Far East Journal of Mathematical Sciences (FJMS)
Volume 28, Issue 1, 2008, Pages 17-36
Published online: February 15, 2008
This paper is available online at http://www.pphmj.com

THE TRANSFORMATION METHOD FOR PARABOLIC EQUATIONS AND VARIATIONAL INEQUALITIES IN NON-CYLINDRICAL DOMAINS

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Abstract

In this paper, parabolic initial-boundary value problems and variational inequalities in non-cylindrical domains are considered. A method is proposed which allows to transfer the problem from a non-cylindrical domain to a cylindrical one.

1. Introduction

In [2], a new method, called the *transformation method*, was introduced and applied. This method allowed to transfer a parabolic initial-boundary value problem in one space variable on some non-

2000 Mathematics Subject Classification: 35K20, 65N40.

Keywords and phrases: parabolic PDEs, numerical methods, time-discretization, method of lines, transformation method, variational inequality.

Received July 10, 2007

rectangular domain Q in the (x, t)-plane into a parabolic problem on a rectangle $\overline{Q}$. The reason was that for a rectangle, Rothe's method of time discretization can be efficiently applied, and the (approximative) solution to the initial problem on Q can be obtained by the inverse transform.

Of course, the application of Rothe's method has had only an auxiliary character. Here, we will show, that the method from [2], used there for linear parabolic problems with uniformly elliptic part, can be applied also to parabolic problems with degeneration and/or singularity, to nonlinear problems and even to problems in more (space) dimensions.

In Section 2, we work in one space dimension. At first, we repeat the considerations from [2] and we will show that the same approach works also for parabolic problems with degeneration and/or singularity in the elliptic part, and for nonlinear problems. In Section 3, we will consider the case of more space dimensions, where we have of course to restrict the admissible domains. Finally, in Section 4, we describe the procedure for the more general case of parabolic *variational inequalities* with a linear operator A.

The results of this paper form a part of the Ph.D. theses of the second and third authors, defended in June 2007 at the University of West Bohemia, Pilsen (see [8] and [9]).

2. The Transformation Method in One Space Dimension

2.1. The linear problem: The uniformly elliptic case

Let Q be a domain in the (x, t)-plane, bounded by the lines $t = 0$, $t = T$, $x = 0$ and by the curve $x = g(t)$, where g is in $C^1[0, T]$ and such that $g(0) = 1$, $g'(t) \geq 0$.

The substitution

$$x = \xi g(t) \tag{2.1}$$

maps the domain Q onto the rectangle $\overline{Q}$ in the (ξ, t)-plane:

$$\overline{Q} = \{(\xi, t) : 0 < \xi < 1, 0 < t < T\}.$$

The parabolic equation

$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(a(x, t)\frac{\partial u}{\partial x}\right) = f(x, t) \text{ on } Q \tag{2.2}$$

with solution $u = u(x, t)$ is then transformed into the parabolic equation

$$\frac{\partial \overline{u}}{\partial t} - \frac{\partial}{\partial \xi}\left(\overline{a}(\xi, t)\frac{\partial \overline{u}}{\partial \xi}\right) - \frac{g'(t)}{g(t)}\xi\frac{\partial \overline{u}}{\partial \xi} = \bar{f}(\xi, t) \text{ on } \overline{Q} \tag{2.3}$$

for the function

$$\overline{u}(\xi, t) := u(\xi g(t), t), \tag{2.4}$$

where

$$\overline{a}(\xi, t) := \frac{a(\xi g(t), t)}{g^2(t)}, \quad \bar{f}(\xi, t) := f(\xi g(t), t).$$

If the "elliptic part" in (2.2) is uniformly elliptic and bounded, i.e., if there exist constants $c_0, c_1 > 0$ such that

$$c_0 \leq a(x, t) \leq c_1 \text{ for all } (x, t) \in Q, \tag{2.5}$$

then the same is true for (2.3) since, due to our assumptions on the function g, we have that

$$0 < \bar{c}_0 \leq \overline{a}(\xi, t) \leq \bar{c}_1 \text{ for all } (\xi, t) \in \overline{Q},$$

with

$$\bar{c}_0 = \frac{c_0}{g^2(T)}, \quad \bar{c}_1 = \frac{c_1}{g^2(0)}.$$

Hence, the character of the problem (2.2) remains after the substitution (2.1) unchanged: the problem (2.3) is of the same type.

Let us denote by S_1, S_2 the left and right "sides" of Q and by S_0 its "bottom", i.e.,

$$S_0 = \{(x, 0) : 0 < x < 1\},$$

$$S_1 = \{(0, t) : 0 < t < T\},$$

$$S_2 = \{(g(t), t) : 0 < t < T\}. \tag{2.6}$$

For the purpose of a (weak) solution of the parabolic initial-boundary value problem

