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GENERATING NEW INEQUALITIES AND EQUALITIES: A STATISTICAL APPROACH

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Abstract

This article gives a set of inequalities based on using known elementary inequalities and probability distributions. The concept depends on replacing the arguments in an elementary inequality with random variables whose domain must be consistent with the constraints the arguments obey. Taking the expectation of both sides of the inequality – after replacement – leads to the result. Seven probability distributions will be used with many elementary inequalities to generate inequalities. The same approach is used to generate a set of equalities too.

Introduction

Famous inequalities that are of great importance have been introduced to the literature. Such inequalities have valuable applications whether in the pure or applied sciences. Examples of these inequalities are Bernoulli's inequality, Chebychev's inequality, Abel's inequality, the Cauchy-Schwarz-Buniakowski inequality, Young's inequality, Jensen's inequality, the Fejer-Jackson inequality, and Jordan's inequality. A

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variety of elementary inequalities, inequalities involving powers and factorials, finite sums and products, trigonometric functions, and integrals are found in the page problems of periodicals such as the American Mathematical Monthly which provides an abundant source of both elementary and advanced problems. In addition, a huge number of inequalities are collected in several references. This for example can be found in Mitrinovic et al. [3] and Hardy et al. [2].

We might differentiate among different approaches in deriving inequalities. One approach is to manipulate algebraically using the elegant techniques of the number theory. Another approach is a statistical approach by making use of the positivety of the variance of a random variable as discussed in Younan [4].

In this article, a simple statistical approach is used to introduce a set of inequalities and equalities to be injected in the literature to be lined up side by side with the huge number of inequalities that already exist for an updated photograph of inequalities. Alternative proofs for these proposed inequalities and equalities may be introduced using algebra. But, the main purpose of the article is to introduce a statistical approach to generate such inequalities and equalities. Alternative proofs of some inequalities have been introduced in the literature. This can be found in El-Neweihi and Proschan [1].

Methodology

The suggested approach combines the use of some known elementary inequalities and particular probability distributions in generating inequalities. Mathematically speaking, assume that $I_1(a's) \leq I_2(a's)$ is a known elementary inequality whose arguments are the fixed numbers $a_1, a_2, \ldots, a_n$ and let $X_1, X_2, \ldots, X_n$ be a random sample taken on continuous random variable X with density function $f(x)$. Let $J_1(X's) \leq J_2(X's)$ be the elementary inequality after replacing the fixed numbers $a_1, a_2, \ldots, a_n$ by the random variables $X_1, X_2, \ldots, X_n$. In this case, it might be convenient to take the expectations of both sides of $J_1(X's)$

$\leq J_2(X's)$ to get a new inequality. The suggested approach can be generalized to include elementary inequalities that have several sides. In other words, the approach is applicable to elementary inequalities of the form $I_1(a's) \leq I_2(a's) \leq \cdots \leq I_m(a's)$. In this article, the fixed numbers of the elementary inequality $I_1(a's) \leq I_2(a's)$ are replaced by a random variable X whose domain must be consistent with the constraints the arguments obey. Seven random variables are used with the following probability distributions:

(1) The Gamma distribution with density

$$f(x) = \beta^{\alpha}[\Gamma(\alpha)]^{-1} \cdot x^{\alpha-1} \exp(-\beta x), \quad 0 < x < \infty, 0 < \alpha < \infty, 0 < \beta < \infty.$$

The scale parameter β will be taken to be 1, unless otherwise stated.

(2) The Weibull distribution with density

$$f(x) = \alpha \cdot x^{\alpha-1} \exp(-x^{\alpha}), \quad 0 < x < \infty, 0 < \alpha < \infty.$$

(3) The Maxwell distribution with density

$$f(x) = \sqrt{2/\pi}\alpha^{3/2} \cdot x^2 \exp(-\alpha x^2/2), \quad 0 < x < \infty, 0 < \alpha < \infty.$$

(4) The Reyleigh distribution with density

$$f(x) = \alpha^{-2} \cdot x \exp(-x^2/2\alpha^2), \quad 0 < x < \infty, 0 < \alpha < \infty.$$

