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COMPONENTS AND QUASICOMPONENTS OF SUBSETS OF CONTINUA

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Abstract

In [Amer. Math. Soc. Colloq. Publ., Vol. 28, Amer. Math. Soc., Providence, R. I., 1942], Whyburn proves that a continuum, M, is hereditarily locally connected if and only if the components and the quasicomponents of any subset of M are identical. We consider continua for which various types of subsets have identical classes of quasicomponents and components, and examine some of the properties of such continua. In particular, we prove that a continuum, M, is locally connected if and only if the components and quasicomponents of any *open* subset of M are identical.

1. Introduction

A *continuum* is a compact, connected metric space; a *Hausdorff continuum* is a compact, connected Hausdorff space. A topological space, S, is said to be *locally connected* provided that, for every $p \in S$, and every neighborhood, U, of p, there exists a connected, open neighborhood,

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V, with $p \in V \subseteq U$. A continuum is said to be *hereditarily locally connected* provided that each of its subcontinua is locally connected. We will occasionally use LC and HLC to denote the classes of locally connected and hereditarily locally connected continua, respectively. Recall that, as global properties, local connectedness and connectedness im kleinen are equivalent [2, 5.22]. A *component* of a topological space, S, is a maximal, connected subset of S. A quasicomponent, Q, of S is a maximal subset of S for which no separation of S exists between points of Q; equivalently, for any $p \in S$, the quasicomponent of S that contains p is the intersection of all simultaneously open and closed subsets of S which contain p.

In [3], Whyburn proves that a continuum, M, is hereditarily locally connected if and only if the class of components and the class of quasicomponents of each subset of M are identical. We obtain a similar characterization in 2.5 and 2.6 where we prove that a continuum, M, is locally connected if and only if the class of components and the class of quasicomponents of each *open* subset of M are identical. Related results about aposyndetic continua are obtained in 2.8 and 2.9. Recall that a continuum, M, is *aposyndetic* provided that, for any $p, q \in M$, there is a closed, connected neighborhood of p which misses q. We define a continuum to be *strongly aposyndetic* provided that, for any $p \in M$, and any closed, connected subset, C, of $M - \{p\}$, there is a closed, connected neighborhood of p which misses C. We will occasionally use AP and SAP to denote the classes of aposyndetic and strongly aposyndetic continua, respectively.

Remark 1.1. We note that the notion of strong aposyndesis has been studied under a different guise. In [1], Davis defines a set-valued function, T, on the collection of subsets of a Hausdorff continuum, M, as follows: for any $A \subseteq M$, $T(A)$ is equal to the set of all points, p, for which every subcontinuum of M that contains p in its interior intersects A nonvoidly. It is easy to see that a Hausdorff continuum, M, is strongly aposyndetic if and only if $T(C) = C$ for every closed, connected subset, C, of M. Moreover, M is aposyndetic if and only if $T(\{p\}) = \{p\}$ for every $p \in M$.

We now define the terms upon which our investigation will focus.

Definition 1.2. Let S be a topological space. We will say that $Q = C$ in S to mean that the class of quasicomponents of S is identical to the class of components of S.

Definition 1.3. Let S be a topological space. We will say that S belongs to class QC (QCC, QCS, QCP) provided that, if N is any proper subset (proper closed subset, proper subcontinuum, singleton subset) of S, then $Q = C$ in $S - N$.

Remark 1.4. Following the pattern of definitions in 1.3, it is natural to define a space, S, to belong to class QCO provided that $Q = C$ in $S - N$ for any proper, open subset, N, of S. We note that, since $Q = C$ in every compact, Hausdorff space [2, 5.18], and since every closed subset of a compact, Hausdorff space is also compact Hausdorff, it follows that all compact, Hausdorff spaces belong to class QCO.

We are now prepared to state the results that we will prove in this article. For convenience, these statements are presented together in the following theorem.

Theorem 1.5. *The following containments hold for the indicated classes of nondegenerate continua*:

$$\begin{array}{ccccccc} HLC & \subset & LC & = & SAP & \subset & AP \\ \| & & \| & & \cap & & \cap \\ QC & \subset & QCC & \subset & QCS & \subset & QCP. \end{array}$$

Moreover, the containments which are presented above as one-sided are, in fact, strict.

