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PERIODIC SOLUTIONS FOR DELAY DIFFERENCE EQUATIONS

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Abstract

Based on the fixed point index theory for a Banach space, nontrivial periodic solutions are found for delay difference equations of the form

$$x_{n+1} = a_n x_n + h_n f(x_{n-\tau(n)}), \quad n \in Z.$$

1. Introduction

In this note, we consider the existence of nontrivial solutions for the delay difference equations

$$x_{n+1} = a_n x_n + h_n f(x_{n-\tau(n)}), \quad n \in Z, \tag{1}$$

where $\{a_n\}_{n \in Z}$ is a positive ω-periodic sequence but $\prod_{s=0}^{\omega-1} a_s^{-1} > 1$, $\{h_n\}_{n \in Z}$ is an ω-periodic positive sequence, $\{\tau(n)\}_{n \in Z}$ is integer valued ω-periodic sequence, and $f(u)$ is a real continuous function.

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The study on the existence of periodic positive solutions for (1) have been made extensively by a number of authors (see for example, [1, 3-7]).

2. Main Result

We proceed formerly from (1) and obtain

$$\Delta\left\{x_n \prod_{k=-\infty}^{n-1} \frac{1}{a_k}\right\} = \prod_{k=-\infty}^{n} \frac{1}{a_k} h_n f(x_{n-\tau(n)}).$$

Then summing the above formal equation from n to $n + \omega - 1$, we obtain

$$x_n = \sum_{s=n}^{n+\omega-1} G(n, s) h_s f(x_{s-\tau(s)}), \quad n \in Z, \tag{2}$$

where

$$G(n, s) = \left(\prod_{k=n}^{s} \frac{1}{a_k}\right)\left(\prod_{k=0}^{\omega-1} \frac{1}{a_k} - 1\right)^{-1}.$$

It is not difficult to check that any ω-periodic sequence $\{x_n\}_{n \in Z}$ that satisfies (2) is also an ω-periodic solution of (1). Note that

$$G(n, n) = \left(\frac{1}{a_n}\right)\left(\prod_{k=0}^{\omega-1} \frac{1}{a_k} - 1\right)^{-1} = G(n + \omega, n + \omega),$$

$$G(n, n + \omega - 1) = \left(\prod_{k=0}^{\omega-1} \frac{1}{a_k}\right)\left(\prod_{k=0}^{\omega-1} \frac{1}{a_k} - 1\right)^{-1} = G(0, \omega - 1),$$

and

$$0 < N \equiv \min_{n \le i \le n+\omega-1} G(n, i) \le G(n, s) \le \max_{n \le i \le n+\omega-1} G(n, i) \equiv M,$$

$$n \le s \le n + \omega - 1.$$

Now let X be the set of all real ω-periodic sequences of the form $u =$

$\{u_n\}_{n\in Z}$, endowed with the usual linear structure as well as the norm

$$\| u \| = \max_{0\le n\le \omega-1} | u_n |.$$

Then X is a Banach space with cone

$$\Omega = \{\{u_n\}_{n\in Z} \mid u_n \ge \sigma\| u \|, n \in Z\}, \text{ where } \sigma = \frac{N}{M}.$$

And $X \times X$ is also a Banach space with the norm $\| (u, v) \| = \max\{\| u \|, \| v \|\}$.

Theorem 1. *Let* $f(x) = f_1(x) - f_2(x)$, *where* $f_i(x), (i = 1, 2)$ *are nonnegative continuous functions satisfying* $f_i(0) = 0\,(i = 1, 2)$. *Assume that*

$$\lim_{|x|\to 0} \frac{f_1(x)}{|x|} = +\infty, \tag{3}$$

$$\lim_{|x|\to 0} \frac{f_2(x)}{|x|} < +\infty, \tag{4}$$

$$\lim_{x\to +\infty} \frac{f_1(x)}{x} = 0, \tag{5}$$

and

$$\lim_{|x|\to +\infty} \frac{f_2(x)}{|x|} = 0. \tag{6}$$

Then equation (1) *has at least a nontrivial periodic solution.*

Proof. Set $\Omega = \{u = \{u_n\}_{n\in Z} \in X \mid u_n \ge 0, u_n \ge \sigma\| u \|, n \in Z\}$. It is not difficult to check that $\Omega \subset X$ is a cone and $\Omega \times \Omega \subset X \times X$ also is a cone.

