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SOME APPLICATIONS OF GENERALIZED FRACTIONAL CALCULUS OPERATORS TO A CERTAIN CLASS OF ANALYTIC FUNCTIONS

SHIGEYOSHI OWA, B. SRUTHA KEERTHI,
B. ADOLF STEPHEN and S. SIVASUBRAMANIAN

Department of Mathematics
Kinki University
Higashi-Osaka, Osaka 577-8502, Japan
e-mail: shige21@ican.zaq.ne.jp

Department of Applied Mathematics
Sri Venkateswara College of Engineering
Pennalur, Chennai-602105, India
e-mail: laya@svce.ac.in

Department of Mathematics
Madras Christian College
Chennai-600059, India
e-mail: adolfmcc2003@yahoo.co.in

Department of Mathematics
Easwari Engineering College
Ramapuram, Chennai-600089, India
e-mail: sivasaisastha@rediffmail.com

Abstract

In the present paper, by making use of certain operators of generalized fractional calculus, we introduce a novel class $T_{\lambda, \ell}^{\mu, \phi, \eta}(n; \alpha)$ of functions

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which are analytic and univalent in the open unit disk U. A necessary and sufficient condition for a function to be in the class $T_{\lambda, \ell}^{\mu, \phi, \eta}(n; \alpha)$ is obtained.

1. Introduction and Definitions

Let $T(n)$ denote the class of functions $f(z)$ of the form

$$f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k \quad (a_k \geq 0;\ n \in \mathbb{N} = \{1, 2, 3, \ldots\}), \tag{1}$$

which are analytic and univalent in the unit open disk

$$U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

Let $T_{\lambda, \ell}^{\mu, \phi, \eta}(n; \alpha)$ denote the subclass of $T(n)$ consisting of functions $f(z)$ which satisfy the inequality:

$$\begin{aligned} &|[[\lambda \ell z^3 J_{0,z}^{3+\mu, 3+\phi, 3+\eta}\{f(z)\} + (2\lambda\ell + \lambda - \ell) z^2 J_{0,z}^{2+\mu, 2+\phi, 2+\eta}\{f(z)\} \\ &+ z J_{0,z}^{1+\mu, 1+\phi, 1+\eta}\{f(z)\}]/[\lambda \ell z^2 J_{0,z}^{2+\mu, 2+\phi, 2+\eta}\{f(z)\} \\ &+ (\lambda - \ell) z J_{0,z}^{1+\mu, 1+\phi, 1+\eta}\{f(z)\} + (1 - \lambda + \ell) J_{0,z}^{\mu, \phi, \eta}\{f(z)\}]] - (1 - \phi)\,| < \alpha \end{aligned} \tag{2}$$

$(z \in U;\ 0 < \alpha \leq 1;\ 0 \leq \mu < 1;\ \phi, \eta \in \mathbb{R},\ \phi < 1;\ \eta > \max\{\mu, \phi\} - 2;\ 0 \leq \ell \leq \lambda \leq 1)$.

Definition 1.1. The *fractional integral* of order μ of a function $f(z)$ is defined by

$$D_z^{-\mu}\{f(z)\} = \frac{1}{\Gamma(\mu)} \int_0^z \frac{f(\xi)}{(z - \xi)^{1-\mu}}\, d\xi \quad (\mu > 0), \tag{3}$$

where $f(z)$ is analytic in a simply-connected region of the z-plane containing the origin, and the multiplicity of $(z - \xi)^{\mu - 1}$ is removed by requiring $\log(z - \xi)$ to be real when $(z - \xi) > 0$.

Definition 1.2. The *fractional derivative* of order μ of a function $f(z)$ is defined by

$$D_z^{\mu}\{f(z)\} = \frac{1}{\Gamma(1-\mu)}\frac{d}{dz}\int_0^z \frac{f(\xi)}{(z-\xi)^{\mu}}d\xi \ (0 \leq \mu < 1), \tag{4}$$

where $f(z)$ is constrained, and the multiplicity of $(z-\xi)^{-\mu}$ is removed, as in Definition 1.1.