2. Proof of the Class Containments Stated in 1.5

In this section, we will prove that all of the containments presented in 1.5 hold in the setting of nondegenerate continua. We begin by demonstrating that the class of strongly aposyndetic continua coincides with the class of locally connected continua.

Proposition 2.1. *A Hausdorff continuum is strongly aposyndetic if and only if it is locally connected.*

Proof. In [1], Davis proves that a Hausdorff continuum is locally connected if and only if $T(C) = C$ for every subcontinuum, where T is the set-valued function described in 1.1. So, by 1.1, we have that a Hausdorff continuum is strongly aposyndetic if and only if it is locally connected.

All of the other "horizontal containments" stated in 1.5 are obvious. The following observation will facilitate the proofs of some of the remaining propositions.

Lemma 2.2. *If the components of a topological space*, S, *are open, then* $Q = C$ *in* S.

Proof. Assume that the components of a space, S, are open, and let K be a component of S. Then, $S - K$ is the union of all of the other components of S; that is, $S - K$ is a union of open sets. Therefore, K is closed in S. Since K is also open, it follows that K is a quasicomponent. Thus, $Q = C$ in S.

Remark 2.3. The following example demonstrates that the converse of 2.2 is false: For each $n = 1, 2, \ldots,$ let $I_n = \left\{\frac{1}{n}\right\} \times [-1, 1],$ and let $p = (0, 0).$ Then, $Q = C$ in $S = \{p\} \cup \bigcup_{i=1}^{n} I_n.$ However, $\{p\}$ is a component of S that is not open in S.

Proposition 2.4. *A continuum*, M, *belongs to class QC if and only if* M *is hereditary locally connected*.

Proof. Whyburn proves this result in [3, V. 2.4].

Proposition 2.5. *Every locally connected topological space belongs to class QCC*.

Proof. Let S be a locally connected topological space, and let N be a closed subset of X. Then, since $S - N$ is open, the components of $S - N$ are open, by [2, 5.22(a)]. So, $Q = C$ in $S - N$ by 2.2. Therefore, $S \in QCC$.

Theorem 2.6. *If a continuum, M, belongs to class QCC, then M is locally connected.*

Proof. Suppose that there is some $p \in M$ at which M fails to be connected im kleinen. We will prove that M fails to have property QCC by constructing a set, W, such that

$$W \text{ is an open set containing } p \text{ for which the component of } p \text{ in } W \text{ is different from the quasicomponent of } p \text{ in } W. \quad (*)$$

Since M fails to be connected im kleinen at p, there exists some open set, U, about p which contains a sequence of points, $\{p_i\}_{i=1}^{\infty}$, and a sequence of subsets, $\{K_i\}_{i=0}^{\infty}$, such that

$$p = \lim p_i \tag{1}$$

$$K_0 \text{ is the component of } p \text{ in } U \tag{2}$$

$$K_i \text{ is the component of } p_i \text{ in } U \text{ for each } i \geq 1 \tag{3}$$

$$K_i \cap K_j = \varnothing \text{ for every } i \neq j. \tag{4}$$

Let μ be an open subset of U, such that

$$p \in \mu \subseteq \overline{\mu} \subseteq U. \tag{5}$$

It follows from (1) that there is some integer, $N \geq 1$, such that $p_i \in \mu$ for all $i \geq N$. For each $i \geq N$, let C and C_i be subsets of $\overline{\mu}$ such that

$$C_i \text{ is the component of } p_i \text{ in } \overline{\mu} \tag{6}$$

$$C = \lim C_i. \tag{7}$$

Observe that, since $p_i \in C_i$ for each $i \geq N$, it follows from (1) and (7) that

$$p \in C. \tag{8}$$

Moreover, it follows from (6) that each C_i is a subcontinuum of $\overline{\mu}$; thus, we have, by (7), that C is a subcontinuum of $\overline{\mu}$. So, since C is connected, it follows from (2) and (8) that $C \subseteq K_0$. Moreover, it follows from (3), (6), and (5) that $C_i \subseteq K_i$ for every $i \geq N$. Therefore, it follows from (4) that

$$C_i \cap (C \cup C_j) = \varnothing \text{ for every } i \neq j, \quad i, j \geq N. \tag{9}$$