Set

$$A_1(u, v)_n = \sum_{s=n}^{n+\omega-1} G(n, s) h_s f_1(u_{s-\tau(s)} - v_{s-\tau(s)}),$$

$$A_2(u, v)_n = \sum_{s=n}^{n+\omega-1} G(n, s) h_s f_2(u_{s-\tau(s)} - v_{s-\tau(s)}),$$

and

$$A(u, v)_n = (A_1(u, v)_n, A_2(u, v)_n).$$

Then $A : \Omega \times \Omega \to X \times X$ is completely continuous (on bounded close subset of $\Omega \times \Omega$). For any $n, \check{n} \in Z$, we have

$$A_i(u, v)_n = \sum_{s=n}^{n+\omega-1} G(n, s) h_s f_i(u_{s-\tau(s)} - v_{s-\tau(s)})$$

$$\leq M \sum_{s=0}^{\omega-1} h_s f_i(u_{s-\tau(s)} - v_{s-\tau(s)}),$$

and

$$A_i(u, v)_{\check{n}} = \sum_{s=\check{n}}^{\check{n}+\omega-1} G(\check{n}, s) h_s f_i(u_{s-\tau(s)} - v_{s-\tau(s)})$$

$$\geq m \sum_{s=0}^{\omega-1} h_s f_i(u_{s-\tau(s)} - v_{s-\tau(s)}) \geq \sigma A_i(u, v)_n \quad \text{for } i = 1, 2.$$

Thus, we have $A : \Omega \times \Omega \to \Omega \times \Omega$.

From (4), we know that there exist $\beta > 0$ and $r_1 > 0$ such that

$$h_s f_2(x) \leq \beta |x| \quad \text{for } |x| \leq r_1 \text{ and } s \in Z. \tag{7}$$

Let $0 < \varepsilon < \min\left\{1, \frac{\sigma}{2(1 + M\beta\omega)}\right\}$. Then we have

$$\mu(F_0(s)) = \mu\{s \leq n \leq s + \omega - 1 \,|\, |u_n - v_n| \geq \varepsilon r\} \geq \min\left\{\omega, \frac{\sigma}{2M\beta}\right\} \tag{8}$$

for $(u, v) \in \Omega \times \Omega$ and $\|(u, v)\| = r \leq r_1$ and $A_2(u, v) = v$, where $F_0(s) = \{s \leq n \leq s + \omega - 1 \,|\, |u_n - v_n| \geq \varepsilon r\}$ and $\mu(F_0(s))$ is the number of points in $F_0(s)$. In fact, if $|u_n - v_n| \geq \varepsilon r$ for any $n \in Z$, then (8) is obvious. If there exists $n_1 \in Z$ such that $|u_{n_1} - v_{n_1}| < \varepsilon r$, then $\|v\| \geq v_{n_1} > u_{n_1} - \varepsilon r \geq \sigma \|u\| - \varepsilon r$. Thus $\|v\| > (\sigma - \varepsilon) r$. Assume that $v_{n_2} = \|v\|$, then from

$A_2(u, v) = v$ and (7), we have

$$(\sigma - \varepsilon)r \le v_{n_2} = \sum_{s=n_2}^{n_2+\omega-1} G(n_2, s)h_s f_2(u_{s-\tau(s)} - v_{s-\tau(s)})$$

$$= \left(\sum_{s\in F_0(n_2)} + \sum_{s\in F(n_2)\setminus F_0(n_2)}\right) G(n_2, s)h_s f_2(u_{s-\tau(s)} - v_{s-\tau(s)})$$

$$\le M\beta\left(\sum_{s\in F_0(n_2)} + \sum_{s\in F(n_2)\setminus F_0(n_2)}\right) | u_{s-\tau(s)} - v_{s-\tau(s)} |$$

$$\le M\beta r[\mu(F_0(n_2)) + \varepsilon\mu(F(n_2)\setminus F_0(n_2))],$$

where $F(n_2) = \{n \in Z \mid n_2 \le n \le n_2 + \omega - 1\}$. It is not difficult now to check that $\mu(F_0(s)) \ge \frac{\sigma}{2M\beta}$, i.e., (8) holds.