$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(a(x, t)\frac{\partial u}{\partial x}\right) = f(x, t) \text{ on } Q,$$

$$u(x, t) = 0 \text{ on } S_1 \cup S_2,$$

$$u(x, 0) = u_0(x) \text{ on } S_0 \tag{2.7}$$

(i.e., with homogeneous Dirichlet conditions), in [2], there was introduced the (anisotropic Sobolev) space

$$W^{1,0;2}(Q)$$

as the set of all $u = u(x, t) \in L^2(Q)$ such that the (distributional) derivative $\frac{\partial u}{\partial x}$ belongs to $L^2(Q)$, equipped with the norm

$$\| u \|_{W^{1,0;2}(Q)} := \left(\int_Q | u(x, t)|^2 dxdt + \int_Q \left|\frac{\partial u}{\partial x}(x, t)\right|^2 dxdt\right)^{\frac{1}{2}}, \tag{2.8}$$

and it was shown (see [2, Lemma 3.1]) that if $u \in W^{1,0;2}(Q)$, then the function $\overline{u}$ from (2.4) belongs to $W^{1,0;2}(\overline{Q})$ and we have that

$$c_1\| \overline{u} \|_{W^{1,0;2}(\overline{Q})} \le \| u \|_{W^{1,0;2}(Q)} \le c_2\| \overline{u} \|_{W^{1,0;2}(\overline{Q})} \tag{2.9}$$

with $c_1, c_2 > 0$ independent on u (more precisely, $c_1^2 = \min\left\{g(0), \frac{1}{g(T)}\right\}$, $c_2^2 = \max\left\{g(T), \frac{1}{g(0)}\right\}$).

Hence, we can solve the transformed problem

$$\frac{\partial \overline{u}}{\partial t} - \frac{\partial}{\partial \xi}\left(\overline{a}(\xi, t)\frac{\partial \overline{u}}{\partial \xi}\right) - \frac{g'(t)}{g(t)}\,\xi\,\frac{\partial \overline{u}}{\partial \xi} = \bar{f}(\xi, t) \text{ on } \overline{Q},$$

$$\overline{u}(0, t) = \overline{u}(1, t) = 0, \quad t \in (0, T),$$

$$\overline{u}(\xi, 0) = u_0(\xi), \quad \xi \in (0, 1) \tag{2.10}$$

(e.g., by Rothe's method) and then we have the (weak) solution u of the initial problem (2.7) by the inverse transform:

$$u(x, t) = \overline{u}\left(\frac{x}{g(t)}, t\right).$$

2.2. Degeneration and/or singularity

Consider again equation (2.2) on Q, now with a coefficient $a(x, t)$ which fails to satisfy (2.5), but is still non-negative and tends to zero (= degeneration) or to infinity (= singularity) either on the "left side" S_1, or on the "right side" S_2, or on both sides simultaneously. Let us illustrate the approach on hand of a simple example.

Example 2.1. Consider equation (2.2) with $a(x, t)$ replaced by

$$x^{\lambda} | g(t) - x |^{\mu} a_1(x, t)$$

with $\lambda, \mu \in \mathbb{R}$ and a_1 satisfying (2.5). Then the equation

$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(x^{\lambda} | g(t) - x |^{\mu} a_1(x, t) \frac{\partial u}{\partial x}\right) = f(x, t) \text{ on } Q \tag{2.11}$$

is in general no more uniformly elliptic in its "elliptic part" and has a degeneration $(\lambda > 0, \mu > 0)$ or singularity $(\lambda < 0, \mu < 0)$ on the corresponding side. After substitution (2.1) we get the equation

$$\frac{\partial \overline{u}}{\partial t} - \frac{\partial}{\partial \xi}\left(\xi^{\lambda}(1 - \xi)^{\mu} \frac{a_1(\xi g(t), t)}{g^{2-\lambda-\mu}(t)} \frac{\partial \overline{u}}{\partial \xi}\right) - \frac{g'(t)}{g(t)} \xi \frac{\partial \overline{u}}{\partial \xi} = \bar{f}(\xi, t) \text{ on } \overline{Q} \tag{2.12}$$

with the same behavior on $\overline{S}_1 (= S_1)$ and $\overline{S}_2 = \{(1, t) : 0 < t < T\}$, respectively.

Moreover, the appropriate space for a (weak) solution of (2.7) will be a weighted Sobolev space, characterized by the claim that

$$\frac{\partial u}{\partial x} \in L^2(Q; \lambda, \mu),$$

i.e.,

$$\left\| \frac{\partial u}{\partial x} \right\|_{L^2(Q;\lambda,\mu)} := \left(\int_Q \left| \frac{\partial u}{\partial x}(x, t) \right|^2 x^\lambda |\, g(t) - x\,|^\mu dxdt \right)^{\frac{1}{2}}$$

$$= \left(\int_0^T \int_0^{g(t)} \left| \frac{\partial u}{\partial x}(x, t) \right|^2 x^\lambda |\, g(t) - x\,|^\mu dxdt \right)^{\frac{1}{2}} < \infty.$$

(We omit for a while the behavior of the function u itself.) For the transformed function $\overline{u}$ from (2.4) we then have

$$\left\| \frac{\partial u}{\partial x} \right\|^2_{L^2(Q;\lambda,\mu)} = \int_0^T \int_0^{g(t)} \left| \frac{\partial u}{\partial x}(x, t) \right|^2 x^\lambda |\, g(t) - x\,|^\mu dxdt$$

$$= \int_0^T g(t)^{\lambda+\mu-1} \int_0^1 \left| \frac{\partial \overline{u}}{\partial \xi}(\xi, t) \right|^2 \xi^\lambda |\, 1 - \xi\,|^\mu d\xi dt.$$