Definition 1.3. Let $\mu > 0$ and $\eta, \beta \in \mathbb{R}$. Then, in terms of familiar (Gauss's) hypergeometric function ${}_2F_1$, the *generalized fractional integral operator* $I_{0,z}^{\mu,\beta,\eta}$ of a function $f(z)$ is defined by

$$I_{0,z}^{\mu,\beta,\eta}\{f(z)\} = \frac{z^{-\mu-\beta}}{\Gamma(\mu)}\int_0^z (z-\xi)^{\mu-1}f(\xi)\left[{}_2F_1\left(\mu+\beta; -\eta; 1-\frac{\xi}{2}\right)\right]d\xi, \tag{5}$$

where the function $f(z)$ is analytic in a simply-connected region of the z-plane containing the origin, with the order

$$f(z) = O(|z|^{\varepsilon}), \ (z \to 0), \tag{6}$$

for

$$\varepsilon > \max\{0, \beta-\eta\} - 1, \tag{7}$$

and the multiplicity of $(z-\xi)^{\mu-1}$ is removed by requiring $\log(z-\xi)$ to be real when $(z-\xi) > 0$.

Definition 1.4. Let $0 \leq \mu < 1$ and $\eta, \beta \in \mathbb{R}$. Then, the *generalized fractional derivative operator* $J_{0,z}^{\mu,\beta,\eta}$ of a function $f(z)$ is defined by

$$J_{0,z}^{\mu,\beta,\eta}\{f(z)\}$$

$$= \frac{1}{\Gamma(1-\mu)}\frac{d}{dz}\left\{z^{\mu-\beta}\int_0^z (z-\xi)^{-\mu}f(\xi)\left[{}_2F_1\left(\beta-\mu, 1-\eta; 1-\mu; 1-\frac{\xi}{2}\right)\right]d\xi\right\}, \tag{8}$$

where the function $f(z)$ is analytic in a simply-connected region of the z-plane containing the origin, with the order as given by (6), and the multiplicity of $(z-\xi)^{-\mu}$ is removed by requiring $\log(z-\xi)$ to be real when $(z-\xi) > 0$.

By comparing (3) with (5), we obtain the following relationship

$$I_{0,z}^{\mu,-\mu,\eta}\{f(z)\} = D_z^{-\mu}\{f(z)\} \ (\mu > 0). \tag{9}$$

Similarly, by comparing (4) with (8), we find

$$J_{0,z}^{\mu,\mu,\eta}\{f(z)\} = D_z^{\mu}\{f(z)\} \ (0 \leq \mu < 1). \tag{10}$$

From the general class $T_{\lambda,\ell}^{\mu,\phi,\eta}(n;\alpha)$ defined by (2), we obtain the following important subclasses:

$$V_{\ell}^{\mu,\phi,\eta}(n;\alpha) = T_{0,\ell}^{\mu,\phi,\eta}(n;\alpha) \text{ and } V_0^{\mu,\phi,\eta}(n;\alpha) = T_{0,0}^{\mu,\phi,\eta}(n;\alpha), \tag{11}$$

$$W_{\ell}^{\mu,\phi,\eta}(n;\alpha) = T_{1,\ell}^{\mu,\phi,\eta}(n;\alpha) \text{ and } W_0^{\mu,\phi,\eta}(n;\alpha) = T_{1,0}^{\mu,\phi,\eta}(n;\alpha), \tag{12}$$

$$\Omega_{\mu,\ell}(n;\alpha) = V_{\ell}^{\mu,\mu,\eta}(n;\alpha) \text{ and } \Omega_{\mu}(n;\alpha) = V_0^{\mu,\mu,\eta}(n;\alpha), \tag{13}$$

$$\Delta_{\mu,\ell}(n;\alpha) = W_{\ell}^{\mu,\mu,\eta}(n;\alpha) \text{ and } \Delta_{\mu}(n;\alpha) = W_0^{\mu,\mu,\eta}(n;\alpha), \tag{14}$$

$$\mathcal{S}_n(\alpha) = \Omega_0(n;\alpha), \tag{15}$$

$$\mathcal{C}_n(\alpha) = \Delta_0(n;\alpha), \tag{16}$$

$$\mathcal{S}^*(\alpha) = S_1(\alpha), \tag{17}$$

$$\mathcal{C}(\alpha) = C_1(\alpha), \tag{18}$$

$(n \in \mathbb{N};\ 0 < \alpha \leq 1;\ 0 \leq \mu < 1;\ \phi, \eta \in \mathbb{R},\ \phi < 1;\ \eta > \max\{\mu, \phi\} - 2;$

$0 \leq \ell \leq \lambda \leq 1).$

The classes $\mathcal{S}_n(\alpha)$ and $\mathcal{S}^*(\alpha)$ consist of starlike functions, of order $1-\alpha$ and the classes $\mathcal{C}_n(\alpha)$ and $\mathcal{C}(\alpha)$ consist of convex functions, of order $1-\alpha$. These classes are of much importance in the Geometric Function theory. Some other interesting papers involving fractional calculus operators are the ones by Altintaş et al. [1, 2], Chen et al. [3, 4], Irmak [5], and Raina and Srivastava [7].

