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# EQUIVARIANT DEFINABLE MORSE FUNCTIONS ON DEFINABLE $C^rG$ MANIFOLDS

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## Abstract

Let $0 \leq r < \infty$, $G$ be a compact definable $C^r$ group, $X$ be a compact affine definable $C^rG$ manifold and $f$ be an equivariant definable Morse function on $X$. We prove that if $f$ has no critical value in $[a, b]$, then $f^{-1}((-\infty, a])$ is definably $C^rG$ diffeomorphic to $f^{-1}((-\infty, b])$. Moreover we prove that the set of equivariant definable Morse functions on $X$ whose critical loci are finite unions of nondegenerate critical orbits is dense in the set of $G$ invariant $C^r$ functions on $X$ with respect to the $C^r$ Whitney topology.

We also prove that if $G$ is a compact definable group and $X$ is a definable $G$ manifold, then $X$ is definably $G$ homeomorphic to an open definable $G$ $CW$ complex.

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## 1. Introduction

In Morse theory it is proved that the topological data of a given space can be described via data given by Morse functions defined on the space. We refer the reader to the book by Milnor [17] for Morse theory on compact $C^\infty$ manifolds, and to the book by Goresky and MacPherson [4] on singular spaces. Its equivariant versions are studied in Wasserman [23], Mayer [16], Datta and Pandey [1], and its definable versions are studied in Peterzil and Starchenko [19], Loi [15].

Let $\mathcal{M} = (\mathbb{R}, +, \cdot, <, \ldots)$ be an o-minimal expansion of the standard structure $\mathcal{R} = (\mathbb{R}, +, \cdot, <)$ of the field of real numbers. The term "definable" means "definable with parameters in $\mathcal{M}$". General references on o-minimal structures are [2], [3], see also [22]. It is known in [20] that there exist uncountably many o-minimal expansions of $\mathcal{R}$.

In this paper we consider its equivariant definable $C^r$ version of Morse theory. Everything is considered in $\mathcal{M}$, $2 \le r < \infty$, every definable map is continuous and every definable $C^r$ manifold does not have boundary unless otherwise stated. Remark that the condition that $r \ge 2$ is necessary to define Morse functions. Definable $C^rG$ manifolds are studied in [11], [9].

Let $X$ be an $n$-dimensional definable $C^r$ manifold and $f : X \to \mathbb{R}$ be a definable $C^r$ function. We say that a point $p \in X$ is a *critical point* of $f$ if the differential of $f$ at $p$ is zero. If $p$ is a critical point of $f$, then $f(p)$ is called a *critical value* of $f$. Let $p$ be a critical point of $f$ and $(U, u)$ be a definable $C^r$ coordinate system on $X$ at $p$ (i.e., $U$ is a definable open subset of $X$ containing $p$ and $u$ is a definable $C^r$ diffeomorphism from $U$ onto a definable open subset of $\mathbb{R}^n$ with $u(p) = 0$). The critical point $p$ is *nondegenerate* if the Hessian matrix of $f \circ u^{-1}$ at 0 is nonsingular. Direct computations show that the notion of nondegeniricity does not depend on the choice of a local coordinate system. In the non-equivariant setting, Peterzil and Starchenko [19] introduced definable $C^r$ Morse functions in an o-minimal expansion of the standard structure of a real closed field.

Let $G$ be a definable $C^r$ group, $X$ be a definable $C^rG$ manifold and $f : X \to \mathbb{R}$ be a $G$ invariant definable $C^r$ function on $X$. A closed definable $C^rG$ submanifold $Y$ of $X$ is called a *critical manifold* (resp. a *nondegenerate critical manifold*) of $f$ if each $p \in Y$ is a critical point (resp. a nondegenerate critical point) of $f$. We say that $f$ is an *equivariant definable Morse function* if the critical locus of $f$ is a finite union of nondegenerate critical manifolds of $f$ without interior.