If we denote

$$\left\| \frac{\partial \overline{u}}{\partial \xi} \right\|_{L^2(\overline{Q};\lambda,\mu)} := \left(\int_{\overline{Q}} \left| \frac{\partial \overline{u}}{\partial \xi}(\xi, t) \right|^2 \xi^\lambda |\, 1 - \xi\,|^\mu d\xi dt \right)^{\frac{1}{2}},$$

then we have the equivalence relation

$$c_1 \left\| \frac{\partial \overline{u}}{\partial \xi} \right\|_{L^2(\overline{Q};\lambda,\mu)} \le \left\| \frac{\partial u}{\partial x} \right\|_{L^2(Q;\lambda,\mu)} \le c_2 \left\| \frac{\partial \overline{u}}{\partial \xi} \right\|_{L^2(\overline{Q};\lambda,\mu)}$$

with $c_1^2 = \min\limits_{[0,T]} g(t)^{\lambda+\mu-1}$, $c_2^2 = \max\limits_{[0,T]} g(t)^{\lambda+\mu-1}$.

Hence, we can instead of the initial problem, consisting of the equation (2.11) and the conditions

$$u(x, t) = 0 \text{ on } S_1 \cup S_2,$$

$$u(x, 0) = u_0(x) \text{ on } S_0$$

solve the transformed problem consisting of the equation (2.12) and the conditions

$$\overline{u}(0, t) = \overline{u}(1, t) = 0 \text{ for } t \in (0, T),$$

$$\overline{u}(\xi, 0) = u_0(\xi) \text{ for } \xi \in (0, 1). \tag{2.13}$$

If we look for the (weak) solution $\overline{u}$ of the second problem in a space characterized by the claim that

$$\frac{\partial \overline{u}}{\partial \xi} \in L_2(\overline{Q}, \lambda, \mu),$$

then we obtain again a weak solution u of the initial problem by the inverse transform:

$$u(x, t) = \overline{u}\left(\frac{x}{g(t)}, t\right). \qquad \blacksquare$$

More generally, we could consider (2.2) with a coefficient $a(x, t)$ which behaves like $\omega(d(x, t))$,

$$a(x, t) \approx \omega(d(x, t)),$$

where $\omega = \omega(s)$ is a positive function on $(0, \infty)$ and $d(x, t)$ is the distance of the point $(x, t) \in Q$ from the "sides", i.e., from $S_1 \cup S_2$.

For simplicity, we have chosen a very special degeneration (singularity) characterized by the distance to the "sides" S_1, S_2 of Q, but a similar approach can be used for more general degenerations/ singularities, and even for the case when the singularity or degeneration appears at the term $\frac{\partial \overline{u}}{\partial t}$, i.e., for equations of the form

$$d^{\kappa}(x, t)\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(a(x, t)\frac{\partial u}{\partial x}\right) = f(x, t) \text{ on } Q$$

with $\kappa \in \mathbb{R}$ (see [7]).

2.3. The nonlinear problem

In this subsection we consider a *nonlinear* parabolic problem, namely

$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(a\left(x, t; \frac{\partial u}{\partial x}\right)\right) = f(x, t) \text{ on } Q,$$

$$u(x, t) = 0 \ \text{ on } \ S_1 \cup S_2,$$

$$u(x, 0) = u_0(x) \ \text{ on } \ S_0 \tag{2.14}$$

with $a = a(x, t; \eta)$ defined on $Q \times \mathbb{R}$.

At first we start with a simple example.

Example 2.2. Let us consider the following problem:

$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(a(x, t)\left|\frac{\partial u}{\partial x}\right|^{p-2}\frac{\partial u}{\partial x}\right) = f(x, t) \ \text{on} \ Q,$$

$$u(x, t) = 0 \ \text{ on } \ S_1 \cup S_2,$$

$$u(x, 0) = u_0(x) \ \text{ on } \ S_0, \tag{2.15}$$

where $1 < p < \infty$. Suppose that the coefficient $a(x, t)$ satisfies condition (2.5).

After the substitution (2.1), we have for

$$\overline{u}(\xi, t) = u(\xi g(t), t)$$

the following problem:

$$\frac{\partial \overline{u}}{\partial t} - \frac{\partial}{\partial \xi}\left(\overline{a}(\xi, t)\left|\frac{\partial \overline{u}}{\partial \xi}\right|^{p-2}\frac{\partial \overline{u}}{\partial \xi}\right) - \frac{g'(t)}{g(t)}\xi\frac{\partial \overline{u}}{\partial \xi} = \bar{f}(\xi, t) \ \text{ on } \ \overline{Q},$$

$$\overline{u}(0, t) = \overline{u}(1, t) = 0 \ \text{ for } \ t \in (0, T),$$

$$\overline{u}(\xi, 0) = \overline{u}_0(\xi) \ \text{ for } \ \xi \in (0, 1), \tag{2.16}$$

where

$$\overline{a}(\xi, t) := \frac{a(\xi g(t), t)}{(g(t))^p}, \quad \bar{f}(\xi, t) := f(\xi g(t), t).$$

