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HOMOTOPY GROUPS OF THE HOMOGENEOUS SPACES F_4/G_2 AND $F_4/\mathrm{Spin}(9)$

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Abstract

In this paper, we calculate 2-primary components of homotopy groups of homogeneous spaces F_4/G_2 and $F_4/\mathrm{Spin}(9)$.

1. Introduction

Let G_2 and F_4 be the compact, connected, simply connected, simple, exceptional Lie groups of rank 2 and 4, respectively. We consider the two homogeneous spaces of F_4 : F_4/G_2 and $F_4/\mathrm{Spin}(9) = \Pi$, where Π

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denotes the Cayley projective plane. The last author has determined $\pi_i(F_4/G_2 : 2)$ and $\pi_i(\Pi)$ for $i \leq 23$ in [7] and the first author has determined $\pi_i(F_4/G_2 : 2)$ for $i \leq 37$ in his master's thesis [3] under the supervision of the second author, where we denote by $\pi_i(X : p)$ the p-primary component of $\pi_i(X)$. In this paper, we calculate homotopy groups $\pi_i(F_4/G_2 : 2)$ and $\pi_i(\Pi : 2)$ for $i \leq 45$ and $i \leq 38$, respectively, by using the results in [2]. Summing up these results, we obtain the following tables.

Theorem 1. *We have the following results on* $\pi_i(F_4/G_2 : 2)$ *for* $i \leq 45$.

i	$i \leq 14$	15	16	17	18	19, 20	21	22
$\pi_i(F_4/G_2 : 2)$	0	∞	2	2	8	0	2	0

23	24	25	26	27, 28	29	30	31
$\infty + 2$	$(2)^2$	2	64	0	$(2)^2$	$128 + 2$	$(2)^4$

32	33	34	35	36	37	38
$(2)^6$	$(8)^2 + 2$	$64 + (2)^2$	8	$(2)^2$	$8 + 4 + (2)^2$	$256 + 8 + (2)^4$

39	40	41	42	43	44	45
$(2)^5$	$(2)^5$	$8 + 4 + 2$	$64 + 2$	$8 + 2$	$(2)^3$	$8 + (2)^4$

Theorem 2. *We have the following results on* $\pi_i(\Pi : 2)$ *for* $i \leq 38$.

i	$i \leq 7$	8	9	10	11	12, 13	14	15
$\pi_i(\Pi : 2)$	0	∞	2	2	8	0	2	8

16	17	18	19	20	21	22	23
$(2)^3$	$(2)^4$	$8+2$	$8+2$	0	2	4	$\infty+8+(2)^2$

24	25	26	27	28	29	30
$(2)^3$	$(2)^4$	$64+2$	$8+2$	8	$(2)^3$	$128+(2)^3$

31	32	33	34	35	36
$(8)^2+(2)^5$	$(2)^8$	$8+4+(2)^3$	$64+(2)^6$	$(8)^2+2$	$(2)^4$

37	38
$16+4+(2)^3$	$256+8+(2)^3$

Here an integer 'n' indicates a cyclic group $\mathbf{Z}_n$ of order n, the symbol '∞' an infinite cyclic group $\mathbf{Z}$, the symbol '+' the direct sum of the group and '$(n)^k$' indicates the direct sum of k-copies of $\mathbf{Z}_n$. These results are stated in Theorems 4.4 and 5.3 respectively, in which we also give their generators.

The calculation will be done by making use of the homotopy exact sequences associated with the 2-local fibration

$$S^{15} \to F_4/G_2 \to S^{23}$$

and the fibration

$$S^7 \to \Omega \prod \to \Omega S^{23}$$

given by Davis-Mahowald [2].

For the case F_4/G_2, from the long exact sequence

$$\cdots \to \pi_{i+1}(S^{23}:2) \xrightarrow{\Delta_{i+1}} \pi_i(S^{15}:2) \xrightarrow{i_*} \pi_i(F_4/G_2:2) \xrightarrow{p_*} \pi_i(S^{23}:2) \xrightarrow{\Delta_i} \cdots$$

we obtain a short exact sequence

$$0 \to \operatorname{Coker} \Delta_{i+1} \to \pi_i(F_4/G_2 : 2) \to \operatorname{Ker} \Delta_i \to 0.$$

Then we determine the group extension by using a formula of Toda brackets in homotopy groups of a fiber space which is proved by Mimura-Toda (Theorem 2.1 of [9]). By virtue of the formula, we can determine the group extension by investigating the corresponding Toda bracket.

For the case $\Omega \prod$, we consider the long exact sequence

$$\cdots \to \pi_{i+1}(\Omega S^{23} : 2) \xrightarrow{\Delta_{i+1}} \pi_i(S^7 : 2) \xrightarrow{i_*} \pi_i(\Omega \prod : 2) \xrightarrow{p_*} \pi_i(\Omega S^{23} : 2) \xrightarrow{\Delta_i} \cdots.$$

From this long exact sequence, we obtain a short exact sequence

$$0 \to \operatorname{Coker} \Delta_{i+1} \to \pi_i(\Omega \prod : 2) \to \operatorname{Ker} \Delta_i \to 0.$$

Then we calculate $\pi_i(\Omega \prod : 2)$ by an argument similar to the case F_4/G_2.

In Section 2, we summarize for later use the results on the 2-primary components of the homotopy groups of S^7, S^{15} and S^{23}. We show in the same section some relations among the elements which are necessary to calculate the boundary homomorphisms Δ_i in the above sequence. In Section 3, we calculate some Toda brackets which are used to determine the group extensions. In Sections 4 and 5, we determine 2-components of homotopy groups of F_4/G_2 and $\prod$, respectively.

The notations and the terminologies in [5], [6], [8], [9], [10], [11], [13], [15] will be freely used in the present paper, and we also omit for simplicity the notation 'o' indicating composition.

The results in the present paper shall be used to deduce $\pi_i(F_4)$ from $\pi_i(G_2)$ and $\pi_i(\mathrm{Spin}(9))$ in the forthcoming paper.

2. Homotopy Groups of Spheres

Let π^n_{n+i} denote the 2-primary component of the homotopy group

$\pi_{n+i}(S^n)$. Toda [15], Mimura-Toda [8], Mimura [6], Mimura-Mori-Oda [10], and Oda [13] have determined π^n_{n+i} for $i \leq 30$. We recall here some necessary results on it from these papers. We denote by G_i the 2-primary component of the i-th stable homotopy groups of spheres.

Table 2.1.

i	21	22	23
π_i^7	$\mathbf{Z}_8 \oplus \mathbf{Z}_4$	$\mathbf{Z}_8 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$	$\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$
generator	$\sigma'\sigma_{14}$, κ_7	ρ'', $\sigma'\bar{\nu}_{14}$, $\sigma'\varepsilon_{14}$, $\bar{\varepsilon}_7$	$\sigma'\mu_{14}$, $E\zeta'$, $\mu_7\sigma_{16}$, $\eta_7\bar{\varepsilon}_8$
π_i^{15}	$\mathbf{Z}_2$	$\mathbf{Z}_{16}$	$\mathbf{Z}_2 \oplus \mathbf{Z}_2$
generator	ν_{15}^2	σ_{15}	$\bar{\nu}_{15}$, ε_{15}
π_i^{23}	0	0	$\mathbf{Z}$
generator			ι_{23}

24	25	26	27	28
$\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$	$\mathbf{Z}_8 \oplus \mathbf{Z}_2$	$\mathbf{Z}_8 \oplus \mathbf{Z}_2$	$\mathbf{Z}_8$	$\mathbf{Z}_2 \oplus \mathbf{Z}_2$
$\sigma'\eta_{14}\mu_{15}$, $\nu_7\kappa_{10}$, $\bar{\mu}_7$, $\eta_7\mu_8\sigma_{17}$	$\zeta_7\sigma_{18}$, $\eta_7\bar{\mu}_8$	$\bar{\zeta}_7$, $\bar{\sigma}_7$	$\bar{\kappa}_7$	$\eta_7\bar{\kappa}_8$, $\sigma'\kappa_{14}$
$\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_8$	0	0
ν_{15}^3, $\eta_{15}\varepsilon_{16}$, μ_{15}	$\eta_{15}\mu_{16}$	ζ_{15}		
$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_8$	0	0
η_{23}	η_{23}^2	ν_{23}		

29	30
$\mathbf{Z}_8 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$	$\mathbf{Z}_8 \oplus \mathbf{Z}_8 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$
$\sigma'\rho_{14},\ \bar{\nu}_7\kappa_{15},\ \varepsilon_7\kappa_{15},\ \nu_7\bar{\sigma}_{10}$	$\nu_7\bar{\kappa}_{10}, \bar{\rho}', \phi_7, \bar{\kappa}_7\nu_{27} - \nu_7\bar{\kappa}_{10}, \sigma'\sigma_{14}\mu_{21}, \sigma'\omega_{14}$
$\mathbf{Z}_4 \oplus \mathbf{Z}_2$	$\mathbf{Z}_{32} \oplus \mathbf{Z}_2$
$\sigma_{15}^2,\ \kappa_{15}$	$\rho_{15},\ \bar{\varepsilon}_{15}$
$\mathbf{Z}_2$	$\mathbf{Z}_{16}$
ν_{23}^2	σ_{23}

31	32
$\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$	$(\mathbf{Z}_4 \oplus \mathbf{Z}_8) \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$
$\delta_7, \bar{\mu}_7\sigma_{24}, \nu_7\sigma_{10}\kappa_{17}, \bar{\zeta}'_7, \bar{\sigma}'_7, \sigma'\bar{\mu}_{14}, \sigma'\omega_{14}\eta_{30}$	$\{\phi''', E^2\phi''\}, \sigma'\eta_{14}\bar{\mu}_{15}, \mu_{3,7}, \eta_7\bar{\mu}_8\sigma_{25}$
$\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$	$\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$
$\eta^{*\prime},\ \omega_{15},\ \sigma_{15}\mu_{22}$	$\sigma_{15}\eta_{22}\mu_{23},\ \eta^{*\prime}\eta_{31},\ \varepsilon^*_{15},\ \nu_{15}\kappa_{18},\ \bar{\mu}_{15}$
$\mathbf{Z}_2 \oplus \mathbf{Z}_2$	$\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$
$\bar{\nu}_{23},\ \varepsilon_{23}$	$\nu_{23}^3,\ \varepsilon_{23}\eta_{31},\ \mu_{23}$

33	34
$\mathbf{Z}_8 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$	$\mathbf{Z}_8 \oplus \mathbf{Z}_8 \oplus \mathbf{Z}_2$
$\bar{\zeta}_7\sigma_{26},\ \sigma'\omega_{14}\nu_{30},\ \bar{\kappa}_7\nu_{27}^2,\ \bar{\sigma}_7\sigma_{26},\ \phi_7\nu_{30},\ \nu_7^2\bar{\kappa}_{13},\ \eta_7\mu_{3,8}$	$\sigma'\bar{\kappa}_{14},\ \zeta_{3,7},\ \bar{\nu}_7\bar{\sigma}_{15}$
$\mathbf{Z}_8 \oplus \mathbf{Z}_8 \oplus \mathbf{Z}_2$	$\mathbf{Z}_8 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$
$\xi_{15},\ E^2\lambda,\ \eta_{15}\bar{\mu}_{16}$	$\bar{\zeta}_{15},\ \omega_{15}\nu_{31},\ \bar{\sigma}_{15}$
$\mathbf{Z}_2$	$\mathbf{Z}_8$
$\eta_{23}\mu_{24}$	ζ_{23}