**Theorem 1.1.** *Let $G$ be a compact definable $C^r$ group and $f$ be an equivariant definable Morse function on a compact affine definable $C^rG$ manifold $X$. If $f$ has no critical value in $[a, b]$, then $f^a := f^{-1}((-\infty, a])$ is definably $C^rG$ diffeomorphic to $f^b := f^{-1}((-\infty, b])$.*

Theorem 1.1 is an equivariant definable $C^r$ version of Theorem 4.3 [23].

In the non-equivariant definable case, Loi [15] proved the density of definable Morse functions.

Let $Def^r(\mathbb{R}^n)$ denote the set of definable $C^r$ functions on $\mathbb{R}^n$. For each $f \in Def^r(\mathbb{R}^n)$ and for each positive definable function $\varepsilon : \mathbb{R}^n \to \mathbb{R}$, the $\varepsilon$-neighborhood $N(f; \varepsilon)$ of $f$ in $Def^r(\mathbb{R}^n)$ is defined by $\{h \in Def^r(\mathbb{R}^n) \,|\, |\partial^\alpha (h - f)| < \varepsilon, \ \forall \alpha \in \mathbb{N}^n, \ |\alpha| \le r\}$, where $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\partial^\alpha F = \dfrac{\partial^{|\alpha|} F}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$. We call the topology defined by these $\varepsilon$-neighborhoods the *definable $C^r$ topology*.

**Theorem 1.2** [15]**.** *Let $X$ be a definable $C^r$ submanifold of $\mathbb{R}^n$. Then the set of definable $C^r$ functions on $\mathbb{R}^n$ which are Morse functions on $X$ and have distinct critical values are open and dense in $Def^r(\mathbb{R}^n)$ with respect to the definable $C^r$ topology.*

Remark that the definable $C^r$ topology and the $C^r$ Whitney topology do not coincide in general. If $X$ is compact, then these topologies of the set $Def^r(X)$ of definable $C^r$ functions on $X$ are the same ([22, p.156]).

A nondegenerate critical manifold of an equivariant Morse function on a definable $C^rG$ manifold is called a *nondegenerate critical orbit* if it is an orbit. The following is the density of equivariant definable Morse functions.

**Theorem 1.3.** *Let G be a compact definable $C^r$ group and X be a compact affine definable $C^rG$ manifold. Then the set $Def_{equi-Morse,o}(X)$ of equivariant definable Morse functions on X whose critical loci are finite unions of nondegenerate critical orbits is dense in the set $C^r_{inv}(X)$ of G invariant $C^r$ functions on X with respect to the $C^r$ Whitney topology. Moreover $Def_{equi-Morse,o}(X)$ is open and dense in the set $Def^r_{inv}(X)$ of G invariant definable $C^r$ functions with respect to the definable $C^r$ topology.*

Definable *G CW* complexes are introduced in [7]. Moreover it is proved that if *G* is a compact definable group, then every definable *G* set is definably *G* homeomorphic to a *G* invariant definable subset of a definable *G CW* complex obtained by removing some open *G* cells ([7, 1.1]). Here definable *G* set means a *G* invariant definable subset of some representation of *G*.

In this paper we consider *open definable G CW complexes* (See Definition 4.1) which are more general than 2.2 [7].

We say that a definable $C^0G$ manifold is a *definable G manifold*.

**Theorem 1.4.** *Let G be a compact definable group and X be a definable G manifold.*

(1) *X is definably G homeomorphic to an open definable G CW complex in the sense of Definition* 4.1.

(2) *If X is compact*, *then X is definably G homeomorphic to a complete*

*definable G CW complex in the sense of Definition* 4.1. *In particular*, *X is G homeomorphic to a finite G CW complex.*

Theorem 1.4 is somewhat stronger than the following usual equivariant $C^\infty$ version [16].