Moreover, the appropriate space for a weak solution of (2.14) will be the Sobolev space $W^{1, 0; p}(Q)$ characterized by the norm

$$\| u \|_{W^{1, 0; p}(Q)} := \left(\int_Q | u(x, t)|^p \, dxdt + \int_Q \left|\frac{\partial u}{\partial x}(x, t)\right|^p dxdt\right)^{\frac{1}{p}},$$

and it can be shown (similarly as in [2]) that if $u \in W^{1,0;p}(Q)$, then $\overline{u} \in W^{1,0;p}(\overline{Q})$ and we have the equivalence

$$\| \overline{u} \|_{W^{1,0;p}(\overline{Q})} \approx \| u \|_{W^{1,0;p}(Q)}.$$

So, we have a nonlinear parabolic problem on the rectangle $\overline{Q}$. The problem (2.16) can be solved by Rothe's method if we assume additionally, that the derivatives $\frac{\partial a}{\partial t}$, $\frac{\partial^2 a}{\partial t^2}$ and $\frac{\partial^2 a}{\partial x \partial t}$ exist and are bounded on Q (for details see [3], [4]). Hence, we obtain the weak solution $\overline{u} \in W^{1,0;p}(\overline{Q})$ and we get finally the weak solution $u \in W^{1,0;p}(Q)$ of the initial problem (2.15) by the inverse transformation:

$$u(x, t) = \overline{u}\left(\frac{x}{g(t)}, t \right). \qquad \blacksquare$$

Now we consider the more general problem (2.14). After substitution (2.1), we have for $\overline{u}(\xi, t)$ the following parabolic problem in the rectangle $\overline{Q}$:

$$\frac{\partial \overline{u}}{\partial t} - \frac{\partial}{\partial \xi}\left(\overline{a}\left(\xi, t; \frac{\partial \overline{u}}{\partial \xi} \right) \right) - \frac{g'(t)}{g(t)} \xi \frac{\partial \overline{u}}{\partial \xi} = \bar{f}(\xi, t) \text{ on } \overline{Q},$$

$$\overline{u}(0, t) = \overline{u}(1, t) = 0, \quad t \in (0, 1),$$

$$\overline{u}(\xi, 0) = \overline{u}_0(\xi), \quad \xi \in (0, 1), \tag{2.17}$$

where

$$\overline{a}(\xi, t; \eta) = \frac{1}{g(t)} a\left(\xi g(t), t; \frac{\eta}{g(t)} \right),$$

$$\bar{f}(\xi, t) = f(\xi g(t), t).$$

This problem can be solved (in the weak sense) in $W^{1,0;p}(\overline{Q})$, if we make the following assumptions.

Assumption 2.3. Let $t \in (0, T)$ and denote Ω_t the segment $(0, g(t))$.

(A1) $a(x, t; \eta)$ satisfies the *Carathéodory condition*, i.e., $a(\cdot, t; \eta)$ is measurable in Ω_t for every $\eta \in \mathbb{R}$ and $a(x, t; \cdot)$ is continuous in $\mathbb{R}$ for a.e. $x \in \Omega_t$.

(A2) $a(x, t; \eta)$ satisfies the *growth condition*

$$|a(x, t; \eta)| \le C(g(x) + |\eta|^{p-1})$$

for a.e. $x \in \Omega_t$ and every $\eta \in \mathbb{R}$; here C is a given positive constant and g is a given function from $L_{p'}(\Omega_t)$, $p' = \frac{p}{p-1}$.

(A3) $a(x, t; \eta)$ satisfies the *monotonicity condition*, i.e.,

$$(a(x, t; \eta) - a(x, t; \zeta))(\eta - \zeta) > 0$$

for a.e. $x \in \Omega_t$ and for all $\eta, \zeta \in \mathbb{R}$, $\eta \ne \zeta$.

(A4) $a(x, t; \eta)$ satisfies the *coercivity* (*ellipticity*) *condition*

$$a(x, t; \eta)\eta \ge C|\eta|^p$$

for a.e. $x \in \Omega_t$ and for every $\eta \in \mathbb{R}$ with the constant $C > 0$ independent of η and t.

(A5) There exist the partial derivatives

$$\frac{\partial a}{\partial t}(x, t; \eta), \ \frac{\partial^2 a}{\partial t^2}(x, t; \eta), \ \frac{\partial^2 a}{\partial x \partial t}(x, t; \eta), \ \frac{\partial a}{\partial \eta}(x, t; \eta)$$

and satisfy the growth condition (A2).

(A6) The function f satisfies the Lipschitz condition,

$$\| f(\cdot, t) - f(\cdot, t')\|_{L_2(\Omega_{t'})} \le C(t - t') \text{ for all } t, t' \in (0, T), t > t'.$$

Assumption 2.3 guarantees, that if we apply to (2.17) Rothe's method, then we can obtain a uniquely determined solution $u = u(x, t)$ using for the corresponding elliptic problems the theory of monotone operators. (For details, see [3], [4]; the assumptions are caused by the need to use the theory of monotone operators).