35	36
$\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$	$\mathbf{Z}_8 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$
$\phi'''\nu_{32}$, $\sigma'\eta_{14}\bar{\kappa}_{15}$, $\bar{\nu}_7\bar{\kappa}_{15}$, $\varepsilon_7\bar{\kappa}_{15}$	δ'', $\sigma'\varepsilon_{14}\kappa_{22}$, $\sigma'\omega_{14}\nu_{30}^2$, $\phi_7\nu_{30}^2$, $\eta_7\varepsilon_8\bar{\kappa}_{16}$
$\mathbf{Z}_8$	$\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$
$\bar{\kappa}_{15}$	$\eta_{15}\bar{\kappa}_{16}$, σ_{15}^3, $E^2\lambda\nu_{33}$
0	0

37	38
$\mathbf{Z}_8 \oplus \mathbf{Z}_8 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$	$\mathbf{Z}_8 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$
$\sigma'\bar{\rho}_{14}$, $\nu_7\sigma_{10}\bar{\kappa}_{17}$, $\sigma'\phi_{14}$, $\sigma'\psi_{14}$, $\phi_7\sigma_{30}$	α_3''', $\sigma'\bar{\sigma}_{14}'$, $\sigma'\bar{\mu}_{14}\sigma_{31}$, $\bar{\nu}_7\nu_{15}\bar{\kappa}_{18}$, $\delta_7\sigma_{31}$
$\mathbf{Z}_{16} \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$	$\mathbf{Z}_{16} \oplus \mathbf{Z}_8 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$
$\sigma^{*\prime}$, $\omega_{15}\nu_{31}^2$, $\varepsilon_{15}\kappa_{23}$, $\nu_{15}\bar{\sigma}_{18}$	$\bar{\rho}_{15}$, $\nu_{15}\bar{\kappa}_{18}$, ϕ_{15}, ψ_{15}, $\bar{\varepsilon}^{*\prime}$, $\bar{\nu}^{*\prime}$
$\mathbf{Z}_2 \oplus \mathbf{Z}_2$	$\mathbf{Z}_{32} \oplus \mathbf{Z}_2$
σ_{23}^2, κ_{23}	ρ_{23}, $\bar{\varepsilon}_{23}$

39	40
$\mathbf{Z}_8 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$	
$E^2\phi''\sigma_{32}$, $\sigma'\mu_{3,14}$, $\sigma'\eta_{14}\sigma_{15}\bar{\mu}_{22}$, $\mu_{3,7}\sigma_{32}$	
$\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$	$\mathbf{Z}_8 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$
δ_{15}, $\sigma_{15}\bar{\mu}_{22}$, $\bar{\sigma}_{15}'$, $\tilde{\varepsilon}_{15}$, $E\zeta^*$, $\mu^{*\prime}$	$\xi_{15}\sigma_{33}$, $\sigma_{15}\eta_{22}\bar{\mu}_{23}$, $\mu_{3,15}$, $D_1\mu_{31}$
$\mathbf{Z}_2 \oplus \mathbf{Z}_2$	$\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$
ω_{23}, $\sigma_{23}\mu_{30}$	$\sigma_{23}\mu_{30}\eta_{39}$, $\nu_{23}\kappa_{26}$, $\bar{\mu}_{23}$, ε_{23}^*

41	42	43	44
$\mathbf{Z}_8 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$	$\mathbf{Z}_8$	$\mathbf{Z}_2$	$\mathbf{Z}_2 \oplus \mathbf{Z}_2$
$E^3\tau^{IV}$, $\nu_{15}^2\bar{\kappa}_{21}$, $\eta_{15}\mu_{3,16}$	$\zeta_{3,15}$	$\varepsilon_{15}\bar{\kappa}_{23}$	$\sigma_{15}\sigma_{22}^*$, L_1
$\mathbf{Z}_8 \oplus \mathbf{Z}_2$	$\mathbf{Z}_8 \oplus \mathbf{Z}_2$	$\mathbf{Z}_8$	$\mathbf{Z}_2 \oplus \mathbf{Z}_2$
ν_{23}^*, $\eta_{23}\bar{\mu}_{24}$	$\bar{\zeta}_{23}$, $\bar{\sigma}_{23}$	$\bar{\kappa}_{23}$	$\bar{\kappa}_{23}\eta_{43}$, σ_{23}^3

45	46
$\mathbf{Z}_{64} \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$	$\mathbf{Z}_{64} \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$
θ^V, $\omega_{15}\kappa_{31}$, $\psi_{15}\sigma_{38}$	$\rho_{3,15}$, D_1^{II}, $D_1^{(1)}\sigma_{39}$, $\kappa^{*\prime}$, ω_{15}^*, κ_{15}^*
$\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$	$\mathbf{Z}_{16} \oplus \mathbf{Z}_8 \oplus \mathbf{Z}_2$
σ_{23}^*, $\varepsilon_{23}\kappa_{31}$, $\nu_{23}\bar{\sigma}_{26}$	$\bar{\rho}_{23}$, $\nu_{23}\bar{\kappa}_{26}$, ϕ_{23}

Table 2.2.

(1) $\sigma_n\eta_{n+7} = \bar{\nu}_n + \varepsilon_n$ for $n \geq 10$ by Lemma 6.4 of [15].

(2) $\bar{\nu}_n\eta_{n+8} = \nu_n^3$ for $n \geq 6$ by Lemma 6.3 of [15].

(3) $\sigma_n\nu_{n+7} = 0$ for $n \geq 12$ by (7.20) of [15].

(4) $\sigma_n\varepsilon_{n+7} = 0$ for $n \geq 11$ by Lemma 10.7 of [15].

(5) $\sigma_n\bar{\nu}_{n+7} = 0$ for $n \geq 11$ by (10.18) of [15].

(6) $\sigma_n\zeta_{n+7} = 0$ for $n \geq 13$ by (12.23) of [15].

(7) $\sigma_n \kappa_{n+7} = 0$ for $n \geq 15$ by Proposition 7.2 of [6].

(8) $\sigma_n \bar{\varepsilon}_{n+7} = 0$ for $n \geq 10$ by p. 317 of [6].

Table 2.3.

(1) $\sigma_n \omega_{n+7} \equiv \phi_n \bmod\{4\nu_n \bar{\kappa}_{n+3}\}$ for $n \geq 11$ by Part I, Proposition 3.4.(4) of [13].

(2) $\sigma_n^2 \mu_{n+7} = 0$ for $n \geq 10$ by (2.3) of [10].

(3) $\sigma_n \bar{\zeta}_{n+7} = 0$ for $n \geq 13$ by Part I, Proposition 3.1(9) of [13].

(4) $\sigma_n \bar{\sigma}_{n+7} = 0$ for $n \geq 14$ by Part I, Proposition 6.4(10) of [13].

(5) $\sigma_n^3 = \nu_n \xi_{n+3}$ for $n \geq 9$ by Part II, Proposition 2.1(2) of [13].

(6) $\sigma_n \phi_{n+7} = 0$ for $n \geq 11$ by Part III, Proposition 2.5(6) of [13].

(7) $\sigma_n \psi_{n+7} = 0$ for $n \geq 13$ by Part III, Proposition 2.5(6) of [13].

Table 2.4.

(1) $\sigma' \sigma_{14}^2 = 0$ by Part I, Proposition 3.1(7) of [13].

(2) $\sigma' \sigma_{14} \rho_{21} = 0$ by Part III, Proposition 2.4(4) of [13].

We prepare some lemmas which will be needed later.

Lemma 2.5. *We have the following relations*:

(1) $\sigma_{15} \rho_{22} = 8\sigma^{*\prime}$.

(2) $\sigma_{15} \varepsilon_{22}^* \equiv \delta_{15} \bmod\{\bar{\mu}_{15} \sigma_{32}\}$.

(3) $\sigma_{15} \nu_{22}^* \equiv 2\xi_{15} \sigma_{33} \bmod\{2\sigma_{15} \nu_{22}^*\}$.

(4) $\sigma_{15} \bar{\kappa}_{22} = 0$.

(5) $\sigma_{12}^4 = 0$.

(6) $\sigma_{15} \bar{\rho}_{22} \equiv 8x\theta^{\mathrm{V}} \bmod\{16\theta^{\mathrm{V}}\}$ *for some odd integer x.*

Proof. (1) By Part II, Proposition 2.1(4) of [13], we have

$$2\sigma_{15}\rho_{22} = 0, \quad 4\rho_{15}\sigma_{30} = 0 \quad \text{and} \quad \sigma_{15}\rho_{22} = 2\rho_{15}\sigma_{30}.$$

Now, from Lemma 6.2(2) of [5], we have

$$2\rho_{15}\sigma_{30} \equiv 4E\sigma^{*\prime\prime} \bmod\{2\sigma_{15}\rho_{22}, 4\rho_{15}\sigma_{30}\},$$

where $\{2\sigma_{15}\rho_{22}, 4\rho_{15}\sigma_{30}\} = 0$ by the above observation. Also, from Lemma 6.2(3) of [5], we have

$$4E\sigma^{*\prime\prime} \equiv 8\sigma^{*\prime} \bmod\{4\sigma_{15}\rho_{22}, 4\rho_{15}\sigma_{30}\},$$

where $\{4\sigma_{15}\rho_{22}, 4\rho_{15}\sigma_{30}\} = 0$ by the above observation. Thus we obtain the relation $\sigma_{15}\rho_{22} = 8\sigma^{*\prime}$.

(2) By Lemma 12.15(ii) of [15], we have

$$\sigma_{15}\varepsilon^*_{22} = \sigma_{15}\omega_{22}\eta_{38}.$$

Here, by Table 2.3(1), we have

$$\sigma_{15}\omega_{22}\eta_{38} \equiv \phi_{15}\eta_{38} \bmod\{4\nu_{15}\overline{\kappa}_{18}\eta_{38}\},$$

where $\{4\nu_{15}\overline{\kappa}_{18}\eta_{38}\} = 0$ by the fact $2\eta_{38} = 0$. That is $\sigma_{15}\omega_{22}\eta_{38} = \phi_{15}\eta_{38}$. By Part I, Proposition 3.5(9) of [13], we have

$$\phi_{15}\eta_{38} \equiv \delta_{15} \bmod\{\overline{\mu}_{15}\sigma_{32}, \nu_{15}\eta_{18}\overline{\kappa}_{19}\}.$$

Since $\nu_{15}\eta_{18} = 0$, by (5.9) of [15], the above relation reduces to

$$\phi_{15}\eta_{38} \equiv \delta_{15} \bmod\{\overline{\mu}_{15}\sigma_{32}\}.$$

Thus we obtain the relation $\sigma_{15}\varepsilon^*_{22} = \sigma_{15}\omega_{22}\eta_{38} \equiv \delta_{15} \bmod\{\overline{\mu}_{15}\sigma_{32}\}$.

(3) By Part II, Proposition 2.1(6) of [13], we have

$$\sigma_{15}\nu^*_{22} \equiv E^4\xi'\sigma_{33} \bmod\{2\sigma_{15}\nu^*_{22}\}.$$

By Lemma 12.19 of [15], we have

$$E^4\xi'\sigma_{33} = 2\xi_{15}\sigma_{33}.$$

Thus we obtain the relation $\sigma_{15}\nu^*_{22} \equiv 2\xi_{15}\sigma_{33} \bmod\{2\sigma_{15}\nu^*_{22}\}$.

(4) By Proposition 3.1 of [15], we have

$$\sigma_{15}\overline{\kappa}_{22} = \overline{\kappa}_{15}\sigma_{35}.$$

By Part III, Proposition 2.2(5) of [13], we have

$$\overline{\kappa}_{15}\sigma_{35} \equiv 0 \bmod\{2E^8\sigma'\overline{\kappa}_{22},\ \overline{\nu}_{15}\overline{\sigma}_{23}\} \quad \text{and} \quad 2\overline{\kappa}_{15}\sigma_{35} = 0.$$

So, by Lemma 5.14 of [15], we have

$$2E^8\sigma'\overline{\kappa}_{22} = 4\sigma_{15}\overline{\kappa}_{22} = 4\overline{\kappa}_{15}\sigma_{35} = 0.$$

Now by Part II, Proposition 2.2(7) of [13], we have $\overline{\nu}_{15}\overline{\sigma}_{23} = \overline{\sigma}'_{15}\nu_{39}$ and $\nu_{15}\overline{\sigma}'_{18} = 0$. By Proposition 3.1 of [15], we have $\overline{\sigma}'_{15}\nu_{39} = \nu_{15}\overline{\sigma}'_{18}$. Therefore, we have

$$\overline{\nu}_{15}\overline{\sigma}_{23} = 0.$$

Thus we obtain the relation $\sigma_{15}\overline{\kappa}_{22} = 0$.