**Theorem 1.5** [16]**.** *Let G be a compact Lie group and f be a special equivariant Morse function on a* $C^\infty G$ *manifold X such that every* $f^a$ *is compact*. *Then X is G homotopy equivalent to a G CW complex. If X is compact*, *then X is G homotopy equivalent to a finite G CW complex.*

The following is a definable version of a well-known topological result (e.g. [5, 6.2.4]).

**Theorem 1.6.** *Let X be an n-dimensional compact definable* $C^r$ *manifold admitting a definable Morse function* $f : X \to \mathbb{R}$ *with only two critical points*. *Then X is definably homeomorphic to the n-dimensional unit sphere* $S^n$. *If* $n \leq 6$, *then X is definably* $C^r$ *diffeomorphic to* $S^n$.

Remark that if $n = 7$, then Milnor [18] found a $C^\infty$ manifold which is homeomorphic to $S^7$, but not $C^\infty$ diffeomorphic to $S^7$. Since every $C^r$ manifold admits a unique $C^\infty$ manifold structure up to $C^\infty$ diffeomorphism (e.g. [5, 2.3.4]), this result holds in the $C^r$ setting.

## 2. Preliminaries and Proof of Theorem 1.1

A *definable* $C^r$ *manifold* is a $C^r$ manifold with a finite system of charts whose transition functions are definable, and definable $C^r$ maps, definable $C^r$ diffeomorphisms and definable $C^r$ imbeddings are defined similarly ([11], [9]). A definable $C^r$ manifold is *affine* if it is definably $C^r$ imbeddable into some $\mathbb{R}^n$. If $\mathcal{M} = \mathcal{R}$, a definable $C^\omega$ manifold (resp. an affine definable $C^\omega$ manifold) is called a *Nash manifold* (resp. an *affine Nash manifold*). By [10], every definable $C^r$ manifold is affine. The definable $C^\omega$ case is complicated. Even if $\mathcal{M} = \mathcal{R}$, it is known that for every compact or compactifiable $C^\omega$ manifold of positive dimension admits a continuum number of distinct nonaffine Nash manifold structures [21], and its equivariant version is proved in [12].

A group $G$ is a *definable* $C^r$ *group* if $G$ is a definable $C^r$ manifold such that the group operations $G \times G \to G$ and $G \to G$ are definable $C^r$ maps. Let $G$ be a definable $C^r$ group. A *definable* $C^r G$ *manifold* is a pair $(X, \phi)$ consisting of a definable $C^r$ manifold $X$ and a group action $\phi : G \times X \to X$ such that $\phi$ is a definable $C^r$ map. For simplicity, we write $X$ instead of $(X, \phi)$.

Let $G$ be a definable $C^r$ group. A *representation map* of $G$ means a group homomorphism from $G$ to some $O_n(\mathbb{R})$ which is of class definable $C^r$ and the *representation* of this representation map is $\mathbb{R}^n$ with the orthogonal action induced by the representation map. In this paper, we always assume that every representation is orthogonal. A *definable* $C^r G$ *submanifold* of a representation $\Omega$ of $G$ is a $G$ invariant definable $C^r$ submanifold of $\Omega$. We say that a definable $C^r G$ manifold is *affine* if it is definably $C^r G$ diffeomorphic to a definable $C^r G$ submanifold of some representation of $G$.

**Theorem 2.1** [8]. *Let $X$ and $Y$ be compact affine definable $C^r G$ manifolds possibly with boundary and $2 \leq r < \infty$. Then the following three conditions are equivalent.*

(1) *$X$ and $Y$ are $C^1 G$ diffeomorphic.*

(2) *$X$ and $Y$ are definably $C^r G$ diffeomorphic.*

(3) *The interior of $X$ is definably $C^r G$ diffeomorphic to that of $Y$.*

**Proof of Theorem 1.1.** By the proof of Theorem 4.3 [23], $f^a = f^{-1}((-\infty, a])$ is $C^{r-1} G$ diffeomorphic to $f^b = f^{-1}((-\infty, b])$. Since $X$ is compact and affine, these two manifolds are compact affine definable $C^r G$ manifolds with boundary. Thus Theorem 2.1 proves Theorem 1.1. □

Remark that the method of the proof of Theorem 4.3 [23] is the integration of a $G$ invariant $C^\infty$ vector field. This method does not work in the definable category because the integration of a $G$ invariant definable $C^r$ vector field is not always definable.

