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CHARACTERIZATION OF 2-PRIMAL IDEALS

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Abstract

In this paper we show that an ideal I of a ring R with identity is 2-primal if and only if $N_I(P) = \{a \in R/aRb \subset \mathcal{P}(I)$ for some $b \in R - P\}$ has the IFP (Insertion of Factors Property), where $\mathcal{P}(I)$ is the intersection of all prime ideals which contain I. Also we prove some equivalent conditions for 2-primal ideals.

1. Introduction

Throughout this paper R denotes a ring with identity. Birkenmeier et al. [1] called a ring R 2-*primal* if its prime radical $\mathcal{P}(R)$ coincides with the set $\mathcal{N}(R)$ of all nilpotent elements of R. An ideal I of R is called 2-primal if $\mathcal{P}(R/I) = \mathcal{N}(R/I)$. An ideal I of R is said to have IFP if $xy \in I$ implies $xRy \subseteq I$ for $x, y \in R$. Spec(R) denotes the set of all prime ideals of R. Observe that every completely semiprime ideal of R has IFP. Birkenmeier et al. [1] proved that an ideal I is 2-primal if and only if $\mathcal{P}(I)$ is completely semiprime ideal of R. Kim and Kwak [4] proved that a ring R is 2-primal if and only if the prime radical $\mathcal{P}(R)$ has the IFP if and

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only if $N(P) = \{a \in R/aRb \subseteq \mathcal{P}(R)$ for some $b \in R - P\}$ has the IFP. In this paper we introduce the definition of $N_I(P)$ and N_I^P and also we show that an ideal I of R is 2-primal if and only if $N_I(P)$ has the IFP.

2. Characterization of 2-Primal Ideals

In this section we introduce $N_I(P)$, N_I^P and discuss the characterizations of 2-primal ideals.

Definition 2.1 [2]**.** Let R be a ring with identity and P be a prime ideal of R. Then $N(P) = \{y \in R/yRs \subseteq \mathcal{P}(R)$ for some $s \in R - P\}$, $N_P = \{y \in R/ys \in \mathcal{P}(R)$ for some $s \in R - P\}$, $\overline{N(P)} = \{y \in R/y^n \in N(P)$ for some $n\}$, $\overline{N_P} = \{y \in R/y^n \in N_P$ for some $n\}$.

A ring R is said to satisfy (SI) if for each $a \in R$, $r(a)$ is a two sided ideal of R, where $r(a) = \{b \in R/ab = 0\}$.

Definition 2.2. Let I be any ideal of R and P be a prime ideal of R. Then we define $N_I(P) = \{y \in R/yRs \subseteq \mathcal{P}(I)$ for some $s \in R - P\}$, $N_I^P = \{y \in R/ys \in \mathcal{P}(I)$ for some $s \in R - P\}$, $\overline{N_I(P)} = \{y \in R/y^n \in N_I(P)$ for some $n\}$, $\overline{N_I^P} = \{y \in R/y^n \in N_I^P$ for some $n\}$. Note that $N(P) \subseteq N_I(P)$, $N_P \subseteq N_I^P$, $I \subseteq N_I(P) \subseteq \overline{N_I(P)}$ and $I \subseteq N_I^P \subseteq \overline{N_I^P}$ for any ideal I of R.

Example 2.3. Let $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$, where F is a field and $I = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$, $P = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$ and $Q = \begin{pmatrix} 0 & F \\ 0 & F \end{pmatrix}$. Then clearly I is an ideal of R and P, Q are prime ideals of R. It can be easily checked that $\mathcal{P}(I) = I$, $\mathcal{P}(P) = P$ and $\mathcal{P}(Q) = Q$.