(5) By Table 2.3(5) and Table 2.2(3), we have

$$\sigma_{12}^4 = \sigma_{12}\nu_{19}\xi_{22} = 0.$$

(6) By Part III, Proposition 2.5(2) of [13], we have $\sigma_{15}\overline{\rho}_{22} \equiv 2x\rho_{15}^2 \bmod\{\sigma_{15}\phi_{22},\ \sigma_{15}\psi_{22}\}$, where x is an odd integer. So, by (6) and (7) of Table 2.3, we have

$$\sigma_{15}\overline{\rho}_{22} = 2x\rho_{15}^2.$$

Moreover, by Part III, (8.22) of [13], we have

$$2x\rho_{15}^2 \equiv 8x\theta^{\mathrm{V}} \bmod\{16\theta^{\mathrm{V}}\}.$$

Thus we obtain the relation $\sigma_{15}\overline{\rho}_{22} \equiv 8x\theta^{\mathrm{V}} \bmod\{16\theta^{\mathrm{V}}\}$.

Lemma 2.6. *We have the following relations*:

(1) $\sigma'\sigma_{14}\eta_{21}^2 = \eta_7\bar{\varepsilon}_8 + E\zeta'$.

(2) $\sigma'\sigma_{14}\eta_{21}\mu_{22} = \bar{\zeta}'_7$.

(3) $\sigma'\sigma_{14}\kappa_{21} = 0$.

(4) $\sigma'\sigma_{14}\omega_{21} \equiv \sigma'\phi_{14} \bmod\{4\nu_7\sigma_{10}\bar{\kappa}_{17}\}$.

Proof. (1) By Table 2.2(1), we have

$$\sigma'\sigma_{14}\eta_{21}^2 = \sigma'\bar{\nu}_{14}\eta_{22} + \sigma'\varepsilon_{14}\eta_{22}.$$

Here, we have

$$\sigma'\bar{\nu}_{14}\eta_{22} = \eta_7\bar{\varepsilon}_8$$

by Proposition 2.1(2) of [14] and also

$$\sigma'\varepsilon_{14}\eta_{22} = \sigma'\eta_{14}\varepsilon_{15} = E\zeta'$$

by Proposition 3.1 of [15] and (12.4) of [15]. Thus we obtain the relation $\sigma'\sigma_{14}\eta_{21}^2 = \eta_7\bar{\varepsilon}_8 + E\zeta'$.

(2) By Table 2.2(1), we have

$$\sigma'\sigma_{14}\eta_{21}\mu_{22} = \sigma'\bar{\nu}_{14}\mu_{22} + \sigma'\varepsilon_{14}\mu_{22}.$$

By Theorem 14.1(iv) of [15], we have $\bar{\nu}\mu = 0$. Since $E^\infty : \pi_{31}^{14} \to G_{17}$ is an isomorphism, we have

$$\bar{\nu}_{14}\mu_{22} = 0.$$

By (5.10) of [10], we have

$$\sigma'\varepsilon_{14}\mu_{22} = \bar{\zeta}'_7.$$

Thus we obtain the relation $\sigma'\sigma_{14}\eta_{21}\mu_{22} = \bar{\zeta}'_7$.

(3) By Proposition 7.2 of [6], we have

$$E\sigma'\sigma_{15}\kappa_{22} = 0.$$

Since $E : \pi^7_{36} \to \pi^8_{37}$ is a monomorphism, we obtain the relation $\sigma'\sigma_{14}\kappa_{21} = 0$.

(4) By Part I, Proposition 3.4(4) of [13], we have

$$\sigma'\sigma_{14}\omega_{21} \equiv \sigma'\phi_{14} \bmod\{4\sigma'\nu_{14}\overline{\kappa}_{17}\}.$$

By (7.19) of [15] and by the relation $8\nu_7\sigma_{10}\overline{\kappa}_{17} = 0$, we have

$$4\sigma'\nu_{14}\overline{\kappa}_{17} = 4\nu_7\sigma_{10}\overline{\kappa}_{17}.$$

Thus we obtain the relation $\sigma'\sigma_{14}\omega_{21} \equiv \sigma'\phi_{14} \bmod\{4\nu_7\sigma_{10}\overline{\kappa}_{17}\}$.

3. Determination of Toda Brackets

In this section, we calculate some Toda brackets which will be used later to determine the group extension. Throughout this section, we work in the 2-primary components of the homotopy groups of spheres. We use freely the results on the order of an element which are given by [6], [10], [13] and [15].

Lemma 3.1. *We have the following relations*:

(1) $\{\sigma_{13}, \nu_{20}, 8\iota_{23}\} = x\zeta_{13}$ *for some odd integer x.*

(2) $\{\sigma_{15}, 4\sigma_{22}, 4\iota_{29}\} \ni \rho_{15}$.

(3) $\{\sigma_{11}, \varepsilon_{18}, 2\iota_{26}\} \ni 0$.

(4) $\{\sigma_{14}, \zeta_{21}, 8\iota_{32}\} = x\overline{\zeta}_{14} + y\omega_{14}\nu_{30}$

$\{\sigma_{15}, \zeta_{22}, 8\iota_{33}\} = -x\overline{\zeta}_{15}$ *for some odd integer x and even integer y.*

(5) $\{\sigma_{15}, \kappa_{22}, 2\iota_{36}\} \ni \nu_{15}\overline{\sigma}_{18} + x\omega_{15}\nu^2_{31}$ *for some integer x.*

(6) $\{\sigma_{14}, \overline{\varepsilon}_{21}, 2\iota_{36}\} \ni 0$.

Proof. (1) We can define a Toda bracket $\{\sigma_{13}, \nu_{20}, 8\iota_{23}\}$ by Table 2.2(3) and the fact $8\nu_{20} = 0$. By (5.5) of [15] and Lemma 9.1 of [15], we have a relation on the stable Toda bracket

$$\langle \sigma, \nu, 8\iota \rangle \subset \langle \sigma, 4\nu, 2\iota \rangle = \langle \sigma, \eta^3, 2\iota \rangle = \zeta + 2G_{11}.$$

Since $G_4 = 0$ and $G_{11} \cong \mathbf{Z}_8$, the indeterminacy of the stable Toda bracket $\langle \sigma, \nu, 8\iota \rangle$ is trivial. So we may put

$$\langle \sigma, \nu, 8\iota \rangle = x\zeta$$

for some odd integer x. Since $E^\infty : \pi_{24}^{13} \to G_{11}$ is an isomorphism, we have

$$\{\sigma_{13}, \nu_{20}, 8\iota_{23}\} = x\zeta_{13}.$$

(2) By Lemma 10.9 of [15], we have

$$\langle \sigma, 4\sigma, 4\iota \rangle \supset \langle \sigma, 2\sigma, 8\iota \rangle \ni \rho.$$

Since $E^\infty : \pi_{30}^{15} \to G_{15}$ is an isomorphism, we have

$$\{\sigma_{15}, 4\sigma_{22}, 4\iota_{29}\} \ni \rho_{15}.$$

(3) We can define a Toda bracket $\{\sigma_{11}, \varepsilon_{18}, 2\iota_{26}\}$ by Table 2.2(4) and the fact that $2\varepsilon_{18} = 0$. By (2) and (5) of Table 2.2, we have

$$\sigma_{11}\nu_{18}^3 = \sigma_{11}\bar{\nu}_{18}\eta_{26} = 0.$$

By Table 2.2(4), we have

$$\sigma_{11}\varepsilon_{18}\eta_{26} = 0.$$

Therefore the indeterminacy of $\{\sigma_{11}, \varepsilon_{18}, 2\iota_{26}\}$ is given by

$$\sigma_{11}\pi_{27}^{18} + 2\pi_{27}^{11} = \{\sigma_{11}\nu_{18}^3, \sigma_{11}\varepsilon_{18}\eta_{26}, \sigma_{11}\mu_{18}\} = \{\sigma_{11}\mu_{18}\}.$$

Since $\pi_{27}^{11} = \{\sigma_{11}\mu_{18}\} \cong \mathbf{Z}_2$, we have $\{\sigma_{11}, \varepsilon_{18}, 2\iota_{26}\} \ni 0$.

(4) We can define a Toda bracket $\{\sigma_{14}, \zeta_{21}, 8\iota_{32}\}_1$ by Table 2.2(6) and the fact that $8\zeta_{20} = 0$. By Part I, Proposition 3.3(2) of [13], we have

$$\langle \sigma, \zeta, 8\iota \rangle \subset \langle \sigma, 4\zeta, 2\iota \rangle = \bar{\zeta} + 2G_{19}.$$

Since $G_{12} = 0$ and $G_{19} \cong \mathbf{Z}_8 \oplus \mathbf{Z}_2$, the indeterminacy of the stable Toda bracket $\langle \sigma, \zeta, 8\iota \rangle$ is trivial. Thus we may put

$$\langle \sigma, \zeta, 8\iota \rangle = x\overline{\zeta}$$

for some odd integer x. We compare $\pi_{33}^{14} = \{\overline{\zeta}_{14}, \omega_{14}\nu_{30}, \overline{\sigma}_{14}\} \cong \mathbf{Z}_8 \oplus \mathbf{Z}_4 \oplus \mathbf{Z}_2$ with $G_{19} = \{\overline{\zeta}, \overline{\sigma}\} \cong \mathbf{Z}_8 \oplus \mathbf{Z}_2$. By the fact that $P(\nu_{35}) = \omega_{17}\nu_{33}$ (p. 170 of [15]), we have

$$\omega_{18}\nu_{34} = 0,$$

where $P : \pi_{38}^{35} \to \pi_{36}^{17}$ is the boundary homomorphism of the EHP-exact sequence. So we may put

$$\{\sigma_{14}, \zeta_{21}, 8\iota_{32}\}_1 \ni -x\overline{\zeta}_{14} + y\omega_{14}\nu_{30}$$

for some integer y. We consider the generalized Hopf homomorphism $H : \pi_{33}^{14} \to \pi_{33}^{27}$. We have

$$\begin{aligned} H(\{\sigma_{14}, \zeta_{21}, 8\iota_{32}\}_1) &\subset \{H(\sigma_{14}), \zeta_{21}, 8\iota_{32}\}_1 \\ &= \{0, \zeta_{21}, 8\iota_{32}\}_1 \\ &\equiv 0 \bmod 8\pi_{33}^{27} = 0. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} H(x\overline{\zeta}_{14} + y\omega_{14}\nu_{30}) &= yH(\omega_{14})\nu_{30} \\ &= y\nu_{27}^2 \qquad \text{by Lemma 12.15 of [15].} \end{aligned}$$

Since ν_{27}^2 is of order two, we see that y is an even integer. Since $\pi_{33}^{21} = 0$, the indeterminacy of $\{\sigma_{14}, \zeta_{21}, 8\iota_{32}\}$ is $\sigma_{14}\pi_{33}^{21} + 8\pi_{33}^{14} = 0$. So we have

$$\{\sigma_{14}, \zeta_{21}, 8\iota_{32}\} = \{\sigma_{14}, \zeta_{21}, 8\iota_{32}\}_1 = -x\overline{\zeta}_{14} + y\omega_{14}\nu_{30}.$$

Moreover, we have

$$\{\sigma_{15}, \zeta_{22}, 8\iota_{33}\} \supset -E\{\sigma_{14}, \zeta_{21}, 8\iota_{32}\} = x\overline{\zeta}_{15} - y\omega_{15}\nu_{31}.$$

Since $\omega_{15}\nu_{31}$ is of order two, we have

$$\{\sigma_{15}, \zeta_{22}, 8\iota_{33}\} \ni x\overline{\zeta}_{15}.$$

Since $\pi_{34}^{22} = 0$ and $\pi_{34}^{15} \cong \mathbf{Z}_8 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$, the indeterminacy of $\{\sigma_{15}, \zeta_{22}, 8\iota_{33}\}$ is trivial. Thus we obtain the second formula.