**Example 2.2.** (1) Let $\mathcal{M} = \mathcal{R}$ and $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \dfrac{1}{x^2+1}$. Then $f$ is a definable $C^\omega$ function, but $F(x) := \int_0^x f(t)dt = \tan^{-1}(x)$ is not definable in $\mathcal{M}$.

(2) Let $\mathcal{M} = \mathbf{R}_{\exp} = (\mathbb{R}, +, \cdot, <, e^x)$ and $f : \mathbb{R} \to \mathbb{R}$, $f(x) = e^{x^2}$. Then $f$ is a definable $C^\omega$ function, but $F(x) := \int_0^x f(t)dt$ is not definable in $\mathcal{M}$.

## 3. Proof of Theorem 1.3

Let $G$ be a compact definable $C^r$ group. Let $f$ be a map from a $C^rG$ manifold $X$ to a representation $\Omega$ of $G$. Denote the Haar measure of $G$ by $dg$ and let $C^r(X, \Omega)$ denote the set of $C^r$ maps from $X$ to $\Omega$. Define

$$A : C^r(X, \Omega) \to C^r(X, \Omega), \quad A(f)(x) = \int_G g^{-1} f(gx)dg.$$

We call $A$ the *averaging function*. In particular, if $G = \{g_1, \dots, g_n\}$, then $A(f)(x) = \frac{1}{n}\sum_{i=1}^n g_i^{-1} f(g_i x)$.

Observations similar to 2.6 [13], 4.3 [9] and 2.35 [14] show the following proposition.

**Proposition 3.1** ([13], [9], [14])**.** *Let $G$ be a compact definable $C^r$ group.*

(1) *$A(f)$ is equivariant, and $A(f) = f$ if $f$ is equivariant.*

(2) *If $0 \le r \le \infty$ and $f \in C^r(X, \Omega)$, then $A(f) \in C^r(X, \Omega)$.*

(3) *If $f$ is a polynomial map, then so is $A(f)$.*

(4) *If* $0 \leq r < \infty$ *and X is compact, then* $A : C^r(X, \Omega) \to C^r(X, \Omega)$ *is continuous in the* $C^r$ *Whitney topology.*

(5) *If* $G$ *is a finite group and* $0 \leq r \leq \omega$, $X$ *is a definable* $C^r G$ *manifold and f is a definable* $C^r$ *map, then* $A(f)$ *is a definable* $C^r G$ *map.*

We say that a $C^r$ manifold $G$ is a $C^r$ *group* if $G$ is a group and the group operations $G \times G \to G$ and $G \to G$ are $C^r$ maps. By the proof of Lemma 4.8 [23] proves the following.

**Theorem 3.2** [23]**.** *Let G be a compact* $C^r$ *group and X be a compact* $C^r G$ *manifold. Then the set* $C^r_{equi-Morse,o}(X)$ *of equivariant Morse functions on X whose critical loci are finite unions of nondegenerate critical orbits is open and dense in the set* $C^r_{inv}(X)$ *of G invariant* $C^r$ *functions on X with respect to the* $C^r$ *Whitney topology.*