(5) The beta distribution with density

$$f(x) = \beta^{-1}(p, q) \cdot x^{p-1}(1-x)^{q-1}, \quad 0 < x < 1, 0 < p < \infty, 0 < q < \infty.$$

(6) The one-parameter Pareto distribution with density

$$f(x) = \alpha.x^{-(1+\alpha)}, \quad 1 < x < \infty, 0 < \alpha < \infty.$$

(7) The Y-distribution with density*

$$f(x) = 2\beta^{-1}(1/2, \alpha) \cdot (1-x^2)^{\alpha-1}, \quad 0 < x < 1, 0 < \alpha < \infty.$$

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*This is the distribution of the random variable $X = \frac{|X_1 - X_2|}{X_1 + X_2}$, where (X_1, X_2) is a random sample of two observations taken from the gamma distribution with two parameters α and β. The name Y-distribution is just for abbreviation.

Results

The proof of each inequality first gives the elementary inequality we start with, the replacement of the argument in the elementary inequality by the random variable X, and the probability distribution we assume that the random variable X has. The choice of this random variable is such that its domain must be consistent with the constraints the argument obeys. After replacing the argument by the random variable X, the expectation of both sides of the inequality is taken, a step which is not mentioned since it is common in all proofs. The same process is applicable to equalities. How the inequality is sharp, that is, how the two sides are close to each other is shown. The parameter values of each inequality for which the inequality is sharp are given associated with the ratio of the smaller side to the larger side. The closer this ratio to one, the sharper the inequality. This ratio is denoted by R. The results are demonstrated in three groups that give inequalities, sums of infinite series, and sums of finite series as follows.

Group 1. Inequalities

Inequality 1. If α is a positive value and n is a positive integer, then

$$2^{\alpha}\Gamma(\alpha) \geq \sum_{i=0}^{n} \frac{\Gamma(i+\alpha)}{2^{i}(i!)}.$$

Proof. Since $e^{a} = \sum_{i=0}^{\infty} \frac{a^{i}}{i!}$ for any positive value a, then $e^{a} \geq \sum_{i=0}^{n} \frac{a^{i}}{i!}$ for some positive integer n. Replace a by X, and let X have the gamma distribution with two parameters α and $\beta = 2$.

The smaller the value of α, the sharper the inequality. Table 1 shows the ratio R for $\alpha = 0.3,\ 0.5,$ and 1. The values of n are taken to be 2 to 20.

Table 1

$\alpha = 0.3$		$\alpha = 0.5$		$\alpha = 1$	
n	R	n	R	n	R
2	0.97369	2	0.95017	2	0.87500
4	0.99513	4	0.98988	4	0.96875
6	0.99901	6	0.99781	6	0.99219
8	0.99979	8	0.99951	8	0.99805
10	0.99995	10	0.99989	10	0.99951
12	0.99999	12	0.99997	12	0.99988
14	**1.00000**	14	0.99999	14	0.99997
16	1.00000	16	**1.00000**	16	0.99999
18	1.00000	18	1.00000	18	**1.00000**
20	1.00000	20	1.00000	20	1.00000

It is clear that as α gets larger, the two sides approach to be equal late.

Inequality 2. If p and q are positive values and n is a positive integer, then

$$\beta(p,\, n + q) \geq \beta(p,\, q)\left(1 - \frac{np}{p + q}\right),$$

with equality if $n = 1$.

Proof. The elementary inequality is (Mitrinovic et al. [3, p. 98])

$$\prod_{k=1}^{n}(1 - a_k) \geq 1 - \sum_{k=1}^{n} a_k, \quad (0 < a_k < 1).$$

Replace a_k by X for all k, and let X have the beta distribution with two parameters p and q.

Since the two sides of the inequality are equal if $n = 1$, we expect that the smaller the value of n (such that bigger than 1), the sharper the inequality. Table 2 shows the ratio R for $n = 2$ and the selected values of p and q.

Table 2

p	q	R
1	2	0.66667
3	10	0.89091
2	12	0.96154
3	25	0.98154
5	50	0.98824
7	80	0.99136

It is clear that the larger (q/p), the sharper the inequality.