(5) We can define a Toda bracket $\{\sigma_{15}, \kappa_{22}, 2\iota_{36}\}_1$ by Table 2.2(7) and the fact that $2\kappa_{21} = 0$. Since $E : \pi_{36}^{21} \to \pi_{37}^{22}$ is an isomorphism, we have $\sigma_{15}\pi_{37}^{22} = \sigma_{15}E\pi_{36}^{21}$. So we have

$$\{\sigma_{15}, \kappa_{22}, 2\iota_{36}\} = \{\sigma_{15}, \kappa_{22}, 2\iota_{36}\}_1.$$

By Theorem 2.1(viii) of [11], we have

$$\langle\sigma, \kappa, 2\iota\rangle = \nu\overline{\sigma}.$$

We compare $\pi_{37}^{15} = \{\sigma^{*\prime}, \omega_{15}\nu_{31}^2, \varepsilon_{15}\kappa_{23}, \nu_{15}\overline{\sigma}_{18}\} \cong \mathbf{Z}_{16} \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$ with $G_{22} = \{\varepsilon\kappa, \nu\overline{\sigma}\} \cong \mathbf{Z}_2 \oplus \mathbf{Z}_2$. By Lemma 6.2(4) of [5], we have

$$E^8\sigma^{*\prime} \equiv 2\sigma_{23}^* \bmod \sigma_{23}E^8\pi_{37}^{22} + E^7\pi_{31}^{16}\sigma_{38}.$$

By Lemma 8.3 of [6], we have

$$2\sigma_{23}^* = 0.$$

By Part II, Proposition 2.1(4) of [13], we have $\sigma_{23}\rho_{30} = \rho_{23}\sigma_{38} = 0$. By Proposition 3.1 of [15] and Table 2.2(8), we have $\overline{\varepsilon}_{23}\sigma_{38} = \sigma_{23}\overline{\varepsilon}_{30} = 0$. So we have

$$\sigma_{23}E^8\pi_{37}^{22} + E^7\pi_{31}^{16}\sigma_{38} = \{\sigma_{23}\rho_{30}, \sigma_{23}\overline{\varepsilon}_{30}, \rho_{23}\sigma_{38}, \overline{\varepsilon}_{23}\sigma_{38}\} = 0.$$

Therefore, we have

$$E^8\sigma^{*\prime} = 0.$$

By the argument in the proof of Lemma 3.1(4), we have $\omega_{18}\nu_{34} = 0$. So we have

$$\omega_{18}\nu_{34}^2 = 0.$$

Therefore, we have

$$\{\sigma_{15}, \kappa_{22}, 2\iota_{36}\}_1 \ni \nu_{15}\overline{\sigma}_{18} + x\omega_{15}\nu_{31}^2 + y\sigma^{*\prime}$$

for some integers x and y. We consider the generalized Hopf homomorphism $H : \pi_{37}^{15} \to \pi_{37}^{29}$ for which we have

$$\begin{aligned} H(\{\sigma_{15}, \kappa_{22}, 2\iota_{36}\}_1) &\subset \{H(\sigma_{15}), \kappa_{22}, 2\iota_{36}\}_1 \\ &= \{0, \kappa_{22}, 2\iota_{36}\}_1 \\ &\equiv 0 \bmod 2\pi_{37}^{29} = 0. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} H(\nu_{15}\overline{\sigma}_{18} + x\omega_{15}\nu_{31}^2 + y\sigma^{*\prime}) &= yH(\sigma^{*\prime}) \\ &= y(\overline{\nu}_{29} + \varepsilon_{29}) \text{ by Lemma 6.2(3) of [5].} \end{aligned}$$

Since $\overline{\nu}_{29} + \varepsilon_{29}$ is of order two, we see that y is an even integer. By Lemma 2.5(1) and Table 2.2(8), the indeterminacy of $\{\sigma_{15}, \kappa_{22}, 2\iota_{36}\}$ is given by

$$\sigma_{15}\pi_{37}^{22} + 2\pi_{37}^{15} = \{\sigma_{15}\rho_{22}, \sigma_{15}\overline{\varepsilon}_{22}, 2\sigma^{*\prime}\} = \{2\sigma^{*\prime}\}.$$

Thus we have $\{\sigma_{15}, \kappa_{22}, 2\iota_{36}\} \ni \nu_{15}\overline{\sigma}_{18} + x\omega_{15}\nu_{31}^2$.

(6) We can define a Toda bracket $\{\sigma_{14}, \overline{\varepsilon}_{21}, 2\iota_{36}\}$ by Table 2.2(8) and the fact that $2\overline{\varepsilon}_{21} = 0$. We have

$$\begin{aligned} \{\sigma_{14}, \overline{\varepsilon}_{21}, 2\iota_{36}\} &= \{\sigma_{14}, \kappa_{21}\eta_{35}, 2\iota_{36}\} \text{ by (10.23) of [15]} \\ &\supset \{\sigma_{14}\kappa_{21}, \eta_{35}, 2\iota_{36}\} \\ &= \{2E\lambda\nu_{32}, \eta_{35}, 2\iota_{36}\} \text{ by Proposition 7.2 of [6]} \\ &\supset 2E\lambda\{\nu_{32}, \eta_{35}, 2\iota_{36}\}. \end{aligned}$$

Since $\{\nu_{32}, \eta_{35}, 2\iota_{36}\} \subset \pi_{37}^{32} = 0$, we have $\{\sigma_{14}, \overline{\varepsilon}_{21}, 2\iota_{36}\} \ni 0$.

Lemma 3.2. *We have the following relations*:

(1) $\{\sigma_{15}, 2\rho_{22}, 16\iota_{37}\} \ni x\overline{\rho}_{15} + 4a\nu_{15}\overline{\kappa}_{18} + b\psi_{15} + c\phi_{15}$ *for some odd integer x and integers* a, b, c.

(2) $\{\sigma_{15}, \sigma_{22}\mu_{29}, 2\iota_{38}\} \ni 0$.

(3) $\{\sigma_{15}, 4\nu^*_{22}, 2\iota_{40}\} \ni \nu^2_{15}\overline{\kappa}_{21}$.

(4) $\{\sigma_{15}, \overline{\zeta}_{22}, 8\iota_{41}\} = x\zeta_{3,15}$ *for some odd integer x.*

(5) $\{\sigma_{15}, \overline{\kappa}_{22}, 8\iota_{42}\} = 0$.

(6) $\{\sigma_{15}, \sigma^3_{22}, 2\iota_{43}\} \ni 0$.

Proof. (1) By Lemma 2.5(1), we have $2\sigma_{15}\rho_{22} = 16\sigma^{*\prime} = 0$. Since $32\rho_{21} = 0$, we can define $\{\sigma_{15}, 2\rho_{22}, 16\iota_{37}\}_1$. By Part III, Proposition 2.3(6) of [13], we have

$$\langle\sigma, 2\rho, 16\iota\rangle \subset \langle\sigma, 4\rho, 8\iota\rangle \ni x\overline{\rho} + 4y\nu\overline{\kappa}$$

for some odd integer x and integer y. By (2.3) of [10], we have $\sigma^2\mu = 0$. By Part I, Proposition 3.4(4) of [13], we have $\sigma\omega \equiv \phi \bmod\{4\nu\overline{\kappa}\}$. So we have $\sigma G_{16} = \{\sigma\omega, \sigma^2\mu\} = \{\phi + 4z\nu\overline{\kappa}\}$ where $z = 0$ or 1. Therefore, the indeterminacy of $\langle\sigma, 2\rho, 16\iota\rangle$ is $\{\phi + 4z\nu\overline{\kappa}\}$ and the indeterminacy of $\langle\sigma, 4\rho, 8\iota\rangle$ is $\{\phi + 4z\nu\overline{\kappa}, 8\overline{\rho}\}$. So, for some odd integer x', we have

$$\langle\sigma, 2\rho, 16\iota\rangle \ni x'\overline{\rho} + 4y\nu\overline{\kappa}.$$

We compare $\pi^{15}_{38} = \{\overline{\rho}_{15}, \nu_{15}\overline{\kappa}_{18}, \phi_{15}, \psi_{15}, \overline{\varepsilon}^{*\prime}, \overline{\nu}^{*\prime}\} \cong \mathbf{Z}_{16} \oplus \mathbf{Z}_8 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$ with $G_{23} = \{\overline{\rho}, \nu\overline{\kappa}, \phi\} \cong \mathbf{Z}_{16} \oplus \mathbf{Z}_8 \oplus \mathbf{Z}_2$. By Part I, Proposition 2.7(2) of [13], we have

$$\psi \equiv 0 \bmod\{4\nu\overline{\kappa}, \phi\}.$$

By (3.4) of [10], we have

$$E^2\overline{\varepsilon}^{*\prime} = E^2\overline{\nu}^{*\prime} = 0.$$

So, for some integers $a, b, c, d, e,$ we may put

$$\{\sigma_{15}, 2\rho_{22}, 16\iota_{37}\}_1 \ni x'\overline{\rho}_{15} + 4a\nu_{15}\overline{\kappa}_{18} + b\phi_{15} + c\psi_{15} + d\overline{\varepsilon}^{*\prime} + e\overline{\nu}^{*\prime}.$$

We consider the generalized Hopf homomorphism $H : \pi_{38}^{15} \to \pi_{38}^{29}$. We have

$$\begin{aligned} H(\{\sigma_{15}, 2\rho_{22}, 16\iota_{37}\}_1) &\subset \{H(\sigma_{15}), 2\rho_{22}, 16\iota_{37}\}_1 \\ &= \{0, 2\rho_{22}, 16\iota_{37}\}_1 \\ &\equiv 0 \bmod 16\pi_{38}^{29} = 0. \end{aligned}$$

On the other hand, by (3.4) of [10], we have

$$\begin{aligned} H(x'\overline{\rho}_{15} + 4a\nu_{15}\overline{\kappa}_{18} + b\phi_{15} + c\psi_{15} + d\overline{\varepsilon}^{*\prime} + e\overline{\nu}^{*\prime}) &= dH(\overline{\varepsilon}^{*\prime}) + eH(\overline{\nu}^{*\prime}) \\ &= d\eta_{29}\varepsilon_{30} + e\nu_{29}^3. \end{aligned}$$

Since $\eta_{29}\varepsilon_{30}$ and ν_{29}^3 are of order two, we see that d and e are even integers. This leads to the required result.

(2) We can define a Toda bracket $\{\sigma_{15}, \sigma_{22}\mu_{29}, 2\iota_{38}\}$ by Table 2.3(2) and the fact that $2\mu_{29} = 0$. We have

$$\begin{aligned} \{\sigma_{15}, \sigma_{22}\mu_{29}, 2\iota_{38}\} &= \{\sigma_{15}, \rho_{22}\eta_{37}, 2\iota_{38}\} && \text{by Proposition 12.20 of [15]} \\ &\supset \{\sigma_{15}\rho_{22}, \eta_{37}, 2\iota_{38}\} \\ &= \{8\sigma^{*\prime}, \eta_{37}, 2\iota_{38}\} && \text{by Lemma 2.5(1)} \\ &\supset 4\sigma^{*\prime}\{2\iota_{37}, \eta_{37}, 2\iota_{38}\}. \end{aligned}$$

Since $\pi_{39}^{37} = \{\eta_{37}^2\} \cong \mathbf{Z}_2$, we have $4\sigma^{*\prime}\{2\iota_{37}, \eta_{37}, 2\iota_{38}\} = 0$. Thus we have $\{\sigma_{15}, \sigma_{22}\mu_{29}, 2\iota_{38}\} \ni 0$.