**Proof of Theorem 1.3.** Let $f \in C^r_{inv}(X)$ and $\mathcal{N} \subset C^r_{inv}(X)$ be an open neighborhood of *f* in $C^r_{inv}(X)$. By Theorem 3.2, there exists an open subset $\mathcal{N}' \subset \mathcal{N}$ such that each $h \in \mathcal{N}'$ is an equivariant Morse function whose critical locus is a finite union of nondegenerate critical orbits. Let $C^r(X)$ denote the set of $C^r$ functions on *X*. Since $A : C^r(X) \to C^r(X)$ is continuous and $A(C^r(X)) = C^r_{inv}(X)$, $A : C^r(X) \to C^r_{inv}(X)$ is continuous. Fix $h \in \mathcal{N}'$. Since $A(h) = h$, $A^{-1}(\mathcal{N}')$ is an open neighborhood of $h$ in $C^r(X)$. Applying the polynomial approximation theorem, we have a polynomial function $h'$ lies in $A^{-1}(\mathcal{N}')$. Applying the averaging function, we have a *G* invariant polynomial function $F := A(h')$ lies in $\mathcal{N}'$. Since *F* is a $G$ invariant polynomial function, it is a $G$ invariant definable $C^r$ function. Thus *F* is an equivariant definable Morse function lies in $\mathcal{N}$.

We now prove the second part. Since *X* is compact, the definable $C^r$ topology and the $C^r$ Whitney topology coincide [22, p. 156]. By the first part, $Def_{equi-Morse,o}(X)$ is dense in $C^r_{inv}(X)$. Thus it is dense in $Def^r_{inv}(X)$.

Let $h \in Def_{equi-Morse,o}(X)$. By Theorem 3.2, there exists an open neighborhood $\mathcal{V}$ of $h$ in $C^r_{inv}(X)$ such that each $h \in \mathcal{V}$ is an equivariant Morse function whose critical locus is a finite union of nondegenerate critical orbits. Thus $\mathcal{V} \cap Def_{inv}(X)$ is the required open neighborhood of $h$ in $Def_{inv}(X)$. □

## 4. Proof of Theorem 1.4

Let $G$ be a definable group. A *definable set with a definable $G$ action* is a pair $(X, \theta)$ consisting of a definable set $X$ and a group action $\theta : G \times X \to X$ such that $\theta$ is a definable map. This action is not necessarily linear (orthogonal). We simply write $X$ instead of $(X, \theta)$.

A definable map between definable sets with definable $G$ actions is a *definable $G$ map* if it is a $G$ map. A definable $G$ map is a *definable $G$ homeomorphism* if it is bijective and its inverse is a definable $G$ map.

We consider the definition of open definable $G$ $CW$ complexes which is more general than 2.2 [7].

**Definition 4.1.** Let $G$ be a compact definable group.

(1) An *open definable $G$ CW complex* is a pair of $(X, \{c_i \mid i \in I\})$ consisting of a Hausdorff definable $G$ space $X$ and a finite family of open $G$ cells $\{c_i \mid i \in I\}$ such that

(a) The underlying space $|X|$ of $X$ is a definable set with a definable $G$ action.

(b) The orbit space $X/G$ is a definable subset of some $\mathbb{R}^n$.

(c) For each open $G$ $n$-cell $c_i$, there exist a definable subgroup $H_{c_i}$ of $G$ and the characteristic map $f_{c_i} : G/H_{c_i} \times \Delta \to \overline{c_i} \subset X$ such that $f_{c_i}|G/H_{c_i} \times Int\,\Delta \to c_i$ is a definable $G$ homeomorphism and the boundary $\partial c_i$ is equal to $f_{c_i}(G/H_{c_i} \times \partial\Delta)$, where $\Delta$ is a subset of the standard compact $n$-simplex $\Delta^n$ obtained by removing some open lower dimensional faces of $\Delta^n$, $\overline{c_i}$ denotes the closure of $c_i$ in $X$, $Int\,\Delta$ means the interior of $\Delta$ and $\partial\Delta = \Delta - Int\,\Delta$.

(d) For each $c_i$, $\overline{c_i} - c_i$ is a finite union of open $G$ cells.

(2) An open definable $G$ $CW$ complex is called a *complete definable G CW complex* if every $\Delta$ is a standard compact simplex.