Inequality 3. If α and n are positive integers, then

$$\frac{n}{\alpha} + 2\sum_{i=1}^{\alpha-1}\binom{\alpha-1}{i}(-1)^i \beta(2i+1,\ n+1) \geq \beta\left(\frac{1}{2},\ \alpha\right),$$

with equality if $n = 1$.

Proof. The elementary inequality is the same as in Inequality 2. Replace a_k by X for all k, and let X have the Y-distribution with parameter α as positive integer.

Since the two sides of the inequality are equal if $n = 1$, we expect that the smaller the value of n (such that bigger than 1), the sharper the inequality. Table 3 shows the ratio R in two cases: (1) if both of n and α are equal where both equal 1 to 7, and (2) if n and α are unequal where $n = 3$ and $\alpha = 1$ to 31.

Table 3

n and α equal 1 to 7		$n = 3$ and $\alpha = 1$ to 31	
n and α	R	α	R
7	0.55771	1	0.57143
6	0.58953	6	0.83435
5	0.62810	11	0.89572
4	0.67649	16	0.92364
3	0.74050	21	0.93967
2	0.83333	26	0.95010
1	1.00000	31	0.95743

As shown from the table, if n and α are equal, then the smaller the values of n and α, the sharper the inequality. However, at fixed n, the larger the value of α, the sharper the inequality.

Inequality 4. If α is a positive value and n is a positive integer such that $\alpha > n$, then

$$\alpha \sum_{i=0}^{n} \binom{n}{i} (\alpha - i)^{-1} \geq \frac{2^n}{n+1}\left(1 + \frac{n\alpha}{\alpha - 1}\right),$$

with equality if $n = 1$.

Proof. The elementary inequality is (Mitrinovic et al. [3, p. 99])

$$\prod_{k=1}^{n} (1 + a_k) \geq \frac{2^n}{n+1}\left(1 + \sum_{i=1}^{k} a_k\right) \quad (a_k \geq 1).$$

Replace a_k by X for all k, and let X have the one-parameter Pareto distribution with parameter α.

Since the two sides of the inequality are equal if $n = 1$, we expect that the smaller the value of n (such that bigger than 1), the sharper the inequality. Table 4 gives the ratio R when $n = 1$ to 20 for $\alpha = 30$, 50, and 300.

Table 4

n	$\alpha = 30$	$\alpha = 50$	$\alpha = 300$
	R	R	R
20	0.67974	0.81299	0.96969
15	0.76843	0.86446	0.97801
10	0.85599	0.91556	0.98629
5	0.94144	0.96562	0.99442
2	0.98830	0.99313	0.99888
1	1.00000	1.00000	1.00000

The table shows that the larger the value of α than n, the sharper the inequality.

Inequality 5. If p and q are positive values, then

$$\frac{q}{p+q}\beta(p, q) < \sum_{i=0}^{\infty} \beta(i + p, q).$$

Proof. The elementary inequality is (Mitrinovic et al. [3, p. 99])

$$\prod_{k=1}^{n}(1 - a_k) < \left(1 + \sum_{k=1}^{n} a_k\right)^{-1}, \quad (0 < a_k < 1).$$

Let $n = 1$, replace a_1 by X, and let X have the beta distribution with two parameters p and q.

Table 5 shows the ratio R when $p = 0.5$ and $q = 1, 5,$ and 20.

Table 5

(p, q)	R
(0.5, 1)	0.18766
(0.5, 5)	0.80809
(0.5, 20)	0.95059

As shown from the table, the larger the difference between p and q, the sharper the inequality.

Group 2. Sum of Infinite Series

Series 1. If $\alpha > 1$, then

$$\sum_{i=0}^{\infty}(-1)^i \beta\left(\frac{i+1}{2}, \alpha\right) = \beta\left(\frac{1}{2}, \alpha - 1\right) - \frac{1}{\alpha - 1}.$$

Proof. For $|a| < 1$ we have $(1 + a)^{-1} = \sum_{i=0}^{\infty}(-1)^i a^i$. Replace a by X, and let X have the Y-distribution with parameter $\alpha > 1$.