(3) We can define a Toda bracket $\{\sigma_{12}, 4\nu_{19}^*, 2\iota_{37}\}_1$ by Part I, Proposition 5.1(3) of [13] and the fact $8\nu_{18}^* = 0$. We consider the generalized Hopf homomorphism $H : \pi_{38}^{12} \to \pi_{38}^{23}$, where

$$\pi_{38}^{12} = \{\tau^{IV}, E\tau''' - 8\tau^{IV}, A_1\kappa_{24}, EC_1\kappa_{24}, \sigma_{12}\overline{\sigma}_{19}, \nu_{12}^2\overline{\kappa}_{18}, \eta_{12}\mu_{3,13}\}$$

$$\cong \mathbf{Z}_{64} \oplus \mathbf{Z}_4 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$$

and

$$\pi_{38}^{23} = \{\rho_{23}, \overline{\varepsilon}_{23}\} \cong \mathbf{Z}_{32} \oplus \mathbf{Z}_2.$$

We have

$$H(\{\sigma_{12}, 4\nu_{19}^*, 2\iota_{37}\}_1) \subset \{H(\sigma_{12}), 4\nu_{19}^*, 2\iota_{37}\}_1$$

$$= \{0, 4\nu_{19}^*, 2\iota_{37}\}_1$$

$$\equiv 0 \bmod 2\pi_{38}^{23} = \{2\rho_{23}\}.$$

By Part I, Proposition 4.2(1) of [13], we have

$$H(\tau^{IV}) \equiv \rho_{23} \bmod\{2\rho_{23}\}.$$

By Part I, Proposition 6.2(5) of [13], we have

$$H(A_1\kappa_{24}) = \overline{\varepsilon}_{23}.$$

Therefore, for some integers x, y, z, a, b, c, we have

$$\{\sigma_{12}, 4\nu_{19}^*, 2\iota_{37}\} \ni 2x\tau^{IV} + y(E\tau''' - 8\tau^{IV})$$

$$+ zEC_1\kappa_{24} + a\sigma_{12}\overline{\sigma}_{19} + b\nu_{12}^2\overline{\kappa}_{18} + c\eta_{12}\mu_{3,13}.$$

By Table 2.3(4), we have

$$\sigma_{15}\overline{\sigma}_{22} = 0.$$

From the fact that $8E^3\tau^{\mathrm{IV}} = 0$, we have

$$E^4\tau''' - 8E^3\tau^{\mathrm{IV}} = E^4\tau'''.$$

By Part I, (8.20) of [13] and the fact that $\sigma_{15}\overline{\sigma}_{22} = 0$, we have

$$E^4\tau''' = 2dE^3\tau^{\mathrm{IV}}$$

for some integer d. From the fact that $\pi_{27}^{15} = 0$, we have

$$E^4C_1\kappa_{27} \in \pi_{27}^{15}\kappa_{27} = 0.$$

Therefore, for some integer x', we have

$$\{\sigma_{15}, 4\nu_{22}^*, 2\iota_{40}\} \ni 2x'E^3\tau^{\mathrm{IV}} + b\nu_{15}^2\overline{\kappa}_{21} + c\eta_{15}\mu_{3,16}.$$

By (3) and (4) of Table 2.3, we have $\sigma_{15}\pi_{41}^{22} + 2\pi_{41}^{15} = \{\sigma_{15}\overline{\zeta}_{22}, \sigma_{15}\overline{\sigma}_{22}, 2E^3\tau^{\mathrm{IV}}\} = \{2E^3\tau^{\mathrm{IV}}\}$ which is the indeterminacy of the Toda bracket $\{\sigma_{15}, 4\nu_{22}^*, 2\iota_{40}\}$. So we may put

$$\{\sigma_{15}, 4\nu_{22}^*, 2\iota_{40}\} \ni b\nu_{15}^2\overline{\kappa}_{21} + c\eta_{15}\mu_{3,16}.$$

By Theorem 1 of [4], we have

$$\langle\sigma, 4\nu^*, 2\iota\rangle = \nu^2\overline{\kappa}.$$

Since $G_{26} = \{\nu^2\overline{\kappa}, \eta\mu_{3,*}\} \cong \mathbf{Z}_2 \oplus \mathbf{Z}_2$, we have $b = 1$ and $c = 0$. Thus we have $\{\sigma_{15}, 4\nu_{22}^*, 2\iota_{40}\} \ni \nu_{15}^2\overline{\kappa}_{21}$.

(4) We apply the formula of Proposition 1.5 of [15]

$$0 \in \{\{\alpha, \beta, \gamma\}, \delta, \varepsilon\} + \{\alpha, \{\beta, \gamma, \delta\}, \varepsilon\} + \{\alpha, \beta, \{\gamma, \delta, \varepsilon\}\}$$

to the case

$$(\alpha, \beta, \gamma, \delta, \varepsilon) = (\sigma_{15}, \zeta_{22}, 8\iota_{33}, 2\sigma_{33}, 8\iota_{40}).$$

Then we have the formula

$$0 \in \{\{\sigma_{15}, \zeta_{22}, 8\iota_{33}\}, 2\sigma_{34}, 8\iota_{41}\} + \{\sigma_{15}, \{\zeta_{22}, 8\iota_{33}, 2\sigma_{33}\}, 8\iota_{41}\}$$
$$+ \{\sigma_{15}, \zeta_{22}, \{8\iota_{33}, 2\sigma_{33}, 8\iota_{40}\}\}.$$

Here we calculate each Toda bracket above. By Lemma 3.1(4), we have

$$\{\{\sigma_{15}, \zeta_{22}, 8\iota_{33}\}, 2\sigma_{34}, 8\iota_{41}\} = x\{\overline{\zeta}_{15}, 2\sigma_{34}, 8\iota_{41}\}$$

for some odd integer x. Then by the definition of $\zeta_{3,5}$ (see Part II, Proposition 3.1(2) of [13]), we have

$$\{\overline{\zeta}_{15}, 2\sigma_{34}, 8\iota_{41}\} \subset \{\overline{\zeta}_{15}, 8\sigma_{34}, 2\iota_{41}\} \supset \{\overline{\zeta}_{15}, 8\iota_{34}, 2\sigma_{34}\} \ni \zeta_{3,15}.$$

By Part II, Proposition 2.2(4) of [13], we have $\overline{\zeta}_{15}\overline{\nu}_{34} = 0$. By Proposition 3.1 of [15] and Part II, Proposition 2.2(5) of [13], we have $\overline{\zeta}_{15}\varepsilon_{34} = \varepsilon_{15}\overline{\zeta}_{23} = 0$. So the indeterminacy of $\{\overline{\zeta}_{15}, 2\sigma_{34}, 8\iota_{41}\}$ is given by

$$\overline{\zeta}_{15}\pi_{42}^{34} + 8\pi_{42}^{15} = \{\overline{\zeta}_{15}\overline{\nu}_{34}, \overline{\zeta}_{15}\varepsilon_{34}\} = 0.$$

Since the indeterminacy of $\{\overline{\zeta}_{15}, 8\sigma_{34}, 2\iota_{41}\}$ is given by

$$\overline{\zeta}_{15}\pi_{42}^{34} + 2\pi_{42}^{15} = \{2\zeta_{3,15}\},$$

we have

$$\{\overline{\zeta}_{15}, 2\sigma_{34}, 8\iota_{41}\} = x'\zeta_{3,15}$$

for some odd integer x'. Therefore, for some odd integer x'', we have

$$\{\{\sigma_{15}, \zeta_{22}, 8\iota_{33}\}, 2\sigma_{34}, 8\iota_{41}\} = x''\zeta_{3,15}.$$

Next, we consider $\{\sigma_{15}, \{\zeta_{22}, 8\iota_{33}, 2\sigma_{33}\}, 8\iota_{41}\}$. By the definition of $\overline{\zeta}_5$ (see p. 137 of [15]), we have

$$\{\zeta_{22}, 8\iota_{33}, 2\sigma_{33}\} \ni -\overline{\zeta}_{22}.$$

By Part II, Proposition 2.2(3) of [13], we have $\zeta_{22}\overline{\nu}_{33} = \zeta_{22}\varepsilon_{33} = 0$. Since $\pi^{22}_{34} = 0$, the indeterminacy of $\{\zeta_{22}, 8\iota_{33}, 2\sigma_{33}\}$ is given by

$$\zeta_{22}\pi^{33}_{41} + \pi^{22}_{34}2\sigma_{34} = \{\zeta_{22}\overline{\nu}_{33}, \zeta_{22}\varepsilon_{33}\} = 0.$$

So we have

$$\{\zeta_{22}, 8\iota_{33}, 2\sigma_{33}\} = -\overline{\zeta}_{22}.$$

Therefore, we have

$$\{\sigma_{15}, \{\zeta_{22}, 8\iota_{33}, 2\sigma_{33}\}, 8\iota_{41}\} = -\{\sigma_{15}, \overline{\zeta}_{22}, 8\iota_{41}\}.$$

Finally, we consider $\{\sigma_{15}, \zeta_{22}, \{8\iota_{33}, 2\sigma_{33}, 8\iota_{40}\}\}$. We have

$$\{8\iota_{33}, 2\sigma_{33}, 8\iota_{40}\} \subset \{4\iota_{33}, 16\sigma_{33}, 2\iota_{40}\} = \{4\iota_{33}, 0, 2\iota_{40}\} \equiv 0 \bmod 2\pi^{33}_{41} = 0.$$

So we have

$$\{\sigma_{15}, \zeta_{22}, \{8\iota_{33}, 2\sigma_{33}, 8\iota_{40}\}\} = \{\sigma_{15}, \zeta_{22}, 0\} \equiv 0 \bmod \sigma_{15}\pi^{22}_{42}.$$

By Lemma 2.5(4), we have $\sigma_{15}\pi^{22}_{42} = \{\sigma_{15}\overline{\kappa}_{22}\} = 0$. Therefore, we have

$$\{\sigma_{15}, \zeta_{22}, \{8\iota_{33}, 2\sigma_{33}, 8\iota_{40}\}\} = 0.$$

Substituting these results into the first formula, we have

$$x''\zeta_{3,15} - \{\sigma_{15}, \overline{\zeta}_{22}, 8\iota_{41}\} + 0 \ni 0.$$

Since the indeterminacy of $\{\sigma_{15}, \overline{\zeta}_{22}, 8\iota_{41}\}$ is given by $\sigma_{15}\pi^{22}_{42} + 8\pi^{15}_{42} = 0$, we have

$$\{\sigma_{15}, \overline{\zeta}_{22}, 8\iota_{41}\} = x''\zeta_{3,15}.$$

(5) We can define a Toda bracket $\{\sigma_{15}, \overline{\kappa}_{22}, 8\iota_{42}\}$ by Lemma 2.5(4) and the fact that $8\overline{\kappa}_{22} = 0$. We have

$$\{\sigma_{15}, \bar{\kappa}_{22}, 8\iota_{42}\} \subset \{\sigma_{15}, 4\bar{\kappa}_{22}, 2\iota_{42}\}$$

$$= \{\sigma_{15}, \nu_{22}^2\kappa_{28}, 2\iota_{42}\} \quad \text{by Lemma 15.4 of [8]}$$

$$\supset \{\sigma_{15}, \nu_{22}^2, 0\}$$

$$\ni 0.$$

We consider $\sigma_{15}\pi_{43}^{22} + 2\pi_{43}^{15} = \{\sigma_{15}P(\iota_{45}), \sigma_{15}\eta_{22}\bar{\kappa}_{23}, \sigma_{15}^4\}$ which is the indeterminacy of $\{\sigma_{15}, 4\bar{\kappa}_{22}, 2\iota_{42}\}$. By Lemma 2.5(5), we have

$$\sigma_{15}^4 = 0.$$

So, by Part I, Proposition 2.8 of [13] and (10.10) of [15], we have

$$\sigma_{15}P(\iota_{45}) = \pm P(\iota_{31})\sigma_{29}^2 = \pm 2\sigma_{15}^4 = 0,$$

where P denotes the boundary homomorphism of the EHP-exact sequence. By Proposition 3.1 of [15] and Lemma 2.5(4), we have

$$\sigma_{15}\eta_{22}\bar{\kappa}_{23} = \sigma_{15}\bar{\kappa}_{22}\eta_{42} = 0.$$