We observe that

(i) $N_I(P) = \overline{N_I(P)} = N_P(P) = \overline{N_P(P)} = \overline{N_I^P} = N_I^P = N_P^P = \overline{N_P^P} = P$,

(ii) $N_I(Q) = \overline{N_I(Q)} = N_Q(Q) = \overline{N_Q(Q)} = \overline{N_I^Q} = N_I^Q = N_Q^Q = \overline{N_Q^Q} = Q$,

(iii) $N_P(Q) = \overline{N_P(Q)} = N_Q(P) = \overline{N_Q(P)} = \overline{N_P^Q} = N_P^Q = N_Q^P = \overline{N_Q^P} = R.$

Proposition 2.4. *Let R be a ring with identity. Then we have the following*:

(i) *$I \subseteq P$ if and only if $N_I(P) \subseteq P$ for any ideal I and prime ideal P of R.*

(ii) *$N_I(P) \subseteq N_I^P$ for any prime ideal P and ideal I of R.*

(iii) *If $I = P$, then $N_I(P) = P$ for any ideal I and any prime ideal P of R.*

(iv) *$P = Q$ if and only if $N_Q(P) = P$ for any prime ideals P and Q of R.*

Proof. (i) Suppose $I \subseteq P$. Then $\mathcal{P}(I) \subseteq P$. So, for any element $x \in N_I(P)$, there exists $b \in R - P$ such that $xRb \subseteq P$. From the primeness of P, we have $x \in P$. Therefore $N_I(P) \subseteq P$. Converse part is obvious.

(ii) Let $x \in N_I(P)$. Then there exists $b \in R - P$ such that $xRb \subseteq \mathcal{P}(I)$. Since P is prime, there exists $r \in R$ such that $brb \in R - P$. Thus we get $xbrb \in xRb \subseteq \mathcal{P}(I)$ and $brb \notin P$. Therefore $x \in N_I^P$ and consequently $N_I(P) \subseteq N_I^P$.

(iii) Since $P = I \subseteq \mathcal{P}(I)$, any element $x \in P$, we get $xR1 \subseteq \mathcal{P}(I)$. So that $P \subseteq N_I(P)$. Thus $N_I(P) = P$ by (i).

(iv) Suppose that $P = Q$, then we obtain $N_Q(P) = P$ by (iii). On the other hand, since $Q \subseteq N_Q(P) = P$, $Q \subseteq P$. Also let $x \in P$, then there exists $b \in R - P$ such that $xRb \subseteq \mathcal{P}(Q)$, because $N_Q(P) = P$. Since Q is prime, $\mathcal{P}(Q) = Q$. Hence we have $x \in Q$, since $b \in R - P \subseteq R - Q$. Therefore $P = Q$. □

Remark 2.5. 1. Converse of Proposition 2.4 (iii) is not true. In example 2.3 $N_I(Q) = Q$, but $I \neq Q$.

2. For any ideal I and J and any prime ideals P, Q if $I \subseteq J$ and $Q \subseteq P$, then $N_I(P) \subseteq N_J(Q)$. But converse is not true. In example 2.3 $N_Q(Q) \subseteq N_Q(P)$ but P is not contained in Q.

Theorem 2.6. *Let R be a ring with identity and I be an ideal of R. Then the following are equivalent*:

(i) *I is a 2-primal ideal*;

(ii) *$\mathcal{P}(I)$ has the IFP*;

(iii) *$N_I(P)$ has the IFP for each $P \in Spec(R)$*;

(iv) *$N_I(P) = \overline{N_I^P}$ for each $P \in Spec(R)$*;

(v) *$N_I(P) = N_I^P$ for each $P \in Spec(R)$*;

(vi) *$N_I^P \subseteq P$ for each prime ideal P which contains I*;

(vii) *$N_{P/\mathcal{P}(I)} \subseteq P/\mathcal{P}(I)$ for each prime ideal P which contains I*;

(viii) *$\overline{N_J^Q} \subseteq N_I(P)$ for any ideal $J \subseteq I$ and prime ideals P, Q such that $P \subseteq Q$*;

(ix) *$N_J^Q \subseteq N_I(P)$ for any ideal $J \subseteq I$ and prime ideals P, Q such that $P \subseteq Q$*;

(x) *$\overline{N_J^Q} \subseteq P$ for any ideal J and prime ideals P, Q such that $J \subseteq I \subseteq P \subseteq Q$*;

(xi) *$N_J^Q \subseteq P$ for any ideal J and prime ideals P, Q such that $J \subseteq I \subseteq P \subseteq Q$*;

(xii) *$N_{Q/\mathcal{P}(I)} \subseteq P/\mathcal{P}(I)$ for each prime ideal P, Q such that $I \subseteq P \subseteq Q$.*

