(2+\alpha)}r^2 \leq |h(z)| \leq r + \frac{(2+\alpha)|\gamma| + \beta(1-3\alpha)}{2(\alpha-\beta)(2+\alpha)}r^2, \tag{3.9}$$

where $0 < \alpha \leq 1/3$, $0 \leq \beta < \alpha$, $\gamma \in \mathbb{C}$ *and* $|z| = r < 1$. *The result is sharp.*

Also, by letting $\mu \to 1$ in Theorem 3, we get

Corollary 3. *If the function* $f = h + \overline{g}$ *(h and g being given by* (1.3)) *belongs to the class* $\mathcal{MA}_{\mathcal{H}}(\alpha, \beta, \gamma)$, *then*

$$1 - \frac{(2+\alpha)|\gamma| + \beta(1-3\alpha)}{(\alpha-\beta)(2+\alpha)}r \leq |h'(z)| \leq 1 + \frac{(2+\alpha)|\gamma| + \beta(1-3\alpha)}{(\alpha-\beta)(2+\alpha)}r, \tag{3.10}$$

where $0 < \alpha \leq 1/3$, $0 \leq \beta < \alpha$, $\gamma \in \mathbb{C}$ *and* $|z| = r < 1$. *The result is sharp.*

4. Further Distortion Theorems

We begin by recalling the fractional integral operator $I_{0,z}^{\lambda,\xi,\delta}$ and the fractional derivative operator $\mathcal{J}_{0,z}^{\lambda,\xi,\delta}$ of Saigo type as follows [6] (see also [2] and [5]):

The fractional integral operator of order λ for a function $h(z)$ is defined by

$$I_{0,z}^{\lambda,\xi,\delta}h(z) = \frac{z^{-\lambda-\xi}}{\Gamma(\lambda)}\int_0^z (z-t)^{\lambda-1}\,{}_2F_1\left(\lambda+\xi, -\delta; \lambda; 1-\frac{t}{z}\right)h(t)dt, \tag{4.1}$$

where $\lambda > 0$, $k > \max\{0, \xi - \delta\} - 1$. The function $h(z)$ is analytic in a simply-connected region of the z-plane containing the origin with the order $h(z) = O(|z|^k)$, $z \to 0$, the function ${}_2F_1(a, b; c; z)$ is the well known Gauss hypergeometric function defined by

$${}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n(1)_n} z^n,$$

where

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} a(a+1)\cdots(a+n-1) & \text{if } n \in \mathbb{N} \\ 1 & \text{if } n = 0, \end{cases}$$

and the multiplicity of $(z-t)^{\lambda-1}$ is removed by requiring $\log(z-t)$ to be real when $(z-t) > 0$.

The fractional derivative operator of a function $h(z)$ of order λ $(0 \leq \lambda < 1)$ is defined by

$$\mathcal{J}_{0,z}^{\lambda,\xi,\delta}h(z) = \frac{1}{\Gamma(1-\lambda)}\frac{d}{dz}\left\{z^{\lambda-\xi}\int_0^z (z-t)^{-\lambda}\,{}_2F_1\left(\xi-\lambda, 1-\delta; 1-\lambda; 1-\frac{t}{z}\right)\right\}h(t)d(t)$$

$$(0 \leq \lambda < 1) \tag{4.2}$$

which holds under similar constraints as mentioned for the operator $I_{0,z}^{\lambda,\xi,\delta}$ defined above by (4.1).

We observe the following relationships of the operators (4.1) and (4.2):

$$I_{0,z}^{\lambda,-\lambda,\delta}h(z) = D_z^{-\lambda}h(z) \quad \text{and} \quad J_{0,z}^{\lambda,\lambda,\delta}h(z) = D_z^{\lambda}h(z).$$

Corresponding to the fractional derivative operator (4.2), we make use of the operator $\Delta_{0,z}^{\lambda,\xi,\delta}$ which is defined by ([2])

$$\Delta_{0,z}^{\lambda,\xi,\delta}h(z) = \frac{\Gamma(2-\xi)\Gamma(2-\lambda+\xi)}{\Gamma(2-\xi+\delta)} z^{\xi} \mathcal{J}_{0,z}^{\lambda,\xi,\delta}h(z) \quad (0 \le \lambda < 1, \xi < 2, \delta > 0). \tag{4.3}$$

In view of (1.3) and (4.3) we obtain the series representation

$$\Delta_{0,z}^{\lambda,\xi,\delta}h(z) = z - \sum_{n=2}^{\infty} \frac{(2-\xi+\delta)_{n-1}(2)_{n-1}}{(2-\xi)_{n-1}(2-\lambda+\delta)_{n-1}} |\, a_n \,| z^n. \tag{4.4}$$

Theorem 4. *Let the function* $f = h + \overline{g}$ *such that* h *and* g *are given by* (1.3). *If* $f \in \mathcal{MA}_{\mathcal{H}}(\alpha, \beta, \gamma)$, *then*

$$|\, \Delta_{0,z}^{\nu,\sigma,s}h(z) | \ge |\, z \,| - |\, z \,|^2 \frac{(2-\sigma+s)[(2+\alpha)|\,\gamma\,| + \beta(1-3\alpha)]}{(2-\sigma)(2-\nu+s)(\alpha-\beta)(2+\alpha)} \tag{4.5}$$

and

$$|\, \Delta_{0,z}^{\nu,\sigma,s}h(z) | \le |\, z \,| + |\, z \,|^2 \frac{(2-\sigma+s)[(2+\alpha)|\,\gamma\,| + \beta(1-3\alpha)]}{(2-\sigma)(2-\nu+s)(\alpha-\beta)(2+\alpha)}, \tag{4.6}$$

where $0 < \alpha \le 1/3$, $0 \le \beta < \alpha$, $\gamma \in \mathbb{C}$, $0 \le \nu < 1$, $\sigma < 2$, $s \in \mathbb{R}_+$, *provided that*

$$\frac{\sigma(\nu - s)}{\nu} \le 3.$$

The results are sharp.