Since each C_i is a component of $\overline{\mu}$, there is some $q_i \in C_i \cap Bd(\mu)$ for every $i \geq N$ [2, 5.6]. Moreover, it follows from (9) that the points, q_i, are distinct. Therefore, $\{q_i : i \geq N\}$ has at least one cluster point, q. Since $Bd(\mu)$ is closed in M, it follows from (7) that

$$q \in C \cap Bd(\mu). \tag{10}$$

Thus, since μ is open in M, it follows from (5) that $p \neq q$. Let V be an open subset of μ such that $p \in V$ and $q \notin \overline{V}$. Then, $K_0 \cap Bd(V)$ separates p and q in K_0. So, if K denotes the component of p in $K_0 - (K_0 \cap Bd(V))$, then

$$q \notin K. \tag{11}$$

Moreover, since K is the component of p in $K_0 - (K_0 \cap Bd(V))$, it follows from (2) that

$$K \text{ is the component of } p \text{ in } U - (K_0 \cap Bd(V)). \tag{12}$$

Since $C_i \subseteq U$ for every $i \geq N$, and $C \subseteq K_0$, it follows from (4) that $C_i \subseteq U - (K_0 \cap Bd(V))$ for every $i \geq N$. So, by (7) and (8), every open neighborhood of p in $U - (K_0 \cap Bd(V))$ meets infinitely many C_i; thus, by (6), every simultaneously open and closed neighborhood of p in $U - (K_0 \cap Bd(V))$ contains infinitely many C_i. Therefore, since $q \in U$ and $q \notin \overline{V}$, it follows from (7) and (10) that

$$q \text{ is in the quasicomponent of } U - (K_0 \cap Bd(V)) \text{ containing } p. \tag{13}$$

Moreover, since $Bd(V)$ is closed, it follows from (2) that $K_0 \cap Bd(V)$ is closed in U. Thus, since U is open in M, we have that

$$U - (K_0 \cap Bd(V)) \text{ is open in } M. \tag{14}$$

So, by (11), (12), (13) and (14), we have that $U - (K_0 \cap Bd(V))$ has all of the properties discussed in (∗). Therefore, $M \notin QCC$. This proves the theorem.

Corollary 2.7. *A continuum, M, is locally connected if and only if the class of components and the class of quasicomponents of each open subset of M are identical.*

Proof. This follows immediately from 2.5 and 2.6.

Proposition 2.8. *Every aposyndetic continuum belongs to class QCP.*

Proof. Let M be an aposyndetic continuum with $p \in M$, and let K be a component of $M - \{p\}$. Since M is aposyndetic and $p \notin K$, we have that for any $x \in K$, there exists a closed, connected neighborhood, J, about x such that $p \notin J$. Since K is a maximal connected subset of $M - \{p\}$, and J is a connected subset of $M - \{p\}$ which shares a point with K, it follows that $J \subseteq K$. Thus, $\mathrm{int}(J)$ is an open subset about x which is contained in K. Therefore, K is an open subset of $M - \{p\}$, and so $Q = C$ in $M - \{p\}$ by 2.2.

Proposition 2.9. *Every strongly aposyndetic continuum belongs to class QCS.*

Proof. Let M be a strongly aposyndetic continuum, let N be a subcontinuum of M, and let K be a component of $M - N$. By substituting N for $\{p\}$ in the proof of 2.8, it follows that K is an open subset of $M - N$. Therefore, by 2.2, $Q = C$ in $M - N$.

3. Counterexamples Showing Strict Class Containments in 1.5

Now that we have proven the class containments presented in 1.5, we wish to demonstrate that those containments which were presented as being one-sided are, in fact, strict. The following observation will be useful towards this end:

Lemma 3.1. *If a topological space, S, has at most finitely many components, then $Q = C$ in S.*

Proof. If S has at most finitely many components, then each of these components is open in S. Therefore, $Q = C$ in S by 2.2.