2. Some Properties of the Class $T_{\lambda,\ell}^{\mu,\phi,\eta}(n;\alpha)$

Theorem 2.1. *Let a function* $f(z) \in T(n)$. *Then, the function* $f(z)$ *belongs to the class* $T_{\lambda,\ell}^{\mu,\phi,\eta}(n;\alpha)$ *if and only if*

$$\sum_{k=n+1}^{\infty} \frac{\Gamma(k+1)\Gamma(k-\phi+\eta+1)}{\Gamma(k-\mu+\eta+1)\Gamma(k-\phi+1)}[G(k,\phi)]a_k$$

$$\leq \alpha \frac{\Gamma(2-\phi+\eta)[\lambda\ell\phi^2-\lambda\ell\phi+1-\lambda\phi+\ell\phi]}{\Gamma(2-\mu+\eta)\Gamma(2-\phi)}, \tag{19}$$

$(n \in \mathbb{N};\ 0 < \alpha \leq 1;\ 0 \leq \mu < 1;\ \phi, \eta \in \mathbb{R},\ \phi < 1;\ \eta > \max\{\mu, \phi\} - 2;\ 0 \leq \ell \leq \lambda \leq 1)$.

The result is sharp for the function $f(z)$ *given by*

$$f(z) = z - \frac{\alpha\Gamma(2-\phi+\eta)(\lambda\ell\phi^2-\lambda\ell\phi+1-\lambda\phi+\ell\phi)}{\Gamma(2-\mu+\eta)\Gamma(2-\phi)\Gamma(n+2)\Gamma(n-\phi+\eta+2)}$$

$$\frac{\Gamma(n-\mu+\eta+2)\Gamma(n-\phi+2)}{G(n+1,\phi)} z^{n+1}, \tag{20}$$

where

$$G(k,\phi) = \{\lambda\ell(k-\phi)^3 + [\alpha\lambda\ell+\phi\lambda\ell-2\lambda\ell+\lambda-\ell](k-\phi)^2 + [\alpha(\lambda-\ell-\lambda\ell)$$

$$+ (1-2\lambda+2\ell-\phi\lambda\ell+\lambda\ell+\lambda\phi-\ell\phi)](k-\phi) + [\alpha(1-\lambda+\ell)$$

$$- (\lambda\phi-\ell\phi+1-\lambda+\ell-\phi)]\}.$$

Proof. Suppose that the function $f(z)$ is defined by (1), and the inequality (19) holds true and take

$$|\lambda\ell z^3 J_{0,z}^{3+\mu,3+\phi,3+\eta}\{f(z)\} + (2\lambda\ell + \lambda - \ell)z^2 J_{0,z}^{2+\mu,2+\phi,2+\eta}\{f(z)\}$$

$$+ zJ_{0,z}^{1+\mu,1+\phi,1+\eta}\{f(z)\} - (1-\phi)\lambda\ell z^2 J_{0,z}^{2+\mu,2+\phi,2+\eta}\{f(z)\}$$

$$+ (\lambda - \ell)zJ_{0,z}^{1+\mu,1+\phi,1+\eta}\{f(z)\} + (1 - \lambda + \ell)J_{0,z}^{\mu,\phi,\eta}\{f(z)\}|$$

$$- \alpha|\lambda\ell z^2 J_{0,z}^{2+\mu,2+\phi,2+\eta}\{f(z)\} + (\lambda - \ell)zJ_{0,z}^{1+\mu,1+\phi,1+\eta}\{f(z)\}$$

$$+ (1 - \lambda + \ell)J_{0,z}^{\mu,\phi,\eta}\{f(z)\}|$$

$$= \left|\sum_{k=n+1}^{\infty} \frac{\Gamma(k+1)\Gamma(k-\phi+\eta+1)}{\Gamma(k-\mu+\eta+1)\Gamma(k-\phi+1)}\{[\lambda\ell(k-\phi)^3 + (\phi\lambda\ell - 2\lambda\ell\right.$$