The solution of the previous problem (2.14) is taken with applying the inverse transform

$$u(x, t) = \overline{u}\left(\frac{x}{g(t)}, t\right).$$

3. Generalization to More Space Dimensions

3.1. The transformation method in more space dimensions

Let Q be a non-cylindrical domain in $\mathbb{R}^{N+1}$,

$$Q = \{(x, t) : x \in \Omega_t, t \in (0, T)\}, \tag{3.1}$$

where $(0, T)$ is a finite interval, Ω_t is a domain in $\mathbb{R}^N$ with Lipschitzian boundary $\partial\Omega_t$ and for every $t, s \in (0, T)$, $t < s$, it is

$$\varnothing \neq \Omega_0 \subset \Omega_t \subset \Omega_s \subset \Omega_T,$$

where $\Omega_0 = \operatorname{int} \bigcap_{t \in (0, T)} \Omega_t$, and $\Omega_T = \bigcup_{t \in (0, T)} \Omega_t$ is bounded domain. Let $\overline{Q} = \Omega_0 \times (0, T)$ be a cylindrical domain in $\mathbb{R}^{N+1}$ and consider the mapping

$$\pi : Q \to \overline{Q},$$

which is defined as $\pi(x, t) = (H(x, t), t) = (\xi, t)$, where

$$\xi = H(x, t) = (H_1(x, t), H_2(x, t), \ldots, H_N(x, t)), \quad x \in \Omega_t. \tag{3.2}$$

Let us assume that the mapping H satisfies the following conditions:

(A1) There exist positive constants C_1, C_2 such that for any fixed $t \in (0, T)$ the mapping $H(\cdot, t) : \Omega_t \to \Omega_0$ satisfies the condition:

$$C_1 | x - y | \leq | H(x, t) - H(y, t) | \leq C_2 | x - y |, \text{ for all } x, y \in \Omega_t.$$

(A2) If we denote by $G(\cdot, t) : \Omega_0 \to \Omega_t$ the inverse transform to $H(\cdot, t)$, for any fixed $t \in (0, T)$, i.e.,

$$G(\cdot, t) = (G_1(\cdot, t), G_2(\cdot, t), \ldots, G_N(\cdot, t)) = H^{-1}(\cdot, t),$$

then the derivatives $\dfrac{\partial H_i}{\partial t}(x, t)$, $\dfrac{\partial G_i}{\partial t}(\xi, t)$ $(i = 1, \ldots, N)$ exist and are bounded in Q, $\overline{Q}$, respectively.

Let u be a function defined in Q and let us define the transform $\mathcal{T}$ as

$$\overline{u}(\xi, t) = (\mathcal{T}u)(\xi, t) = u(G(\xi, t), t). \tag{3.3}$$

Then the inverse transform to $\mathcal{T}$ takes the form

$$u(x, t) = (\mathcal{T}^{-1}\overline{u})(x, t) = \overline{u}(H(x, t), t). \tag{3.4}$$

Lemma 3.1. *Let* $u \in W^{1,2}(Q)$ *and let* $\overline{u}$ *be given by* (3.3). *Then there exist positive constants* C_3 *and* C_4 *such that*

$$C_3\| \overline{u} \|_{W^{1,2}(\overline{Q})} \le \| u \|_{W^{1,2}(Q)} \le C_4\| \overline{u} \|_{W^{1,2}(\overline{Q})}.$$

To prove this lemma we need the following assertion.

Lemma 3.2. *Let* Ω_0 *and* Ω_t *be two bounded open subsets of* $\mathbb{R}^N$. *Let* $t \in (0, T)$ *and* $G(\cdot, t)$ *be a mapping from* Ω_0 *onto* Ω_t *satisfying the following condition*:

There exist $c, d > 0$ *such that*

$$d|\xi - \eta| \le |G(\xi, t) - G(\eta, t)| \le c|\xi - \eta| \tag{3.5}$$

for all $\xi, \eta \in \Omega_t$.

Let $p \in [1, \infty)$. *Then the mapping* $\mathcal{T}$ *defined by* (3.3) *is a continuous linear operator from* $W^{1,p}(\Omega_0)$ *into* $W^{1,p}(\Omega_t)$.

Proof. The proof is similar to the proof of Lemma 5.7.3 in [6] so we omit the details.

Proof of Lemma 3.1. Due to Lemma 3.2, we have

$$C_1^*\| \overline{u}(\cdot, t) \|^2_{W^{1,2}(\Omega_0)} \le \| u(\cdot, t) \|^2_{W^{1,2}(\Omega_t)} \le C_2^*\| \overline{u}(\cdot, t) \|^2_{W^{1,2}(\Omega_0)},$$

since $G(\cdot, t)$ satisfies the condition (3.5) with the constants $d = \frac{1}{C_2}$, $c = \frac{1}{C_1}$, which immediately follows from the condition (A1). The constants C_1^*, C_2^* are independent on t, due to the conditions (A1), (A2). From these and from the definition of the norm of the space $W^{1,2}(Q)$, i.e.,

$$\| u \|^2_{W^{1,2}(Q)} = \int_0^T \| u(\cdot, t) \|^2_{W^{1,2}(\Omega_t)} dt + \int_0^T \left\| \frac{\partial u}{\partial t}(\cdot, t) \right\|^2_{L_2(\Omega_t)} dt$$

we have the proof. ■

4. Parabolic Variational Inequalities

4.1. Variational inequalities in non-cylindrical domains

For Ω a domain in $\mathbb{R}^N$, we denote by $K(\Omega)$ a closed convex subset of the Sobolev space $W_0^{1,2}(\Omega)$ and define

$$K_t := K(\Omega_t)$$

as subset of $W_0^{1,2}(\Omega_t)$.