Remark 3.2. We note that the space constructed in 2.3 demonstrates that the converse of 3.1 is false.

Example 3.3 ($QCP \not\subset QCS$)**.** Let $I_n = \left\{\frac{1}{n}\right\} \times [-1, 1]$ for each $n = 1, 2, \ldots,$ and let $J_1 = \{0\} \times (0, 1)$ and $J_2 = \{0\} \times [-1, 0)$. Define $N = ([-1, 1] \times \{1\}) \cup (\{-1\} \times [0, 1]) \cup ([-1, 0] \times \{0\})$. Then, $M = \left(\bigcup_{n=1}^{\infty} I_n\right) \cup J_1 \cup J_2 \cup N$ is a continuum. Observe that for any $p \in M$, $M - \{p\}$ has, at most, three components; thus, $Q = C$ in $M - \{p\}$ by 3.1. Therefore, $M \in QCP$. However, J_1 and J_2 are components of $M - N$, while $J_1 \cup J_2$ is a quasicomponent of $M - N$. Since N is a subcontinuum of M, it follows that $M \notin QCS$.

Example 3.4 ($QCS \not\subset QCC$)**.** Let M be the usual $\sin \frac{1}{x}$ continuum; specifically, let $M = \left\{\left(x, \sin \frac{1}{x}\right) : 0 < x \leq 1\right\} \cup (\{0\} \times [-1, 1])$. Define $J_1 = \{0\} \times (0, 1]$ and $J_2 = \{0\} \times [-1, 0)$. Observe that for any subcontinuum, K, of M, it is true that $M - K$ has, at most, two components; thus, $Q = C$ in $M - K$, by 3.1. Therefore, $M \in QCS$. However, if $N = \{(0, 0)\} \cup \left\{\left(\frac{1}{2n\pi}, 0\right) : n = 1, 2, \ldots\right\}$, then J_1 and J_2 are components of $M - N$, while $J_1 \cup J_2$ is a quasicomponent of $M - N$. Since N is closed in M, it follows that $M \notin QCC$.

Example 3.5 ($QCC \not\subset QC$)**.** Let M be any continuum which is locally connected, but not hereditarily locally connected (see, for example, [2, 10.38]). Then, by 2.5, $M \in QCC$. However, by 2.4, $M \notin QC$.

Example 3.6 ($QCS \not\subset AP$)**.** Let M be as in 3.4. Then, $M \in QCS$ by 3.4. However, M clearly fails to be aposyndetic at each point of J_1; therefore, $M \notin AP$.

Example 3.7 ($QCS \not\subset SAP$)**.** Let M be as in 3.4. Then, $M \in QCS$, by 3.4. However, since M is not locally connected, we have that $M \notin SAP$ by 2.1.

Example 3.8 ($AP \not\subset QCS$)**.** Let M be a unit square with a handle, H, and with unit intervals converging to the bottom interval, I. Specifically, define

$$I = [0, 1] \times \{0\}$$

$$T = (\{0, 1\} \times [0, 1]) \cup ([0, 1] \times \{1\})$$

$$H = \left(\left\{0, \frac{1}{2}\right\} \times \left[-\frac{1}{2}, 0\right]\right) \cup \left(\left[0, \frac{1}{2}\right] \times \left\{-\frac{1}{2}\right\}\right)$$

$$I_k = [0, 1] \times \left\{\frac{1}{k}\right\} \text{ for each } k = 2, 3, \ldots$$

$$M = I \cup T \cup H \cup (\bigcup I_k).$$

Note that M is aposyndetic at each $p \notin I$ since M is locally connected at each $p \notin I$. Moreover, closed and connected neighborhoods of each point $p \in I$ can be constructed by reaching far enough to the left or right of p, and these neighborhoods can be chosen to miss any other $x \in M$. Therefore, $M \in AP$. However, $M - (T \cup H)$ contains a quasicomponent which is not a component. Therefore, $M \notin QCS$.

All remaining relationships amongst the properties discussed in 1.5 follow easily from the relationships proven thus far.

Remark 3.9. We note, in closing, that all of our counterexamples showing strict containment are one-dimensional, planar continua.

References

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