$$+ \lambda - \ell)(k-\phi)^2 + (1 - 2\lambda + 2\ell - \phi\lambda\ell + \lambda\ell + \lambda\phi - \ell\phi)(k-\phi)$$

$$\left. - (1 - \lambda + \ell - \phi + \lambda\phi - \ell\phi)]a_k z^{k-\phi}\}\right|$$

$$- \alpha\left|\frac{\Gamma(2-\phi+\eta)}{\Gamma(2-\mu+\eta)\Gamma(2-\phi)}(\lambda\ell\phi^2 - \lambda\ell\phi - \lambda\phi + \ell\phi + 1)z^{1-\phi}\right.$$

$$\left.- \sum_{k=n+1}^{\infty} \frac{\Gamma(k+1)\Gamma(k-\phi+\eta+1)}{\Gamma(k-\mu+\eta+1)\Gamma(k-\phi+1)}\{\lambda\ell(k-\phi)^2 + (\lambda - \ell - \lambda\ell)(k-\phi) + (1-\lambda+\ell)\}z^{k-\phi}\right|$$

$$\leq \sum_{k=n+1}^{\infty} \frac{\Gamma(k+1)\Gamma(k-\phi+\eta+1)}{\Gamma(k-\mu+\eta+1)\Gamma(k-\phi+1)}[G(k,\phi)]a_k$$

$$- \alpha\frac{\Gamma(2-\phi+\eta)}{\Gamma(2-\mu+\eta)\Gamma(2-\phi)}\{(\lambda\ell\phi^2 - \lambda\ell\phi - \lambda\phi + \ell\phi + 1)\}$$

$$\leq 0$$

$(n \in \mathbb{N};\ 0 < \alpha \leq 1;\ 0 \leq \mu < 1;\ \phi, \eta \in \mathbb{R},\ \phi < 1;\ \eta > \max\{\mu, \phi\} - 2;\ 0 \leq \ell \leq \lambda \leq 1)$.

Hence, by maximum modulus theorem, the function $f(z)$ defined by (1) belongs to the class $T_{\lambda,\ell}^{\mu,\phi,\eta}(n;\alpha)$.

In order to prove the converse, we assume that the function $f(z) \in T_{\lambda, \ell}^{\mu, \phi, \eta}(n; \alpha)$, then

$$|[[\lambda \ell z^3 J_{0,z}^{3+\mu, 3+\phi, 3+\eta}\{f(z)\} + (2\lambda\ell + \lambda - \ell) z^2 J_{0,z}^{2+\mu, 2+\phi, 2+\eta}\{f(z)\}$$

$$+ z J_{0,z}^{1+\mu, 1+\phi, 1+\eta}\{f(z)\}]/[\lambda \ell z^2 J_{0,z}^{2+\mu, 2+\phi, 2+\eta}\{f(z)\} + (\lambda - \ell) z J_{0,z}^{1+\mu, 1+\phi, 1+\eta}\{f(z)\}$$

$$+ (1 - \lambda + \ell) J_{0,z}^{\mu, \phi, \eta}\{f(z)\}]] - (1 - \phi)| < \alpha$$

$$\left|\left[\sum_{k=n+1}^{\infty} \frac{\Gamma(k+1)\Gamma(k-\phi+\eta+1)}{\Gamma(k-\mu+\eta+1)\Gamma(k-\phi+1)} \rho a_k z^{k-\phi} \Big/ \frac{\Gamma(2-\phi+\eta)}{\Gamma(2-\mu+\eta)\Gamma(2-\phi)}(\lambda\ell\phi^2\right.\right.$$

$$\left.\left. - \lambda\ell\phi - \lambda\phi + \ell\phi + 1) z^{1-\phi} - \sum_{k=n+1}^{\infty} \frac{\Gamma(k+1)\Gamma(k-\phi+\eta+1)}{\Gamma(k-\mu+\eta+1)\Gamma(k-\phi+1)} \tau a_k z^{k-\phi}\right] < \alpha\right| \tag{21}$$

where

$$\rho = \{[\lambda\ell(k-\phi)^3 + (\phi\lambda\ell - 2\lambda\ell + \lambda - \ell)(k-\phi)^2$$

$$+ (1 - 2\lambda + 2\ell - \phi\lambda\ell + \lambda\ell + \lambda\phi - \ell\phi)(k-\phi) - (1 - \lambda + \ell - \phi + \lambda\phi - \ell\phi)]\},$$

$$\tau = \{\lambda\ell(k-\phi)^2 + (\lambda - \ell - \lambda\ell)(k-\phi) + (1 - \lambda + \ell)\}.$$

Since $|\operatorname{Re}(z)| \le |z|$ for any z, choosing z to be real and letting $z \to 1^{-1}$ through real values, (21) yields (19).