In the above definition, if $(X, \{c_i \mid i \in I\})$ is complete and $|X|$ is a definable $G$ set, then this coincides 2.2 [7]. Remark that a complete definable $G$ $CW$ complex is a compact standard $G$ $CW$ complex.

The following is a generalization of 1.1 [7].

**Theorem 4.2.** *Let G be a compact definable group and X be a definable set with a definable G action.*

(1) *X is definably G homeomorphic to an open definable G CW complex.*

(2) *If X is compact, then X is definably G homeomorphic to a complete definable G CW complex.*

To prove Theorem 4.2, we first prepare a piecewise equivariant definable trivialization theorem. Its non-equivariant version is proved in 9.1.2 [2].

Let $X$ be a definable set with a definable $G$ action and $Y \subset \mathbb{R}^n$ be a definable set, and $f : X \to Y$ be a $G$ invariant definable map. We say that $f$ is *definably G trivial* if there exist a definable set $F$ with a definable $G$ action and a definable $G$ map $h : X \to F$ such that $(f, h) : X \to Y \times F$ is a definable $G$ homeomorphism. In this case, each fiber $f^{-1}(a)$ of $f$ over $a$ is definably $G$ homeomorphic to $F$.

**Theorem 4.3.** *Let G be a compact definable group and let X be a definable set with a definable G action. Let Y be a definable set in some* $\mathbb{R}^n$ *and let* $f : X \to Y$ *be a G invariant definable map. Then there exists* *a finite partition* $\{A_i\}$ *of A into definable sets such that each* $f | f^{-1}(A_i) : f^{-1}(A_i) \to A_i$ *is definably G trivial.*

Using the following two theorems, the proof of 2.5 [9] proves Theorem 4.3.

**Theorem 4.4** (10.2.18 [2])**.** *Let G be a compact definable group and let X be a definable set with a definable G action. Then the orbit space* $X/G$ *exists as a definable subset of some* $\mathbb{R}^n$ *and the orbit map* $\pi : X \to X/G$ *is G invariant, definable and proper.*

The following is a special case of 1.3 [6]

**Theorem 4.5** [6]**.** *Let G be a compact definable group. Then every definable set with a definable G action has only finitely many orbit types.*

The following is a definable triangulation of a definable set 8.2.9 [2].

**Theorem 4.6** (8.2.9 [2]) (Definable triangulation)**.** *Let* $S_1, \ldots, S_k$ *be definable subsets of a definable set S in* $\mathbb{R}^n$. *Then there exist a finite simplicial complex* $K \subset \mathbb{R}^n$ *and a definable map* $\phi : S \to \mathbb{R}^n$ *such that* $\phi$ *maps S and each* $S_i$ *homeomorphically onto unions of open simplexes of K.*

We call $(\phi, K)$ a *definable triangulation of S compatible with* $S_1, \ldots, S_k$.

**Proof of Theorem 4.2.** Let $\pi : X \to X/G$ be the orbit map. Then $X/G$ is a definable set and $\pi$ is a definable map. By Theorem 4.3, there exists a finite decomposition $\{A_i\}$ of $X/G$ into definable sets such that each $\pi | \pi^{-1}(A_i) : \pi^{-1}(A_i) \to A_i$ is definably $G$ trivial. Using Theorem 4.5, $X$ has only finitely many orbit types $\{(H_1), \ldots, (H_k)\}$. By Theorem 4.6, there exists a definable triangulation $(\phi, K)$ of $X/G$ compatible with $\{A_i\} \cup \{\pi(X(H_1)), \ldots, \pi(X(H_k))\}$, where $X(H_i) = \{x \in X \mid G_x$ is conjugate to $H_i\}$. Then $\phi(X/G)$ is a definable subset of $K$ obtained by removing some open simplexes.