Let us consider the *parabolic variational inequality* in Q:

$$u(t) \in K_t : \quad a(u(t), v - u(t))_t + (b(t)u'(t), v - u(t))_t$$
$$\geq (f(t), v - u(t))_t \text{ for all } v \in K_t, \tag{4.1}$$

for a.e. $t \in (0, T)$, with the initial condition

$$u(0) = u_0, \tag{4.2}$$

where $a(\cdot, \cdot)_t$ is a bilinear form which is defined as

$$a(u(t), v(t))_t = \sum_{|i|,|j| \leq 1} \int_{\Omega_t} a_{i,j}(x, t) \partial^i u(x, t) \partial^j v(x, t) dx,$$

and $(\cdot, \cdot)_t$ is the scalar product in $L_2(\Omega_t)$. The coefficients $a_{i,j}$ of the bilinear form, b and f are functions defined a.e. in Q.

The transform $\mathcal{T}$ defined by (3.3) maps $W^{1,2}(Q)$ into $W^{1,2}(\overline{Q})$, due to Lemma 3.1. Then (4.1) is transformed to the following variational inequality

$$\overline{u}(t) \in K_0: \quad \overline{a}(\overline{u}(t), v - \overline{u}(t))_0 + (\overline{b}(t)\overline{u}'(t), v - \overline{u}(t))_0$$

$$\geq (\bar{f}(t), v - \overline{u}(t))_0 \text{ for all } v \in K_0, \tag{4.3}$$

with the initial condition

$$\overline{u}(0) = \overline{u}_0 = u_0(G(\cdot, 0)). \tag{4.4}$$

Here $\overline{a}(\cdot, \cdot)_0$ is a bilinear form defined as

$$\overline{a}(\overline{u}(t), \overline{v}(t))_0 = a(\mathcal{T}^{-1}\overline{u}(t), \mathcal{T}^{-1}\overline{v}(t))_t$$

$$+ \sum_{k=1}^{N} \int_{\Omega_0} \overline{b}(\xi, t) \frac{\partial H_k}{\partial t}(G(\xi, t), t) \frac{\partial \overline{u}}{\partial \xi_k}(\xi, t)\overline{v}(\xi, t) d\xi, \tag{4.5}$$

where

$$\overline{b}(\xi, t) = (\mathcal{T}b)(\xi, t) | J(G)(\xi, t) |, \quad \bar{f}(\xi, t) = (\mathcal{T}f)(\xi, t) | J(G)(\xi, t) |$$

and $| J(G)(\xi, t) |$ is the Jacobian of the mapping $G(\cdot, t)$.

If the bilinear form $a(u, v)_t$ in (4.1) is bounded and uniformly elliptic, i.e., if there exist constants $C_0, C_1 > 0$ such that

$$| a(u, v)_t | \leq C_1 \| u \|_{W^{1,2}(\Omega_t)} \| v \|_{W^{1,2}(\Omega_t)} \text{ for all } u, v \in W^{1,2}(\Omega_t); \tag{4.6}$$

$$a(u, u)_t \geq C_0 \| u \|^2_{W^{1,2}(\Omega_t)} \text{ for all } u \in V_t = W_0^{1,2}(\Omega_t), \tag{4.7}$$

then the same is true for (4.5) if we, in addition to conditions (A1), (A2) from Section 3, assume that

(A3) The function b is positive and satisfies the Lipschitz condition in x, i.e.,

there exists a constant C_4 such that for any fixed $t \in (0, T)$

$$| b(x, t) - b(y, t) | \leq C_4 | x - y |, \text{ for all } x, y \in \Omega_t;$$

(A4) For any fixed $t \in (0, T)$, the function

$$\overline{b}(\cdot, t)\frac{\partial H_k}{\partial t}(G(\cdot, t), t)$$

belongs to $W^{1,\infty}(\Omega_0)$ and the derivatives

$$\frac{\partial}{\partial \xi_k}\left[\overline{b}(\xi, t)\frac{\partial H_k}{\partial t}(G(\xi, t), t)\right] \leq 0 \text{ a.e. in } \Omega_0, \quad k = 1, 2, \ldots, N.$$

According to (4.6), (A3), (A4), Lemma 3.2 and the Schwarz inequality, we have

- Boundedness:

$$\begin{aligned}
|\overline{a}(\overline{u}(t), \overline{v}(t))_0| &\leq |a(\mathcal{T}^{-1}\overline{u}(t), \mathcal{T}^{-1}\overline{v}(t))_t| \\
&\quad + \sum_{k=1}^{N}\int_{\Omega_0}\left|\overline{b}(\xi, t)\frac{\partial H_k}{\partial t}(G(\xi, t), t)\frac{\partial \overline{u}}{\partial \xi_k}(\xi, t)\overline{v}(\xi, t)\right|d\xi \\
&\leq C_1\|(\mathcal{T}^{-1}\overline{u}(t)\|_{W^{1,2}(\Omega_t)}\|\mathcal{T}^{-1}\overline{v}(t)\|_{W^{1,2}(\Omega_t)} \\
&\quad + C_2\sum_{k=1}^{N}\left\|\frac{\partial \overline{u}(t)}{\partial \xi_k}\right\|_{L_2(\Omega_0)}\|\overline{v}(t)\|_{L_2(\Omega_0)} \\
&\leq C_3\|\overline{u}(t)\|_{W^{1,2}(\Omega_0)}\|\overline{v}(t)\|_{W^{1,2}(\Omega_0)} \text{ for all } \overline{u}(t), \overline{v}(t) \in W^{1,2}(\Omega_0);
\end{aligned}$$