Corollary 2.1. *Let a function* $f(z) \in T(n)$. *Then, the function* $f(z)$ *belongs to the class* $V_{\ell}^{\mu, \phi, \eta}(n; \alpha)$ *if and only if*

$$\sum_{k=n+1}^{\infty} \frac{\Gamma(k+1)\Gamma(k-\phi+\eta+1)}{\Gamma(k-\mu+\eta+1)\Gamma(k-\phi+1)}$$

$$\{-\ell(k-\phi)^2 + (1 + 2\ell - \ell\phi - \alpha\ell)(k-\phi) - 1 - \ell + \phi + \ell\phi + \alpha + \alpha\ell\} a_k$$

$$\le \alpha \left\{\frac{\Gamma(2-\phi+\eta)}{\Gamma(2-\mu+\eta)} \frac{(1+\ell\phi)}{\Gamma(2-\phi)}\right\},$$

$(n \in \mathbb{N};\ 0 < \alpha \le 1;\ 0 \le \mu < 1;\ \phi, \eta \in \mathbb{R},\ \phi < 2;\ \eta > \max\{\mu, \phi\} - 2)$.

Corollary 2.2. *Let a function* $f(z) \in T(n)$. *Then, the function* $f(z)$ *belongs to the class* $W_{\ell}^{\mu, \phi, \eta}(n; \alpha)$ *if and only if*

$$\sum_{k=n+1}^{\infty} \frac{\Gamma(k+1)\Gamma(k-\phi+\eta+1)}{\Gamma(k-\mu+\eta+1)\Gamma(k-\phi+1)}$$

$$\{\ell(k-\phi)^3 + (1-3\ell+\ell\phi+\alpha\ell)(k-\phi)^2 + (\alpha(1-2\ell)-1+3\ell-2\phi\ell+\phi)(k-\phi)$$

$$+ \alpha\ell + \ell\phi - \ell\}a_k \leq \alpha\left\{\frac{\Gamma(2-\phi+\eta)}{\Gamma(2-\mu+\eta)\Gamma(2-\phi)}(\ell\phi^2 - \phi + 1)\right\},$$

$(n \in \mathbb{N};\ 0 < \alpha \leq 1;\ 0 \leq \mu < 1;\ \phi, \eta \in \mathbb{R},\ \phi < 1;\ \eta > \max\{\mu, \phi\} - 2;\ 0 \leq \ell \leq \lambda \leq 1)$.

Corollary 2.3. *Let a function* $f(z) \in T(n)$. *Then, the function* $f(z)$ *belongs to the class* $\Omega_{\mu, \ell}(n; \alpha)$ *if and only if*

$$\sum_{k=n+1}^{\infty} \frac{\Gamma(k+1)}{\Gamma(k-\mu+1)}$$

$$\{-\ell(k-\mu)^2 + (1+2\ell-\ell\mu-\alpha\ell)(k-\mu) - 1 - \ell + \mu + \ell\mu + \alpha + \alpha\ell\}a_k$$

$$\leq \alpha\left[\frac{1+\ell\mu}{\Gamma(2-\mu)}\right],\ (n \in \mathbb{N};\ 0 < \alpha \leq 1;\ 0 \leq \mu < 1).$$

Corollary 2.4. *Let a function* $f(z) \in T(n)$. *Then, the function* $f(z)$ *belongs to the class* $\Delta_{\mu, \ell}(n; \alpha)$ *if and only if*

$$\sum_{k=n+1}^{\infty} \frac{\Gamma(k+1)}{\Gamma(k-\mu+1)}\{\ell(k-\mu)^3 + (1-3\ell+\mu\ell+\alpha\ell)(k-\mu)^2$$

$$+ (\alpha(1-2\ell) - 1 + 3\ell - 2\mu\ell + \mu)(k-\mu) + \alpha\ell + \ell\mu - \ell\}a$$

$$\leq \frac{\alpha}{\Gamma(2-\mu)}\{\ell\mu^2 - \mu + 1\},\ (n \in \mathbb{N};\ 0 < \alpha \leq 1;\ 0 \leq \mu < 1).$$

We obtained Corollaries 2.1-2.4 by using equations (11)-(14).