Note that $a = \min\left\{\omega, \frac{\sigma}{2M\beta}\right\}$, choose α such that $\alpha \ge \frac{1}{ma\varepsilon}$. Then there exists $r \le r_1$, by (3) such that

$$h_s f_1(x) \ge \alpha|x|, \qquad \text{for } |x| \le r,\ s \in Z. \tag{9}$$

Set $H_n = \sum_{s=n}^{n+\omega-1} G(n, s)$, then $H = \{H_n\}_{n\in Z} \in \Omega$, and for any $(u, v) \in \partial(\Omega\times\Omega)_r = \{(u, v) \in \Omega\times\Omega \mid \|(u, v)\| = r\}$, and $t \ge 0$, we have

$$(u, v) - A(u, v) \ne t(H, \theta). \tag{10}$$

In fact, if there exists $(u^0, v^0) = (\{u_n^0\}_{n\in Z}, \{v_n^0\}_{n\in Z}) \in \partial(\Omega\times\Omega)_r$, $t_0 \ge 0$ such that

$$u^0 - A_1(u^0, v^0) = t_0 H, \tag{11}$$

$$v^0 - A_2(u^0, v^0) = \theta. \tag{12}$$

We assume that $t_0 > 0$, otherwise, (u^0, v^0) is a fixed point of A. From (12) we know that (8) holds for the above ε. From (9) we have $u^0 \ge t_0 H (u_n^0 \ge t_0 H_n)$. Note that $t^* = \sup\{t \mid u^0 \ge tH\}$, then $t^* \ge t_0 > 0$, and from (8), (9) and (11) we have

$$u_n^0 = t_0 H_n + A_1(u^0, v^0)_n$$

$$= t_0 H_n + \sum_{s=n}^{n+\omega-1} G(n, s) h_s f_1(u_{s-\tau(s)}^0 - v_{s-\tau(s)}^0)$$

$$\geq t_0 H_n + \sum_{s-\tau(s) \in F_0(n-\tau(n))} G(n, s) h_s f_1(u_{s-\tau(s)}^0 - v_{s-\tau(s)}^0)$$

$$\geq t_0 H_n + \alpha \sum_{s-\tau(s) \in F_0(n-\tau(n))} G(n, s) | u_{s-\tau(s)}^0 - v_{s-\tau(s)}^0 |$$

$$\geq t_0 H_n + m\alpha\varepsilon r \cdot \mu(F_0(n - \tau(n)))$$

$$\geq t_0 H_n + m\alpha\alpha\varepsilon t^* H_n \geq (t_0 + t^*) H_n.$$

Obviously, this does not satisfy the definition of t^*. Thus (10) holds. See [2], we have

$$i(A, (\Omega \times \Omega)_r, \Omega \times \Omega) = 0. \tag{13}$$

Next, we will prove that there exists $R > 0$, such that

$$A(u, v) \not\geq (u, v) \quad \text{for } (u, v) \in \partial(\Omega \times \Omega)_R. \tag{14}$$

In fact, we take c such that $0 < c < \frac{\sigma}{M\omega}$. From (5) and (6), there exists R_0 such that $h_s f_1(u) \leq cu$ and $h_s f_2(v) \leq c| v |$ for $u \geq R_0$ and $| v | \geq R_0$. Note that

$$T_0 = \max\{ \sup_{0 \leq u \leq R_0} h_s f_1(u), \sup_{0 \leq | v | \leq R_0} h_s f_2(v) \}.$$

Then we have

$$h_s f_1(u) \leq cu + T_0 \quad \text{for any } u \geq 0, \tag{15}$$

and

$$h_s f_2(v) \leq c| v | + T_0 \quad \text{for any } v \in R, \tag{16}$$

where $\check{R} > \max\left\{ r, R_0, \frac{\omega M T_0}{\sigma - cM\omega} \right\}$ such that (14) holds. In fact, let $\|(u,v)\|$