- Ellipticity:

$$\begin{aligned}
&\overline{a}(\overline{u}(t), \overline{u}(t))_0 \\
&= a(\mathcal{T}^{-1}\overline{u}(t), \mathcal{T}^{-1}\overline{u}(t))_t + \sum_{k=1}^{N}\int_{\Omega_0}\overline{b}(\xi, t)\frac{\partial H_k}{\partial t}(G(\xi, t), t)\frac{\partial \overline{u}}{\partial \xi_k}(\xi, t)\overline{u}(\xi, t)d\xi \\
&\geq C_0\|\mathcal{T}^{-1}\overline{u}(t)\|^2_{W^{1,2}(\Omega_t)} + \sum_{k=1}^{N}\int_{\Omega_0}\left[\overline{b}(\xi, t)\frac{\partial H_k}{\partial t}(G(\xi, t), t)\right]\left[\frac{\partial \overline{u}}{\partial \xi_k}(\xi, t)\overline{u}(\xi, t)\right]d\xi \\
&\geq C_4\|\overline{u}(t)\|^2_{W^{1,2}(\Omega_0)} - \frac{1}{2}\sum_{k=1}^{N}\int_{\Omega_0}\frac{\partial}{\partial \xi_k}\left[\overline{b}(\xi, t)\frac{\partial H_k}{\partial t}(G(\xi, t), t)\right][\overline{u}(\xi, t)]^2 d\xi
\end{aligned}$$

$\geq C_4 \| \overline{u}(t) \|^2_{W^{1,2}(\Omega_0)}$ for all $\overline{u}(t) \in W_0^{1,2}(\Omega_0)$,

since the expression

$$-\frac{1}{2}\sum_{k=1}^{N}\int_{\Omega_0}\frac{\partial}{\partial \xi_k}\left[\overline{b}(\xi, t)\frac{\partial H_k}{\partial t}(G(\xi, t), t)\right][\overline{u}(\xi, t)]^2 d\xi$$

is nonnegative due to (A4).

Hence, we can solve the transformed parabolic variational inequality (4.3), (4.4) by *Rothe's method*, applying, e.g., the results of Bock and Kačur in [1], Kačur in [5], and finally we have the solution u (in the sense of Definition 4.1, below) of the initial problem (4.1), (4.2) by the inverse transform (3.4).

Now we give the definition of the weak solution of our problem. First we denote

$$K_{\overline{Q}} = \{\overline{v} \in W^{1,2}(\overline{Q}), \quad \overline{v}(t) \in K_0 \text{ a.e. in } I\}.$$

Definition 4.1. A function $u(t)$ is called a *weak solution* of the problem (4.1), (4.2) if the transformed function $\overline{u}(t)$ defined by (3.3) satisfies the following conditions:

(1) $\overline{u} \in K_{\overline{Q}} \cap AC(I, L_2(\Omega_0))$,

(2) $\overline{u}(0) = \overline{u}_0$,

(3)

$$\int_0^T \overline{a}(\overline{u}(t), v(t) - \overline{u}(t))_0 dt + \int_0^T (\overline{b}(t), \overline{u}'(t), v(t) - \overline{u}(t))_0 dt$$

$$\geq \int_0^T (\bar{f}(t), v(t) - \overline{u}(t))_0 dt \text{ for all } v \in K_{\overline{Q}}.$$

Thus, we have proved the following theorem.

Theorem 4.2. *Under the assumptions* (A1)-(A4) *the problem* (4.1), (4.2) *has exactly one weak solution.*

4.2. Some examples and remarks

In this subsection we illustrate all considerations from the previous sections on some concrete cases. In Example 4.3, we extend the approach used in [2] for $N = 1$ and for a parabolic *equation* to the case of a parabolic *variational inequality*. In Example 4.4 a similar approach is used for the more dimensional cases. In Example 4.5, we consider again the case $N = 1$ for a more general domain Q. In all examples, we suppose for simplicity that the function $b(x, t)$ in (4.1) depends only on t.

Example 4.3. Let Q be a domain in the (x, t)-plane, bounded by the lines $t = 0$, $t = T$, $x = 0$ and by the curve $x = g(t)$, where g is in $C^1[0, T]$ and such that $g(0) = 1$, $g'(t) \geq 0$. Let

$$\overline{Q} = \{(\xi, t) : 0 < \xi < 1, 0 < t < T\}.$$

Then we can define the mapping π in the form

$$\pi(x, t) = \left(\frac{x}{g(t)}, t \right) = (\xi, t),$$

i.e.,

$$\xi = H(x, t) = \frac{x}{g(t)}.$$

In this case the transform $\mathcal{T}$ takes the form

$$\overline{u}(\xi, t) := (\mathcal{T} u)(\xi, t) = u(\xi g(t), t), \tag{4.8}$$

since $G(\xi, t) = H^{-1}(\xi, t) = \xi g(t)$.