Remark 2.1. When $\ell = 0$ corollaries from 2.1-2.4 correspond to the known results proved by Irmak and Raina [6, pp. 447-448].

Theorem 2.2. *Let* $\beta \in \mathbb{R}_+$ *and* $\gamma, \eta \in \mathbb{R}$ *such that* $\eta > \max(-\beta, \gamma) - 2.$ *If n is a positive integer such that*

$$n \geq \frac{\gamma(\beta + \eta)}{\beta} - 2$$

and, if $f(z) \in T_{\lambda, \ell}^{\mu, \phi, \eta}(n; \alpha),$ *then*

$$\left| \left| I_{0, z}^{\beta, \gamma, \eta}\{f(z)\} \right| - \frac{\Gamma(2 - \gamma + \eta)}{\Gamma(2 - \gamma)\Gamma(2 + \beta + \eta)} \left| z \right|^{1-\gamma} \right|$$

$$\leq \frac{\alpha\Gamma(2-\phi+\eta)\Gamma(n-\mu+\eta+2)\Gamma(n+2-\phi)\Gamma(n-\gamma+\eta+2)\{\lambda\ell\phi^2 - \lambda\ell\phi + 1 - \lambda\phi + \ell\phi\}}{\Gamma(2-\mu+\eta)\Gamma(2-\phi)\Gamma(n-\phi+\eta+2)\Gamma(n-\gamma+2)\Gamma(n+\beta+\eta+2)G[n+1, \phi]} \left| z \right|^{n-\gamma+1} \tag{22}$$

for $z \in U$ *if* $\gamma \leq 1$ *and* $z \in D$ *if* $\gamma > 1.$ *The result* (22) *is sharp for the function* $f(z)$ *given by* (20).

Proof. Let $f(z) \in T_{\lambda, \ell}^{\mu, \phi, \eta}(n; \alpha).$ It follows from the inequality (19) that

$$\sum_{k=n+1}^{\infty} a_k \leq \frac{\alpha\Gamma(2-\phi+\eta)\Gamma(n-\mu+\eta+2)\Gamma(n+2-\phi)\{\lambda\ell\phi^2 - \lambda\ell\phi + 1 - \lambda\phi + \ell\phi\}}{\Gamma(2-\mu+\eta)\Gamma(2-\phi)\Gamma(n+2)\Gamma(n-\phi+\eta+2)G[n+1, \phi]}. \tag{23}$$

From (1), (8) and a known result due to Srivastava et al. [9, p. 415, equation (2.3)], we have

$$I_{0, z}^{\beta, \gamma, \eta}\{f(z)\} = \frac{\Gamma(2 - \gamma + \eta)}{\Gamma(2 - \gamma)\Gamma(2 + \beta + \eta)} z^{1-\gamma} - \sum_{k=n+1}^{\infty} \psi(k) a_k z^{k-\gamma}, \tag{24}$$

where

$$\psi(k) = \frac{\Gamma(k + 1)\Gamma(k - \gamma + \eta + 1)}{\Gamma(k - \gamma + 1)\Gamma(k + \beta + \eta + 1)}, \ (k \geq n + 1; \ n \in \mathbb{N}).$$

The function $\psi(k)$ is non-increasing for integers $k(k \geq n + 1, n \in \mathbb{N}),$ under the hypothesis of Theorem 2.2. Therefore we have

$$0 < \psi(k) \leq \psi(n + 1) = \frac{\Gamma(n + 2)\Gamma(n - \gamma + \eta + 2)}{\Gamma(n - \gamma + 2)\Gamma(n + \beta + \eta + 2)} \ (n \in \mathbb{N}). \tag{25}$$

Then the desired assertion (22) of Theorem 2.2 follows by substituting (23) and (25) in (24).