$= \check{R}$ and $u_n \geq v_n$ for any $n \in Z$. Then we have

$$\begin{aligned} A_1(u, v)_n &= \sum_{s=n}^{n+\omega-1} G(n, s) h_s f_1(u_{s-\tau(s)} - v_{s-\tau(s)}) \\ &\leq \sum_{s=n}^{n+\omega-1} G(n, s)[c(u_{s-\tau(s)} - v_{s-\tau(s)}) + T_0] \\ &\leq M\check{R}c\omega + MT_0\omega < \check{R} = \| u \| \end{aligned}$$

by (15). Thus $A_1(u, v) \ngeq u$, that is, $A(u, v) \ngeq (u, v)$. If there exists $n_0 \in Z$ such that $u_{n_0} < v_{n_0}$, then $\| v \| \geq \sigma\check{R}$. Hence, we have

$$\begin{aligned} A_2(u, v)_n &= \sum_{s=n}^{n+\omega-1} G(n, s) h_s f_2(u_{s-\tau(s)} - v_{s-\tau(s)}) \\ &\leq \sum_{s=n}^{n+\omega-1} G(n, s)[c\,| u_{s-\tau(s)} - v_{s-\tau(s)} | + T_0] \\ &\leq M\check{R}c\omega + \omega MT_0 < \sigma\check{R} \leq \| v \| \end{aligned}$$

by (16). Thus $A_2(u, v) \ngeq v$, that is, $A(u, v) \ngeq (u, v)$. From (14), we have

$$i(A, (\Omega \times \Omega)_R, \Omega \times \Omega) = 1. \tag{17}$$

From (13) and (17), we have $i(A, (\Omega \times \Omega)_R \setminus (\Omega \times \Omega)_r, \Omega \times \Omega) = 1$. Thus, there exists $(u^*, v^*) \in (\Omega \times \Omega)_R \setminus (\Omega \times \Omega)_r$ such that $A(u^*, v^*) = (u^*, v^*)$, i.e.,

$$u_n^* = \sum_{s=n}^{n+\omega-1} G(n, s) h_s f_1(u_{s-\tau(s)}^* - v_{s-\tau(s)}^*),$$

$$v_n^* = \sum_{s=n}^{n+\omega-1} G(n, s) h_s f_2(u_{s-\tau(s)}^* - v_{s-\tau(s)}^*),$$

from $f_i(0) = 0\,(i = 1, 2)$, we know that $u^* \neq v^*$. (Indeed, if $u^* = v^*$, then $u^* = v^* = \theta$, which is contrary to the fact that $(u^*, v^*) \in (\Omega \times \Omega)_R \setminus$

$(\Omega \times \Omega)_{r^*}$.) Thus $u^* - v^*$ is a nontrivial periodic solution of equation (2), and also a nontrivial periodic solution of equation (1). The proof is complete.

References

[1] R. P. Agarwal and P. Y. H. Pang, On a generalized difference system, Nonlinear Anal. 30 (1997), 365-376.

[2] H. Amann, Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces, SIAM Rev. 18 (1976), 620-709.

[3] I. Katsunori, Asymptotic analysis for linear difference equations, Trans. Amer. Math. Soc. 349 (1997), 4107-4142.

[4] R. Musielak and J. Popenda, On periodic solutions of a first order difference equation, An. Stiint. Univ. "Al. I. Coza" Iasi Sect. I a Mat. 34(2) (1998), 125-133.

[5] Z. Y. Zhang, An algebraic principle for the stability of difference operators, J. Differential Equations 136 (1997), 236-247.

[6] G. Zhang and S. S. Cheng, Positive periodic solutions of a discrete population model, Functional Differential Eqns. 7(3-4) (2000), 223-230.

[7] R. Y. Zhang, Z. C. Wang, Y. Chen and J. Wu, Periodic solutions of a single species discrete population model with periodic harvest/stock, Comput. Math. Appl. 39 (2000), 77-90.

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