Let $Y$ be the minimal simplicial complex of $K$ containing $\phi(X/G)$. For each $n$-simplex $\Delta^n$ of $Y$, there exists a definable section $s : \phi^{-1}(\text{int}\,\Delta^n) \to X$ of $\pi$ because $\phi^{-1}(\text{int}\,\Delta^n)$ is contained in some $A_i$. After replacing its subdivision, if necessary, we can extend the section $s$ to a definable section $\widetilde{s} : \phi^{-1}(\Delta^n \cap \phi(X/G)) \to X$. Let $\sigma = s(\phi^{-1}(\text{int}\,\Delta^n))$. Then $\overline{\sigma} = \widetilde{s}(\phi^{-1}(\Delta^n \cap \phi(X/G)))$ and $\overline{G\sigma} = G\overline{\sigma}$, where $\overline{\sigma}$ (resp. $\overline{G\sigma}$) denotes the closure of $\sigma$ (resp. $G\sigma$) in $X$. Hence there exists a definable $G$ map $f_\sigma$ : $G/H \times (\Delta \cap \phi(X/G)) \cong G(\widetilde{s}(a_i)) \times (\Delta \cap \phi(X/G)) \to \overline{G\sigma}$, $f_\sigma(gH, x) = g(\widetilde{s}(\phi(x)))$ such that $f_\sigma \,|\, G/H \times \text{int}\,\Delta \to G\sigma$ is a definable $G$ homeomorphism, where $a_i$ is any point in $\sigma$.

By collecting open $G$ cells $G\widetilde{s}(\phi(\text{int}\,\Delta^n)) = \pi^{-1}(\phi(\text{int}\,\Delta^n))$ for all open simplices $\text{int}\,\Delta^n$ of $\phi(X/K)$, we have the required open definable $G$ $CW$ complex. □

**Theorem 4.7** [10]. *If* $0 \leq r < \infty$, *then every definable* $C^r$ *manifold is definably* $C^r$ *imbeddable into some* $\mathbb{R}^n$.

**Proof of Theorem 1.4.** By Theorem 4.7, $X$ is definably homeomorphic to a definable subset $Y$ of some $\mathbb{R}^n$. Thus this definable homeomorphism makes $Y$ a definable set with a definable $G$ action. Applying Theorem 4.2, we have Theorem 1.4. □

## 5. Proof of Theorem 1.6

We now prepare two results.

**Lemma 5.1** (A.6 [19]) (Morse's Lemma). *Let* $r \geq 0$, $X$ *be an* $n$*-dimensional definable* $C^{r+2}$ *manifold,* $f : X \to \mathbb{R}$ *be a definable* $C^{r+2}$ *function and* $p \in X$ *be a nondegenerate critical point of f. Then there exists a definable* $C^r$ *coordinate system* $(U, \phi)$ *on* $X$ *at* $p$ *such that* $f \circ \phi^{-1}(y) = f(p) - y_1^2 - \cdots - y_\lambda^2 + y_{\lambda+1}^2 + \cdots + y_n^2$, *where* $\lambda$ *is the index of f at p.*

**Theorem 5.2** (e.g. 6.2.2 [5]). *Let* $f : X \to [a, b]$ *be a* $C^r$ *map on a compact manifold X with boundary. If f has no critical points and* $f(\partial X) = \{a, b\}$, *then there exists a* $C^{r-1}$ *diffeomorphism* $F : f^{-1}(a) \times [a, b] \to X$ *such that* $f \circ F$ *coincides the projection* $f^{-1}(a) \times [a, b] \to [a, b]$.

**Proof of Theorem 1.6.** Using Morse's Lemma and Theorems 2.1, 4.7 and 5.2, a similar proof of 6.2.4 [5] proves the first half of Theorem 1.6.

If $n \leq 6$, $X$ is $C^r$ diffeomorphic to $S^n$. Thus since $X$ is compact and by Theorems 2.1 and 4.7, $X$ is definably $C^r$ diffeomorphic to $S^n$. □

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