Theorem 2.3. *Let* $0 \leq \beta < 1$ *and* $\gamma, \eta \in \mathbb{R}$ *such that* $\gamma < 2$, $\eta > \max(\beta, \gamma) - 2$. *If n is a positive integer such that*

$$n \geq \frac{\gamma(\beta - \eta)}{\beta} - 2, \quad \text{and if } f(z) \in T_{\lambda, \ell}^{\mu, \phi, \eta}(n; \alpha),$$

then

$$\left| |J_{0,z}^{\beta, \gamma, \eta}\{f(z)\}| - \frac{\Gamma(2 - \gamma + \eta)}{\Gamma(2 - \gamma)\Gamma(2 - \beta + \eta)} |z|^{1-\gamma} \right|$$

$$\leq [[\alpha\Gamma(2 - \phi + \eta)\Gamma(n - \mu + \eta + 2)\Gamma(n + 2 - \phi)\Gamma(n - \gamma + \eta + 2)\{\lambda\ell\phi^2 - \lambda\ell\phi$$

$$+ 1 - \lambda\phi + \ell\phi\}]/[\Gamma(2 - \mu + \eta)\Gamma(2 - \phi)\Gamma(n - \phi + \eta + 2)\Gamma(n - \gamma$$

$$+ 2)\Gamma(n - \beta + \eta + 2)G[n + 1, \phi]]]|z|^{n-\gamma+1} \tag{26}$$

for $z \in U$ *if* $\gamma \leq 1$ *and* $z \in D$ *if* $\gamma > 1$. *The result is sharp for the function* $f(z)$ *given by* (20).

Proof. Under the hypothesis of Theorem 2.3, we have from (1) and a known result of Raina and Srivastava [7, p. 15, equation (2.2)].

$$J_{0,z}^{\beta, \gamma, \eta}\{f(z)\} = \frac{\Gamma(2 - \gamma + \eta)}{\Gamma(2 - \gamma)\Gamma(2 - \beta + \eta)} z^{1-\gamma}$$

$$- \sum_{k=n+1}^{\infty} \frac{\Gamma(k + 1)\Gamma(k - \gamma + \eta + 1)}{\Gamma(k - \gamma + 1)\Gamma(k - \beta + \eta + 1)} z^{k-\gamma}. \tag{27}$$

Making use of (27) and the steps are similar to Theorem 2.2, equation (26) of Theorem 2.3 is easily arrived.

Remark 2.2. When $\ell = 0$ theorems from 2.2-2.3 correspond to the known results proved by Irmak and Raina [6, pp. 449-450].

Corollary 2.5. *If* $f(z) \in T_{\lambda, \ell}^{\mu, \phi, \eta}(n;\ \alpha),$ *then*

$$\left| |D_z^{-\beta}\{f(z)\}| - \frac{1}{\Gamma(2+\beta)} |z|^{1+\beta} \right|$$

$$\leq \frac{\alpha\Gamma(2-\phi+\eta)\Gamma(n-\mu+\eta+2)\Gamma(n+2-\phi)\{\lambda\ell\phi^2 - \lambda\ell\phi + 1 - \lambda\phi + \ell\phi\}}{\Gamma(2-\mu+\eta)\Gamma(2-\phi)\Gamma(n-\phi+\eta+2)\Gamma(n+\beta+2)G[n+1,\ \phi]} |z|^{n+\beta+1} \tag{28}$$

for all $\beta\ (\beta > 0),\ z \in U$ *and* $n \in \mathbb{N}$.

Corollary 2.6. *If* $f(z) \in T_{\lambda, \ell}^{\mu, \phi, \eta}(n;\ \alpha),$ *then*

$$\left| |D_z^{\beta}\{f(z)\}| - \frac{1}{\Gamma(2-\beta)} |z|^{1-\beta} \right|$$

$$\leq \frac{\alpha\Gamma(2-\phi+\eta)\Gamma(n-\mu+\eta+2)\Gamma(n+2-\phi)\{\lambda\ell\phi^2 - \lambda\ell\phi + 1 - \lambda\phi + \ell\phi\}}{\Gamma(2-\mu+\eta)\Gamma(2-\phi)\Gamma(n-\phi+\eta+2)\Gamma(n-\beta+2)G[n+1,\ \phi]} |z|^{n-\beta+1} \tag{29}$$

for all $\beta\ (0 \leq \beta < 1),\ z \in U$ *and* $n \in \mathbb{N}$.

Equations (28), (29) are sharp for the function $f(z)$ given by (20). Corollaries (2.5) and (2.6) are obtained by using equations (9) and (10) respectively.

Remark 2.3. We note that the results (28), (29) coincide with the corollaries obtained in [6, p. 450].

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