Proof. (i) implies (ii): Let $x + \mathcal{P}(I) \in \mathcal{N}(R/\mathcal{P}(I))$. Then there exists a positive integer n such that $(x + \mathcal{P}(I))^n = \mathcal{P}(I)$, i.e., $x^n \in \mathcal{P}(I)$. Since I is a 2-primal ideal, $\mathcal{P}(I)$ is completely semiprime ideal. Hence $x \in \mathcal{P}(I)$, i.e., $x + \mathcal{P}(I) = \mathcal{P}(I)$. Hence $R/\mathcal{P}(I)$ has no nilpotent elements and so $R/\mathcal{P}(I)$ satisfies (SI). Thus $xy \in \mathcal{P}(I)$ implies $xRy \subseteq \mathcal{P}(I)$. Therefore $\mathcal{P}(I)$ has the IFP.

(ii) implies (iii): Let $xy \in N_I(P)$ for each $P \in \mathrm{Spec}(R)$. Then $xyRb \subseteq \mathcal{P}(I)$ for some $b \in R - P$. Since $\mathcal{P}(I)$ has IFP, $xRyRb \subseteq \mathcal{P}(I)$ and so $xRy \subseteq N_I(P)$. Therefore $N_I(P)$ has the IFP for each $P \in \mathrm{Spec}(R)$.

(iii) implies (i) : It is enough to prove that if $x^n \in I$, then $x \in \mathcal{P}(I)$ [1]. Suppose $x \notin \mathcal{P}(I)$, then there exists a prime ideal P which contains I such that $x \notin P$. Since P is prime, there exists $r_1, r_2, \ldots, r_{n-1}$ such that $xr_1xr_2 \cdots r_{n-1}x \notin P$. Since $x^n \in I \subseteq N_I(P)$ and $N_I(P)$ has the IFP, $xr_1xr_2 \cdots r_{n-1}x \in N_I(P)$. Since $I \subseteq P$, $N_I(P) \subseteq P$ by Proposition 2.4 (i) and hence $xr_1xr_2 \cdots r_{n-1}x \in P$, a contradiction. Therefore $x \in \mathcal{P}(I)$. Thus I is a 2-primal ideal.

(i) implies (iv): Let $a \in \overline{N_I^P}$ for each $P \in \mathrm{Spec}(R)$. Then $a^n \in N_I^P$ for some positive integer n. Hence there exists $b \in R - P$ such that $a^n b \in \mathcal{P}(I)$. Since I is 2-primal, $\mathcal{P}(I)$ has IFP and also $\mathcal{P}(I)$ is completely semiprime. So that $aRb \subseteq \mathcal{P}(I)$. Thus $a \in N_I(P)$. Again we observe that $N_I(P) \subseteq N_I^P \subseteq \overline{N_I^P}$ by Proposition 2.4 (ii). Therefore $N_I(P) = N_I^P$.

(iv) implies (v): It is obvious.

(v) implies (vi): It follows from Proposition 2.4 (i).

(vi) implies (vii): Let $a + \mathcal{P}(I) \in N_{P/\mathcal{P}(I)}$. Then there exists $b + \mathcal{P}(I) \in R/\mathcal{P}(I) - P/\mathcal{P}(I)$ such that $(a + \mathcal{P}(I))(b + \mathcal{P}(I)) = \mathcal{P}(I)$, i.e., $ab \in \mathcal{P}(I)$. Hence $a \in N_I^P \subseteq P$. Therefore $a + \mathcal{P}(I) \in P/\mathcal{P}(I)$.

(vii) implies (i): It is enough to prove that $R/\mathcal{P}(I)$ is reduced. Suppose for the purpose of contradiction that $R/\mathcal{P}(I)$ is not reduced. Then there exists $a + \mathcal{P}(I) \in R/\mathcal{P}(I)$ such that $(a + \mathcal{P}(I))^2 = \mathcal{P}(I)$ and $a \notin \mathcal{P}(I)$. Hence there is a prime ideal P such that $a \notin P$. So $a + \mathcal{P}(I) \in R/\mathcal{P}(I) - P/\mathcal{P}(I)$. But since $(a + \mathcal{P}(I))^2 = \mathcal{P}(I)$, we obtain $a + \mathcal{P}(I) \in N_{P/\mathcal{P}(I)} \subseteq P/\mathcal{P}(I)$ which is a contradiction. Therefore I is a 2-primal ideal.