So we have $\sigma_{15}\pi_{43}^{22} + 2\pi_{43}^{15} = 0$. Since the indeterminacy of $\{\sigma_{15}, 4\bar{\kappa}_{22}, 2\iota_{42}\}$ is trivial, we have

$$\{\sigma_{15}, \bar{\kappa}_{22}, 8\iota_{42}\} = \{\sigma_{15}, 4\bar{\kappa}_{22}, 2\iota_{42}\} = 0.$$

(6) We can define a Toda bracket $\{\sigma_{14}, \sigma_{21}^3, 2\iota_{42}\}$ by Lemma 2.5(5) and the fact that $2\sigma_{21}^3 = 0$. By Table 2.2(4), we have $\sigma_{14}\varepsilon_{21}\kappa_{29} = 0$. By Table 2.2(3), we have $\sigma_{14}\nu_{21}\bar{\sigma}_{24} = 0$. Hence the indeterminacy of $\{\sigma_{14}, \sigma_{21}^3, 2\iota_{42}\}$ is given by

$$\sigma_{14}\pi_{43}^{21} + 2\pi_{43}^{14} = \{\sigma_{14}\sigma_{21}^*, \sigma_{14}\varepsilon_{21}\kappa_{29}, \sigma_{14}\nu_{21}\bar{\sigma}_{24}, 2\sigma_{14}\sigma_{21}^*\}$$

$$= \{\sigma_{14}\sigma_{21}^*\}.$$

Since $\pi_{43}^{14} = \{\sigma_{14}\sigma_{21}^*\} \cong \mathbf{Z}_2$, we have $\{\sigma_{14}, \sigma_{21}^3, 2\iota_{42}\} \ni 0$. Thus we have $\{\sigma_{15}, \sigma_{22}^3, 2\iota_{43}\} \ni 0$.

Lemma 3.3. *We have the following relations*:

(1) $\{\sigma'\sigma_{14}, \nu_{21}^2, 2\iota_{27}\} = 0$.

(2) $\{\sigma'\sigma_{14}, \sigma_{21}, 16\iota_{28}\} = -\sigma'\rho_{14} + 4x\sigma'\rho_{14}$ *for some integer x.*

(3) $\{\sigma'\sigma_{14}, \bar{\nu}_{21}, 2\iota_{29}\} \ni 0$.

(4) $\{\sigma'\sigma_{14}, \kappa_{21}, 2\iota_{35}\} \ni \phi_7\nu_{30}^2 + x\sigma'\omega_{14}\nu_{30}^2$ *for some integer x.*

(5) $\{\sigma'\sigma_{14}, \sigma_{21}^2, 2\iota_{35}\} \ni \delta''$.

(6) $\{\sigma'\sigma_{14}, \rho_{21}, 32\iota_{36}\} \ni x\sigma'\bar{\rho}_{14} + 4a\nu_7\sigma_{10}\bar{\kappa}_{17} + b\sigma'\phi_{14} + c\sigma'\psi_{14}$ *for some odd integer x and integers a, b, c.*

Proof. (1) We can define a Toda bracket $\{\sigma_{14}, \nu_{21}^2, 2\iota_{27}\}$ by Table 2.2(3) and the fact that $2\nu_{21}^2 = 0$. The indeterminacy of $\{\sigma_{14}, \nu_{21}^2, 2\iota_{27}\}$ is $\sigma_{14}\pi_{28}^{21} + 2\pi_{28}^{14} = \{\sigma_{14}^2\}$. Since $\pi_{28}^{14} = \{\sigma_{14}^2, \kappa_{14}\} \cong \mathbf{Z}_8 \oplus \mathbf{Z}_2$, we may put

$$x\kappa_{14} \in \{\sigma_{14}, \nu_{21}^2, 2\iota_{27}\}$$

for some integer x. We have

$$\begin{aligned} x\kappa_{14}\eta_{28} &\in \{\sigma_{14}, \nu_{21}^2, 2\iota_{27}\}\eta_{28} \\ &\subset \{\sigma_{14}, \nu_{21}^2, 0\} \\ &= 0 \bmod \sigma_{14}\pi_{29}^{21}. \end{aligned}$$

By (4) and (5) of Table 2.2, we have $\sigma_{14}\pi_{29}^{21} = \{\sigma_{14}\varepsilon_{21}, \sigma_{14}\bar{\nu}_{21}\} = 0$. So we have $x\kappa_{14}\eta_{28} = 0$. On the other hand, by (10.23) of [15] we have $\kappa_{14}\eta_{28} = \bar{\varepsilon}_{14}$. Since $\bar{\varepsilon}_{14}$ is a generator of π_{29}^{14}, we have $x = 0$. Hence we have $\{\sigma_{14}, \nu_{21}^2, 2\iota_{27}\} \ni 0$. So we have

$$\{\sigma'\sigma_{14}, \nu_{21}^2, 2\iota_{27}\} \supset \sigma'\{\sigma_{14}, \nu_{21}^2, 2\iota_{27}\} \ni 0.$$

By Table 2.4(1), the indeterminacy of $\{\sigma'\sigma_{14}, \nu_{21}^2, 2\iota_{27}\}$ is $\sigma'\sigma_{14}\pi_{28}^{21} + 2\pi_{28}^7 = \{\sigma'\sigma_{14}^2\} = 0$. Thus we have $\{\sigma'\sigma_{14}, \nu_{21}^2, 2\iota_{27}\} = 0$.

(2) We can define a Toda bracket $\{\sigma'\sigma_{14}, \sigma_{21}, 16\iota_{28}\}$ by Table 2.4(1) and the fact that $16\sigma_{21} = 0$. By (4) and (5) of Table 2.2, we have $\sigma'\sigma_{14}\varepsilon_{21} = \sigma'\sigma_{14}\bar{\nu}_{21} = 0$. Therefore, we have

$$\sigma'\sigma_{14}\pi_{29}^{21} + 16\pi_{29}^7 = \{\sigma'\sigma_{14}\varepsilon_{21}, \sigma'\sigma_{14}\bar{\nu}_{21}\} = 0,$$

which is the indeterminacy of $\{\sigma'\sigma_{14}, \sigma_{21}, 16\iota_{28}\}$. Hence we have

$$\begin{aligned} -E\{\sigma'\sigma_{14}, \sigma_{21}, 16\iota_{28}\} &\in \{E\sigma'\sigma_{15}, \sigma_{22}, 16\iota_{29}\} \\ &\subset \{E\sigma'\sigma_{15}, 4\sigma_{22}, 4\iota_{29}\} \\ &\supset E\sigma'\{\sigma_{15}, 4\sigma_{22}, 4\iota_{29}\} \\ &\ni E\sigma'\rho_{15} \qquad \text{by Lemma 3.1(2).} \end{aligned}$$

By (4) and (5) of Table 2.2, we have $E\sigma'\sigma_{15}\pi_{30}^{22} = \{E\sigma'\sigma_{15}\varepsilon_{22}, E\sigma'\sigma_{15}\bar{\nu}_{22}\} = 0$. Therefore, we have

$$E\sigma'\sigma_{15}\pi_{30}^{22} + 4\pi_{30}^8 = \{4E\sigma'\rho_{15}, 4\sigma_8\rho_{15}\},$$

which is the indeterminacy of $\{E\sigma'\sigma_{15}, 4\sigma_{22}, 4\iota_{29}\}$. Since $4\sigma_8\rho_{15}$ is not in the image of the suspension homomorphism, we have

$$-E\{\sigma'\sigma_{14}, \sigma_{21}, 16\iota_{28}\} = E\sigma'\rho_{15} + 4xE\sigma'\rho_{15}$$

for some integer x. Since $E : \pi_{29}^7 \to \pi_{30}^8$ is a monomorphism, we have

$$\{\sigma'\sigma_{14}, \sigma_{21}, 16\iota_{28}\} = -\sigma'\rho_{14} - 4x\sigma'\rho_{14}.$$

(3) We can define a Toda bracket $\{\sigma_{14}, \bar{\nu}_{21}, 2\iota_{29}\}_1$ by Table 2.2(5) and the fact $2\bar{\nu}_{20} = 0$. By Table 2.2(3), we have $\sigma_{14}\nu_{21}^3 = 0$. By Table 2.2(4), we have $\sigma_{14}\varepsilon_{21}\eta_{29} = 0$. So the indeterminacy of $\{\sigma_{14}, \bar{\nu}_{21}, 2\iota_{29}\}_1$ is given by

$$\sigma_{14}E\pi_{29}^{20} + 2\pi_{30}^{14} = \{\sigma_{14}\nu_{21}^3, \sigma_{14}\varepsilon_{21}\eta_{29}, \sigma_{14}\mu_{21}, 2\omega_{14}\}$$

$$= \{\sigma_{14}\mu_{21}, 2\omega_{14}\}.$$

Since $\pi_{30}^{14} = \{\omega_{14}, \sigma_{14}\mu_{21}\} \cong \mathbf{Z}_8 \oplus \mathbf{Z}_2$, we may put

$$\{\sigma_{14}, \bar{\nu}_{21}, 2\iota_{29}\}_1 \ni x\omega_{14}$$

for some integer x. We consider the generalized Hopf homomorphism $H : \pi_{30}^{14} \to \pi_{30}^{27}$. We have

$$H(\{\sigma_{14}, \bar{\nu}_{21}, 2\iota_{29}\}_1) \subset \{H(\sigma_{14}), \bar{\nu}_{21}, 2\iota_{29}\}_1$$

$$= \{0, \bar{\nu}_{21}, 2\iota_{29}\}_1$$

$$\equiv 0 \bmod 2\pi_{30}^{27} = \{2\nu_{27}\}.$$

On the other hand, by Lemma 12.15 of [15], we have

$$H(x\omega_{14}) = x\nu_{27}.$$

Hence, we see that x is an even integer. Since the indeterminacy of $\{\sigma_{14}, \bar{\nu}_{21}, 2\iota_{29}\}_1$ is $\{\sigma_{14}\mu_{21}, 2\omega_{14}\}$, we have $\{\sigma_{14}, \bar{\nu}_{21}, 2\iota_{29}\}_1 \ni 0$. Thus we have

$$\{\sigma'\sigma_{14}, \bar{\nu}_{21}, 2\iota_{29}\} \supset \sigma'\{\sigma_{14}, \bar{\nu}_{21}, 2\iota_{29}\} \ni 0.$$

(4) By Lemma 3.1(5), we have

$$\{E\sigma'\sigma_{15}, \kappa_{22}, 2\iota_{36}\} \supset E\sigma'\{\sigma_{15}, \kappa_{22}, 2\iota_{36}\}$$

$$\ni E\sigma'\nu_{15}\bar{\sigma}_{18} + xE\sigma'\omega_{15}\nu_{31}^2$$

for some integer x. By Proposition 3.1 of [15] and Part II, Proposition 2.1(8) of [13], we have

$$E\sigma'\nu_{15}\bar{\sigma}_{18} = E\sigma'\bar{\sigma}_{15}\nu_{34} = \phi_8\nu_{31}^2.$$