Proof. From (2.1) of Theorem 1, we obtain

$$\sum_{n=2}^{\infty} |\, a_n \,| \le \frac{(2+\alpha)|\,\gamma\,| + \beta(1-3\alpha)}{2(\alpha-\beta)(2+\alpha)},$$

and in view of (4.4) the operator $\Delta_{0,z}^{\nu,\sigma,s}$ applied to $h(z)$ gives

$$\Delta_{0,z}^{\nu,\sigma,s} h(z) = z - \sum_{n=2}^{\infty} \frac{(2-\sigma+s)_{n-1}(2)_{n-1}}{(2-\sigma)_{n-1}(2-\nu+s)_{n-1}} |\, a_n \,| z^n. \tag{4.7}$$

Under the assumptions stated with Theorem 4, we observe that the function

$$\psi(n) = \frac{(2-\sigma+s)_{n-1}(2)_{n-1}}{(2-\sigma)_{n-1}(2-\nu+s)_{n-1}} \quad (n \geq 2)$$

is non-increasing and, therefore, we get

$$0 < \psi(n) \leq \psi(2) = \frac{(2-\sigma+s)(2)}{(2-\sigma)(2-\nu+s)}.$$

Consequently, we find that

$$|\, \Delta_{0,z}^{\nu,\sigma,s} h(z) | \geq |\, z \,| - |\, z \,|^2 \psi(2) \sum_{n=2}^{\infty} |\, a_n \,|$$

$$\geq |\, z \,| - |\, z \,|^2 \frac{(2-\sigma+s)[(2+\alpha)|\, \gamma \,| + \beta(1-3\alpha)]}{(2-\sigma)(2-\nu+s)(\alpha-\beta)(2+\alpha)}$$

and

$$|\, \Delta_{0,z}^{\nu,\sigma,s} h(z) | \leq |\, z \,| + |\, z \,|^2 \frac{(2-\sigma+s)[(2+\alpha)|\, \gamma \,| + \beta(1-3\alpha)]}{(2-\sigma)(2-\nu+s)(\alpha-\beta)(2+\alpha)}.$$

The inequalities (4.5) and (4.6) are sharp, and the equalities are attained for the function $h(z)$ given by

$$h(z) = z - \frac{[(2+\alpha)|\, \gamma \,| + \beta(1-3\alpha)]}{2(\alpha-\beta)(2+\alpha)} z^2. \tag{4.8}$$

Evidently, in the special case when $\sigma = \nu$, Theorem 4 would correspond to Theorem 3. In a similar manner, one can obtain the distortion inequalities involving the Saigo type fractional integral operator (4.1).

5. Radii of Starlikeness, Close-to-Convexity and Convexity

We merely state here the following results (Theorems 5 to 7) giving the radii of starlikeness, close-to-convexity and convexity for the analytic part of the function f belonging to the class $\mathcal{MA}_{\mathcal{H}}(\alpha, \beta, \gamma)$. The proofs of these results are omitted here as these results can be established by following [6] (see also [5]).

Theorem 5. *Let* $f \in \mathcal{MA}_{\mathcal{H}}(\alpha, \beta, \gamma)$, *where* $f = h + \overline{g}$ *with* h *and* g *given by* (1.3). *Then* $h(z)$ *is starlike of order* ρ $(0 \le \rho < 1)$ *in* $|z| < r_1$, *where*

$$r_1 = \inf_n \left[\frac{2(\alpha - \beta)(2 + \alpha)(1 - \rho)}{[(2 + \alpha)|\gamma| + \beta(1 - 3\alpha)](n - \rho)}\right]^{\frac{1}{n-1}}, \quad (n \ge 2). \tag{5.1}$$

Theorem 6. *Let* $f \in \mathcal{MA}_{\mathcal{H}}(\alpha, \beta, \gamma)$, *where* $f = h + \overline{g}$ *with* h *and* g *given by* (1.3). *Then* $h(z)$ *is close-to-convex of order* ρ $(0 \le \rho < 1)$ *in* $|z| < r_2$, *where*

$$r_2 = \inf_n \left[\frac{(1 - \rho)(\alpha - \beta)(2 + \alpha)}{(2 + \alpha)|\gamma| + \beta(1 - 3\alpha)}\right]^{\frac{1}{n-1}}, \quad n \ge 2. \tag{5.2}$$

Theorem 7. *Let* $f \in \mathcal{MA}_{\mathcal{H}}(\alpha, \beta, \gamma)$, *where* $f = h + \overline{g}$ *with* h *and* g *given by* (1.3). *Then* $h(z)$ *is convex of order* ρ $(0 \le \rho < 1)$ *in* $|z| < r_3$, *where*

$$r_3 = \inf_n \left[\frac{(1 - \rho)(\alpha - \beta)(2 + \alpha)}{[(2 + \alpha)|\gamma| + \beta(1 - 3\alpha)](n - \rho)}\right]^{\frac{1}{n-1}}, \quad n \ge 2. \tag{5.3}$$

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