(i) implies (viii): Let $x \in \overline{N_J^Q}$. Then there exists n such that $x^n b \in \mathcal{P}(J)$ for some $b \in R - Q$. Since $P \subseteq Q$, and $J \subseteq I$, $x^n b \in \mathcal{P}(I)$ for some $b \in R - P$. Since I is 2-primal, $xRb \subseteq \mathcal{P}(I)$ for some $b \in R - P$. Thus $x \in N_I(P)$.

(viii) implies (ix) implies (xi) implies (vi) and (viii) implies (x) implies (xi) are obvious.

(vi) implies (xii): Let $a + \mathcal{P}(I) \in N_{Q/\mathcal{P}(I)}$. Then $(a + \mathcal{P}(I))(b + \mathcal{P}(I)) = \mathcal{P}(I)$ for some $b + \mathcal{P}(I) \in R/\mathcal{P}(I) - Q/\mathcal{P}(I)$. Hence $ab \in \mathcal{P}(I)$ for some $b \notin Q$. Since $P \subseteq Q$, $b \notin P$. Thus we have $a \in N_I^P$. Since $I \subseteq P$, $N_I^P \subseteq P$ by (vi). Therefore $a + \mathcal{P}(I) \in P/\mathcal{P}(I)$,

(xii) implies (vii): It is obvious. □

Corollary 2.7. *Let R be a ring with identity and I be a 2-primal ideal of R. Then for any prime ideal P of R,* $I \subseteq P$ *if and only if* $N_I^P \subseteq P$.

Proof. Suppose $I \subseteq P$, then $N_I^P \subseteq P$ by Theorem 2.6. Conversely, let $x \in I$. Then $x \in \mathcal{P}(I)$, because $I \subseteq \mathcal{P}(I)$. Since R has the identity, $\mathcal{P}(I) \subseteq N_I^P$ and hence $x \in P$. Thus $I \subseteq P$. □

Corollary 2.8. $I = P$ *if and only if* $N_I^P = P$ *for any completely prime ideal I and prime ideal P of R.*

Proof. Assume that $I = P$. Since I is completely prime ideal of R, I is 2-primal ideal. By Corollary 2.7, $N_I^P \subseteq P$. Hence $P = I \subseteq N_I^P \subseteq P$. Therefore $P = N_I^P$. On the other hand, clearly $I \subseteq P$ by Corollary 2.7. Let $x \in P$. Then there exists $b \in R - P$ such that $xb \in \mathcal{P}(I)$, because

$P = N_I^P$. Since I is completely prime, $\mathcal{P}(I) = I$ and $b \notin I$, we get $x \in I$. Therefore $I = P$. □

Remark 2.9. Theorem 2.6 is more generalization of Theorem 2.1 in [4].

Corollary 2.10. *Let R be a ring with identity. Then the following are equivalent*:

(i) *R is a 2-primal ring*;

(ii) *$\mathcal{P}(R)$ has the IFP*;

(iii) *$N(P)$ has the IFP for each $P \in Spec(R)$*;

(iv) *$N(P) = \overline{N_P}$ for each $P \in Spec(R)$*;

(v) *$N(P) = N_P$ for each $P \in Spec(R)$*;

(vi) *$N_P \subseteq P$ for each $P \in Spec(R)$*;

(vii) *$N_{P/\mathcal{P}(R)} \subseteq P$ for each $P \in Spec(R)$*;

(viii) *$\overline{N_Q} \subseteq N(P)$ for each $Q, P \in Spec(R)$ such that $P \subseteq Q$*;

(ix) *$N_Q \subseteq N(P)$ for each $Q, P \in Spec(R)$ such that $P \subseteq Q$*;

(x) *$\overline{N_Q} \subseteq P$ for each $Q, P \in Spec(R)$ such that $P \subseteq Q$*;

(xi) *$N_Q \subseteq P$ for each prime ideal P and Q such that $P \subseteq Q$*;

(xii) *$N_{Q/\mathcal{P}(R)} \subseteq P/\mathcal{P}(R)$ for each $Q, P \in Spec(R)$ such that $P \subseteq Q$.*

Proof. It follows from Theorem 2.6. □

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