Therefore, we have

$$\{E\sigma'\sigma_{15}, \kappa_{22}, 2\iota_{36}\} \ni \phi_8\nu_{31}^2 + xE\sigma'\omega_{15}\nu_{31}^2.$$

We can define a Toda bracket $\{\sigma'\sigma_{14}, \kappa_{21}, 2\iota_{35}\}$ by Lemma 2.6(3) and the fact that $2\kappa_{21} = 0$. For an element α of $\{\sigma'\sigma_{14}, \kappa_{21}, 2\iota_{35}\}$, we have

$$-E(\alpha) \in \{E\sigma'\sigma_{15}, \kappa_{22}, 2\iota_{36}\} \ni \phi_8\nu_{31}^2 + xE\sigma'\omega_{15}\nu_{31}^2.$$

By Table 2.4(2), we have $\sigma'\sigma_{14}\rho_{21} = 0$ and $E\sigma'\sigma_{15}\rho_{22} = 0$. By Table 2.2(8), we have $\sigma'\sigma_{14}\bar{\varepsilon}_{21} = 0$ and $E\sigma'\sigma_{15}\bar{\varepsilon}_{22} = 0$. Therefore, we have

$$\begin{aligned} E\sigma'\sigma_{15}\pi_{37}^{22} + 2\pi_{37}^{8} &= \{E\sigma'\sigma_{15}\rho_{22}, E\sigma'\sigma_{15}\bar{\varepsilon}_{22}, 2\sigma_8\sigma^*, 2E\delta''\} \\ &= \{2\sigma_8\sigma^*, 2E\delta''\}, \end{aligned}$$

which is the indeterminacy of $\{E\sigma'\sigma_{15}, \kappa_{22}, 2\iota_{36}\}$. Since $2\sigma_8\sigma^*$ is not in the image of the suspension homomorphism, we have

$$-E(\alpha) = \phi_8\nu_{31}^2 + xE\sigma'\omega_{15}\nu_{31}^2 + 2yE\delta''$$

for some integer y. Since $E : \pi_{36}^{7} \to \pi_{37}^{8}$ is a monomorphism, we have

$$\alpha = -\phi_7\nu_{30}^2 - x\sigma'\omega_{14}\nu_{30}^2 - 2y\delta''.$$

By the above argument, we have

$$\sigma'\sigma_{14}\pi_{36}^{21} + 2\pi_{36}^{7} = \{\sigma'\sigma_{14}\rho_{21}, \sigma'\sigma_{14}\bar{\varepsilon}_{21}, 2\delta''\} = \{2\delta''\},$$

which is the indeterminacy of $\{\sigma'\sigma_{14}, \kappa_{21}, 2\iota_{35}\}$. Since ν_{30}^2 is of order two, we have

$$\{\sigma'\sigma_{14}, \kappa_{21}, 2\iota_{35}\} \ni \phi_7\nu_{30}^2 + x\sigma'\omega_{14}\nu_{30}^2.$$

(5) By the definition of δ'' (see p. 215 of [12]), we have

$$\{\sigma'\sigma_{14}, \sigma_{21}^2, 2\iota_{35}\} \supset \{\sigma'\sigma_{14}, \sigma_{21}, 2\sigma_{28}\} \ni \delta''.$$

(6) We can define a Toda bracket $\{\sigma'\sigma_{14}, \rho_{21}, 32\iota_{36}\}$ by Table 2.4(2) and the fact that $32\rho_{21} = 0$. By Lemma 3.2(1), we have

$$\begin{aligned} -E\{\sigma'\sigma_{14}, \rho_{21}, 32\iota_{36}\} &\subset \{E\sigma'\sigma_{15}, \rho_{22}, 32\iota_{37}\} \\ &\subset \{E\sigma'\sigma_{15}, 2\rho_{22}, 16\iota_{37}\} \\ &\supset E\sigma'\{\sigma_{15}, 2\rho_{22}, 16\iota_{37}\} \\ &\ni xE\sigma'\overline{\rho}_{15} + 4aE\sigma'\nu_{15}\overline{\kappa}_{18} + bE\sigma'\phi_{15} + cE\sigma'\psi_{15} \end{aligned}$$

for some odd integer x and integers a, b, c. By (7.19) of [15] and the fact that $\nu_8\sigma_{11}\overline{\kappa}_{18}$ is of order 8, we have $4E\sigma'\nu_{15}\overline{\kappa}_{18} = 4\nu_8\sigma_{11}\overline{\kappa}_{18}$. So we have

$$\begin{aligned} -E\{\sigma'\sigma_{14}, \rho_{21}, 32\iota_{36}\} &\subset \{E\sigma'\sigma_{15}, 2\rho_{22}, 16\iota_{37}\} \\ &\ni xE\sigma'\overline{\rho}_{15} + 4a\nu_8\sigma_{11}\overline{\kappa}_{18} + bE\sigma'\phi_{15} + cE\sigma'\psi_{15}. \end{aligned}$$

By Part I, Proposition 3.4(4) of [13] and the fact $4E\sigma'\nu_{15}\overline{\kappa}_{18} = 4\nu_8\sigma_{11}\overline{\kappa}_{18}$, we have

$$E\sigma'\sigma_{15}\omega_{22} \equiv E\sigma'\phi_{15} \bmod\{4\nu_8\sigma_{11}\overline{\kappa}_{18}\}.$$

By Table 2.3(2), we have $E\sigma'\sigma_{15}^2\mu_{29} = 0$. Therefore, the indeterminacy of $\{E\sigma'\sigma_{15}, 2\rho_{22}, 16\iota_{37}\}$ is given by

$$\begin{aligned} E\sigma'\sigma_{15}\pi_{38}^{22} + 16\pi_{38}^{8} &= \{E\sigma'\sigma_{15}\omega_{22}, E\sigma'\sigma_{15}^2\mu_{29}\} \\ &= \{E\sigma'\phi_{15} + 4y\nu_8\sigma_{11}\overline{\kappa}_{18}\}, \end{aligned}$$

where $y = 0$ or 1. Therefore, for an element α of $\{\sigma'\sigma_{14}, \rho_{21}, 32\iota_{36}\}$, we have

$$-E(\alpha) = xE\sigma'\overline{\rho}_{15} + 4a'\nu_8\sigma_{11}\overline{\kappa}_{18} + b'E\sigma'\phi_{15} + cE\sigma'\psi_{15},$$

where a' and b' are some integers. Since $E : \pi_{37}^{7} \to \pi_{38}^{8}$ is a monomorphism, we have

$$\alpha = -x\sigma'\overline{\rho}_{14} + 4a'\nu_7\sigma_{10}\overline{\kappa}_{17} + b'\sigma'\phi_{14} + c\sigma'\psi_{14}.$$

Thus we obtain the required result.

4. The Homotopy Groups $\pi_i(F_4/G_2 : 2)$ for $i \leq 45$

We consider the 2-local fibration

$$S^{15} \xrightarrow{i} F_4/G_2 \xrightarrow{p} S^{23},$$

which is given by Davis and Mahowald [2]. Then we have the long homotopy exact sequence:

$$\cdots \to \pi_{i+1}(S^{23} : 2) \xrightarrow{\Delta_{i+1}} \pi_i(S^{15} : 2) \xrightarrow{i_*} \pi_i(F_4/G_2 : 2) \xrightarrow{p_*} \pi_i(S^{23} : 2) \xrightarrow{\Delta_i} \cdots \quad (1)$$

associated with the above 2-local fibration. By making use of this exact sequence, we will calculate homotopy groups $\pi_i(F_4/G_2 : 2)$. First we calculate the boundary homomorphisms $\Delta_i : \pi_i^{23} \to \pi_{i-1}^{15}$.

Lemma 4.1. *We have the formulas*

$$\Delta_{23}(\iota_{23}) = \sigma_{15} \quad \text{and} \quad \Delta_i(E\alpha) = \sigma_{15}\alpha,$$

where α *is an arbitrary element of* π_{i-1}^{22} *and* E *is the suspension homomorphism.*

Proof. By Borel [1], we have $H^*(F_4/G_2; \mathbf{Z}_2) \cong \Lambda(x_{15}, x_{23})$, $Sq^8 x_{15} = x_{23}$. Since $Sq^8 x_{15} = x_{23}$, we have $\Delta_{23}(\iota_{23}) = \sigma_{15}$. Then we have the second formula from the familiar property of the boundary homomorphism of a fibration.

Using Tables 2.2 and 2.3 and Lemmas 2.5 and 4.1, we calculate the boundary homomorphism $\Delta_i : \pi_i^{23} \to \pi_{i-1}^{15}$ for $i \leq 45$. The results are stated in the following.

Lemma 4.2. (1) *The homomorphisms* $\Delta_{i+1} : \pi_{i+1}^{23} \to \pi_i^{15}$ *are epimorphisms for* $i = 22, 27, 28$. *For other values of* $i\,(23 \leq i \leq 45)$, *we have the following table of the cokernels of* Δ_{i+1}.

i	23	24	25	26	29	30
Coker Δ_{i+1}	$\mathbf{Z}_2$	$\mathbf{Z}_2 \oplus \mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_8$	$\mathbf{Z}_2$	$\mathbf{Z}_{32} \oplus \mathbf{Z}_2$
generator	$\overline{\nu}_{15}$	ν_{15}^3, μ_{15}	$\eta_{15}\mu_{16}$	ζ_{15}	κ_{15}	ρ_{15}, $\overline{\varepsilon}_{15}$

31	32	33	34
$\mathbf{Z}_2 \oplus \mathbf{Z}_2$	$\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$	$\mathbf{Z}_8 \oplus \mathbf{Z}_8 \oplus \mathbf{Z}_2$	$\mathbf{Z}_8 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$
$\eta^{*\prime}$, ω_{15}	$\eta^{*\prime}\eta_{31}$, ε_{15}^*, $\nu_{15}\kappa_{18}$, $\overline{\mu}_{15}$	ξ_{15}, $E^2\lambda$, $\eta_{15}\overline{\mu}_{16}$	$\overline{\zeta}_{15}$, $\omega_{15}\nu_{31}$, $\overline{\sigma}_{15}$

35	36	37
$\mathbf{Z}_8$	$\mathbf{Z}_2 \oplus \mathbf{Z}_2$	$\mathbf{Z}_8 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$
$\overline{\kappa}_{15}$	$\eta_{15}\overline{\kappa}_{16}$, $E^2\lambda\nu_{33}$	$\sigma^{*\prime}$, $\omega_{15}\nu_{31}^2$, $\varepsilon_{15}\kappa_{23}$, $\nu_{15}\overline{\sigma}_{18}$

38	39	40
$\mathbf{Z}_{16} \oplus \mathbf{Z}_8 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$	$\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$	$\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$
$\overline{\rho}_{15}$, $\nu_{15}\overline{\kappa}_{18}$, ψ_{15}, $\overline{\varepsilon}^{*\prime}$, $\overline{\nu}^{*\prime}$	$\overline{\sigma}'_{15}$, $\tilde{\varepsilon}_{15}$, $E\zeta^*$, $\mu^{*\prime}$	$\xi_{15}\sigma_{33}$, $\mu_{3,15}$, $D_1\mu_{31}$

41	42	43	44	45
$\mathbf{Z}_8 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$	$\mathbf{Z}_8$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_8 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$
$E^3\tau^{\mathrm{IV}}$, $\nu_{15}^2\overline{\kappa}_{21}$, $\eta_{15}\mu_{3,16}$	$\zeta_{3,15}$	$\varepsilon_{15}\overline{\kappa}_{23}$	L_1	θ^{V}, $\omega_{15}\kappa_{31}$, $\psi_{15}\sigma_{38}$

(2) *The homomorphisms* $\Delta_i : \pi_i^{23} \to \pi_{i-1}^{15}$ *are monomorphisms for* $i = 24, 25, 27, 28, 33, 35, 36$. *For other values of* i $(23 \leq i \leq 45)$ *we have the following table of the kernel of* Δ_i.