Series 2. If $\alpha > 1$, then

$$\sum_{i=0}^{\infty}\beta\left(\frac{i+1}{2}, \alpha\right) = \beta\left(\frac{1}{2}, \alpha - 1\right) + \frac{1}{\alpha - 1}.$$

Proof. For $|a| < 1$ we have $(1 - a)^{-1} = \sum_{i=0}^{\infty} a^i$. Replace a by X, and let X have the Y-distribution with parameter $\alpha > 1$.

Series 3. If $\alpha > 1$, then

$$\sum_{i=0}^{\infty}\beta\left(\frac{2i+1}{2}, \alpha\right) = \beta\left(\frac{1}{2}, \alpha - 1\right).$$

Proof. Add series 1 and 2 to each other.

Series 4. If $p > 0$ and $q > 1$, then

$$\sum_{i=0}^{\infty} \beta(i + p, q) = \beta(p, q - 1).$$

Proof. For $|a| < 1$ we have $(1 - a)^{-1} = \sum_{i=0}^{\infty} a^i$. Replace a by X, and let X have the beta distribution with two parameters $p > 0$ and $q > 1$.

Series 5. If θ is a positive integer ≥ 3, then

$$\sum_{i=0}^{\infty} \frac{(-1)^i}{(i+1)(i+\theta)} = \frac{1}{\theta - 1}\left[(1 - (-1)^{\theta-1}) \cdot \ln(2) + \sum_{i=1}^{\theta-1} \frac{(-1)^i}{\theta - i}\right].$$

Proof. For $|a| < 1$ we have $\ln(1 + a) = \sum_{i=0}^{\infty} \frac{(-1)^i}{i+1} \cdot a^{i+1}$. Replace a by X, and let X have the beta distribution with two parameters p as positive integer and $q = 1$, then put $\theta = p + 2$. If $\theta = 3$, then we get the special case

$$\sum_{i=0}^{\infty} \frac{(-1)^i}{(i+1)(i+3)} = \frac{1}{2}\sum_{i=1}^{2} \frac{(-1)^i}{3 - i} = \frac{1}{4}.$$

Group 3. Sum of Finite Series

Series 1. If $p > 0$, $q > 0$, and n is a positive integer, then

$$\sum_{i=0}^{n} {}^nC_i (-1)^i \beta(i + p, q) = \beta(p, n + q).$$

Proof. Write the binomial expansion of $(1 - a)^n$, replace a by X, and let X have the beta distribution with two parameters p and q.

Series 2. If α and n are positive integers, then

$$\sum_{i=0}^{\alpha-1} {}^{\alpha-1}C_i (-1)^i \beta(1 + 2i, n + 1) = \frac{1}{2}\sum_{i=0}^{n} {}^nC_i (-1)^i \beta\left(\frac{i+1}{2}, \alpha\right).$$

Proof. Write the binomial expansion of $(1 - a)^n$, replace a by X, and let X have the Y-distribution with parameter α as a positive integer. The following three special cases are obtained

2.1. If $\alpha = 1$, then

$$\sum_{i=0}^{n} \frac{{}^{n}C_i (-1)^i}{i+1} = \frac{1}{n+1}.$$

2.2. If $n = 1$, then

$$\sum_{i=0}^{\alpha-1} \frac{{}^{\alpha-1}C_i (-1)^i}{(1+i)(1+2i)} = \beta\left(\frac{1}{2}, \alpha\right) - \frac{1}{\alpha}.$$

2.3. If $n = \alpha - 1$, then

$$\sum_{i=0}^{\alpha-1} {}^{\alpha-1}C_i (-1)^i \left[\beta(1+2i, \alpha) - \frac{1}{2}\beta\left(\frac{i+1}{2}, \alpha\right)\right] = 0.$$

Conclusion

The idea of replacing the arguments in an elementary inequality with a random variable whose domain is consistent with the constraints the arguments obey, and taking the expectation of both sides of the elementary inequality after replacement is introduced as an approach to generate new inequalities. This approach might be generalized if multivariate probability distributions are used instead of univariate distributions. An advanced inventiveness is to progress oppositely in the following sense: Given an inequality, can we determine a probability distribution that leads to an elementary inequality whose arguments are real values?

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