If we consider the variational inequality (4.1) (in the case $N = 1$) with

$$a(u(t), v(t))_t = \int_{\Omega_t} a(x, t) \frac{\partial u(x, t)}{\partial x} \frac{\partial v(x, t)}{\partial x} dx, \tag{4.9}$$

then after the transformation (4.8) the transformed bilinear form (4.5) of the variational inequality (4.3) has the form

$$\overline{a}(\overline{u}(t), \overline{v}(t))_0 = \int_{\Omega_0} \overline{a}(\xi, t) \frac{\partial \overline{u}(\xi, t)}{\partial \xi} \frac{\partial \overline{v}(\xi, t)}{\partial \xi} d\xi - g'(t) \int_{\Omega_0} \frac{\partial \overline{u}}{\partial \xi}(\xi, t) \xi \overline{v}(\xi, t) d\xi,$$

and

$$\overline{a}(\xi, t) = \frac{a(\xi g(t), t)}{g(t)}, \quad \bar{f}(\xi, t) = f(\xi g(t), t) g(t).$$

If the bilinear form (4.9) is uniformly elliptic and bounded, then the same is true for the transformed bilinear form $\overline{a}(\overline{u}(t), \overline{u}(t))_0$, since the mapping H and the function b satisfy the conditions (A1)-(A4).

The following example is generalization of the previous one.

Example 4.4. Let us consider the domain Q in $\mathbb{R}^{N+1}$ defined as

$$Q := \left\{(x, t) : \left(\frac{\phi_1(x_1)}{g_1(t)}, \frac{\phi_2(x_2)}{g_2(t)}, \dots, \frac{\phi_N(x_N)}{g_N(t)}\right) \in \Omega_0, t \in (0, T)\right\}, \quad (4.10)$$

where Ω_0 is a bounded domain in $\mathbb{R}^N$ *starshaped at the origin* and $\phi_i \in C^2(\mathbb{R})$, $\phi_i' > 0$, $\phi_i'' \geq 0$ and $g_i \in C^1[0, T]$, $g_i' \geq 0$, $g_i(0) = 1$. Let us denote by Ω_s be the intersection of Q with the hyperplane $t = s$, so that

$$Q := \{(x, t) : t \in (0, T), x \in \Omega_t\}, \quad (4.11)$$

and $\partial\Omega_t$ will denote the boundary of the domain Ω_t in $\mathbb{R}^N$.

Let $\overline{Q} = \Omega_0 \times (0, T)$ and define the mapping $\pi : Q \to \overline{Q}$ as:

$$\pi(x, t) = \left(\frac{\phi_1(x_1)}{g_1(t)}, \frac{\phi_2(x_2)}{g_2(t)}, \dots, \frac{\phi_N(x_N)}{g_N(t)}, t\right) = (\xi, t).$$

Then we have that

$$\xi = H(x, t) = \left(\frac{\phi_1(x_1)}{g_1(t)}, \frac{\phi_2(x_2)}{g_2(t)}, \dots, \frac{\phi_N(x_N)}{g_N(t)}\right).$$

It is easy to show that the mapping H satisfies the conditions (A1)-(A4).

The following example also generalizes Example 4.3.

Example 4.5. Let $Q \subset \mathbb{R}^2$ be a non-cylindrical domain defined by:

$$Q = \{(x, t), x \in (-\psi(t), g(t)), t \in (0, 1)\},$$

where $\psi,\ g \in C^1[0, T]$ are nondecreasing functions on $[0, T]$ and $g(0) = \psi(0) = 1$.

Let $\overline{Q} = (-1, 1) \times (0, T)$ and define the mapping $\pi : Q \to \overline{Q}$ as:

$$\pi(x, t) = (H(x, t), t) = (\xi, t),$$

where

$$\xi = H(x, t) = \begin{cases} \dfrac{x}{g(t)}, & x \in [0, g(t)), \\ \dfrac{x}{\psi(t)}, & x \in (-\psi(t), 0). \end{cases}$$

This mapping satisfies the condition (A1) with the constants

$$C_1 = \frac{1}{\max\{g(T), \psi(T)\}}, \quad C_2 = \frac{1}{\min\{g(0), \psi(0)\}} = 1,$$

since

$$|H(x', t) - H(x'', t)| = \begin{cases} \left|\dfrac{x'}{g(t)} - \dfrac{x''}{g(t)}\right| & x', x'' \in [0, g(t)), \\ \left|\dfrac{x'}{g(t)} - \dfrac{x''}{\psi(t)}\right| & x' \in [0, g(t)), x'' \in (-\psi(t), 0), \\ \left|\dfrac{x'}{\psi(t)} - \dfrac{x''}{\psi(t)}\right| & x', x'' \in (-\psi(t), 0). \end{cases}$$

It is easy to show that this mapping also satisfies the conditions (A2)-(A4).

Remark 4.6. In the foregoing examples, only linear operators A have been considered, but obviously, all considerations can be repeated for nonlinear operators which lead to quasilinear forms as:

$$a(u, v)_t = \sum_{|\alpha| \le k} \int_{\Omega_t} a_\alpha(x, t; \delta_k u(x)) \partial^\alpha v(x) dx,$$

where the coefficients $a_\alpha(x, t; \xi)$ satisfy the growth condition

$$|a_\alpha(x, t; \xi)| \le C_\alpha \left(g_\alpha(x) + \sum_{|\beta| \le k} |\xi_\beta|^{p-1} \right)$$

for a.e. $x \in \Omega_T$ and every $\xi \in \mathbb{R}^m$, with C_α a given positive constant and g_α a given function from $L_{p'}(\Omega_T)$, $p' = \dfrac{p}{p-1}$. Then we have to use the corresponding Sobolev space $W^{k,0;p}(Q)$.

Acknowledgement

The second and third authors have been supported by the Research Plan MSN 497775131301 of the Ministery of Education of the Czech Republic.

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