i	23	26	29	30	31	32
Ker Δ_i	$\mathbf{Z}$	$\mathbf{Z}_8$	$\mathbf{Z}_2$	$\mathbf{Z}_4$	$\mathbf{Z}_2 \oplus \mathbf{Z}_2$	$\mathbf{Z}_2 \oplus \mathbf{Z}_2$
generator	$16\iota_{23}$	ν_{23}	ν_{23}^2	$4\sigma_{23}$	$\bar{\nu}_{23}$, ε_{23}	ν_{23}^3, $\varepsilon_{23}\eta_{31}$

34	37	38	39	40	41
$\mathbf{Z}_8$	$\mathbf{Z}_2$	$\mathbf{Z}_{16} \oplus \mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_2 \oplus \mathbf{Z}_2$	$\mathbf{Z}_2$
ζ_{23}	κ_{23}	$2\rho_{23}$, $\bar{\varepsilon}_{23}$	$\sigma_{23}\mu_{30}$	$\sigma_{23}\mu_{30}\eta_{39}$, $\nu_{23}\kappa_{26}$	$4\nu_{23}^*$

42	43	44	45
$\mathbf{Z}_8 \oplus \mathbf{Z}_2$	$\mathbf{Z}_8$	$\mathbf{Z}_2 \oplus \mathbf{Z}_2$	$\mathbf{Z}_2 \oplus \mathbf{Z}_2$
$\bar{\zeta}_{23}$, $\bar{\sigma}_{23}$	$\bar{\kappa}_{23}$	$\bar{\kappa}_{23}\eta_{43}$, σ_{23}^3	$\varepsilon_{23}\kappa_{31}$, $\nu_{23}\bar{\sigma}_{26}$

From the exact sequence (1), we have the following short exact sequence

$$0 \to \mathrm{Coker}\,\Delta_{i+1} \xrightarrow{i_*} \pi_i(F_4/G_2 : 2) \xrightarrow{p_*} \mathrm{Ker}\,\Delta_i \to 0.$$

From this short exact sequence, we will calculate $\pi_i(F_4/G_2 : 2)$ by making use of the following theorem which is proved by Mimura-Toda.

Theorem 4.3 (Theorem 2.2 of [9]). *Let (X, p, B) be a fibration, F be a fiber $p^{-1}(*)$ and Δ be the boundary homomorphism in the homotopy exact sequence of the fibration. Assume that $\alpha \in \pi_{i+1}(B)$, $\beta \in \pi_j(S^i)$ and $\gamma \in \pi_k(S^j)$ satisfy the conditions $(\Delta(\alpha))\beta = 0$ and $\beta\gamma = 0$. Then for an arbitrary element δ of Toda bracket $\{\Delta(\alpha), \beta, \gamma\} \subset \pi_{k+1}(F)$, there exists an element $\varepsilon \in \pi_{j+1}(X)$ such that*

$$p_*\varepsilon = \alpha E\beta, \qquad i_*\delta = \varepsilon E\gamma.$$

Let us state our first main result.

Theorem 4.4. *We have the following table of the homotopy groups* $\pi_i(F_4/G_2 : 2)$ *for* $i \leq 45$.

i	$i \leq 14$	15	16	17	18
$\pi_i(F_4/G_2 : 2)$	0	$\mathbf{Z}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_8$
generator		$i_*(\iota_{15})$	$i_*(\eta_{15})$	$i_*(\eta_{15}^2)$	$i_*(\nu_{15})$

19, 20	21	22	23	24	25
0	$\mathbf{Z}_2$	0	$\mathbf{Z} \oplus \mathbf{Z}_2$	$\mathbf{Z}_2 \oplus \mathbf{Z}_2$	$\mathbf{Z}_2$
	$i_*(\nu_{15}^2)$		$[16\iota_{23}]$, $i_*(\overline{\nu}_{15})$	$i_*(\nu_{15}^3)$, $i_*(\mu_{15})$	$i_*(\eta_{15}\mu_{16})$

26	27,28	29	30	31
$\mathbf{Z}_{64}$	0	$\mathbf{Z}_2 \oplus \mathbf{Z}_2$	$\mathbf{Z}_{128} \oplus \mathbf{Z}_2$	$\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$
$[\nu_{23}]$		$[\nu_{23}]\nu_{26}$, $i_*(\kappa_{15})$	$[4\sigma_{23}]$, $i_*(\overline{\varepsilon}_{15})$	$[\overline{\nu}_{23}]$, $[\varepsilon_{23}]$, $i_*(\eta^{*\prime})$, $i_*(\omega_{25})$

32
$\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$
$[\nu_{23}]\nu_{26}^2$, $[\varepsilon_{23}]\eta_{31}$, $i_*(\eta^{*\prime}\eta_{31})$, $i_*(\varepsilon_{15}^*)$, $i_*(\nu_{15}\kappa_{18})$, $i_*(\overline{\mu}_{15})$

33	34	35
$\mathbf{Z}_8 \oplus \mathbf{Z}_8 \oplus \mathbf{Z}_2$	$\mathbf{Z}_{64} \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$	$\mathbf{Z}_8$
$i_*(\xi_{15})$, $i_*(E^2\lambda)$, $i_*(\eta_{15}\overline{\mu}_{16})$	$[\zeta_{23}]$, $i_*(\omega_{15}\nu_{31})$, $i_*(\overline{\sigma}_{15})$	$i_*(\overline{\kappa}_{15})$

36	37
$\mathbf{Z}_2 \oplus \mathbf{Z}_2$	$\mathbf{Z}_8 \oplus \mathbf{Z}_4 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$
$i_*(\eta_{15}\bar{\kappa}_{16})$, $i_*(E^2\lambda\nu_{33})$	$i_*(\sigma^{*\prime})$, $[\kappa_{23}]$, $i_*(\omega_{15}\nu_{31}^2)$, $i_*(\varepsilon_{15}\kappa_{23})$

38
$\mathbf{Z}_{256} \oplus \mathbf{Z}_8 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$
$[2\rho_{23}]$, $i_*(\nu_{15}\bar{\kappa}_{18})$, $[\bar{\varepsilon}_{23}]$, $i_*(\psi_{15})$, $i_*(\bar{\varepsilon}^{*\prime})$, $i_*(\bar{\nu}^{*\prime})$

39
$\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$
$[\sigma_{23}\mu_{30}]$, $i_*(\bar{\sigma}'_{15})$, $i_*(\tilde{\varepsilon}_{15})$, $i_*(E\zeta^*)$, $i_*(\mu^{*\prime})$

40
$\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$
$[\sigma_{23}\mu_{30}]\eta_{39}$, $[\nu_{23}]\kappa_{26}$, $i_*(\xi_{15}\sigma_{33})$, $i_*(\mu_{3,15})$, $i_*(D_1\mu_{31})$

41	42	43
$\mathbf{Z}_8 \oplus \mathbf{Z}_4 \oplus \mathbf{Z}_2$	$\mathbf{Z}_{64} \oplus \mathbf{Z}_2$	$\mathbf{Z}_8 \oplus \mathbf{Z}_2$
$i_*(E^3\tau^{\mathrm{IV}})$, $[4\nu_{23}^*]$, $i_*(\eta_{15}\mu_{3,16})$	$[\bar{\zeta}_{23}]$, $[\bar{\sigma}_{23}]$	$[\bar{\kappa}_{23}]$, $i_*(\varepsilon_{15}\bar{\kappa}_{23})$

44	45
$\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$	$\mathbf{Z}_8 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$
$[\bar{\kappa}_{23}]\eta_{43}$, $[\sigma_{23}^3]$, $i_*(L_1)$	$i_*(\theta^{\mathrm{V}})$, $[\varepsilon_{23}]\kappa_{31}$, $[\nu_{23}]\bar{\sigma}_{26}$, $i_*(\omega_{15}\kappa_{31})$, $i_*(\psi_{15}\sigma_{38})$

Here we denote by $[\alpha]$ *an element of* $\pi_i(F_4/G_2 : 2)$ *such that* $p_*([\alpha]) = \alpha \in \pi_i(S^{23} : 2)$. *The following relations hold*:

$$8[\nu_{23}] = i_*(a_1\zeta_{15}),$$

$$4[4\sigma_{23}] = i_*(\rho_{15}),$$

$$8[\zeta_{23}] = i_*(a_2\overline{\zeta}_{15}),$$

$$2[\kappa_{23}] = i_*(\nu_{15}\overline{\sigma}_{18} + b_1\omega_{15}\nu_{31}^2),$$

$$16[2\rho_{23}] = i_*(a_3\overline{\rho}_{15} + 4b_2\nu_{15}\overline{\kappa}_{18} + b_3\psi_{15} + b_4\phi_{15}),$$

$$2[4\nu_{23}^*] = i_*(\nu_{15}^2\overline{\kappa}_{21}),$$

$$8[\overline{\zeta}_{23}] = i_*(a_4\zeta_{3,15}),$$

where a_i *is an odd integer and* $b_j = 0$ *or* 1.

Proof. The homomorphisms $i_* : \mathrm{Coker}\,\Delta_{i+1} \to \pi_i(F_4/G_2 : 2)$ are isomorphisms for $i \le 22$ and $i = 24, 25, 27, 28, 33, 35, 36$.

We remark that $\pi_{27}(F_4/G_2 : 2) = \pi_{28}(F_4/G_2 : 2) = 0$.

Consider the case $i = 23$. By Lemma 4.2, we have an exact sequence

$$0 \to \mathbf{Z}_2 \xrightarrow{i_*} \pi_{23}(F_4/G_2 : 2) \xrightarrow{p_*} \mathbf{Z} \to 0.$$

Since $\mathbf{Z}$ is free, this exact sequence splits.

Consider the case $i = 26$. By Lemma 4.2, we have an exact sequence

$$0 \to \mathbf{Z}_8 \xrightarrow{i_*} \pi_{26}(F_4/G_2 : 2) \xrightarrow{p_*} \mathbf{Z}_8 \to 0,$$

where the first $\mathbf{Z}_8$ is generated by ζ_{15} and the second $\mathbf{Z}_8$ is generated by ν_{23}. By Lemma 3.1(1), we have

$$\{\sigma_{15}, \nu_{22}, 8\iota_{25}\} \ni x\zeta_{15}$$

for some odd integer x. By Theorem 4.3, there exists an element $[\nu_{23}] \in \pi_{26}(F_4/G_2 : 2)$ such that

$$p_*([\nu_{23}]) = \nu_{23} \quad \text{and} \quad i_*(x\zeta_{15}) = 8[\nu_{23}].$$

Therefore, we obtain $\pi_{26}(F_4/G_2 : 2) = \{[\nu_{23}]\} \cong \mathbf{Z}_{64}$.

For $i = 30, 34, 37, 41, 42,$ we obtain the results by an argument similar to the case $i = 26$ from (2), (4) and (5) of Lemma 3.1 and (3) and (4) of Lemma 3.2, respectively[1].

Consider the case $i = 29$. By Lemma 4.2, we have an exact sequence

$$0 \to \mathbf{Z}_2 \xrightarrow{i_*} \pi_{29}(F_4/G_2 : 2) \xrightarrow{p_*} \mathbf{Z}_2 \to 0,$$

where the first $\mathbf{Z}_2$ is generated by κ_{15} and the second $\mathbf{Z}_2$ is generated by ν_{23}^2. We consider $[\nu_{23}]\nu_{26}$. We have

$$\begin{aligned} 2([\nu_{23}]\nu_{26}) &= [\nu_{23}]E^{23}\nu' && \text{by (5.5) of [15]} \\ &\in [\nu_{23}]\{\eta_{26}, 2\iota_{27}, \eta_{27}\} && \text{by the definition of } \nu' \text{ (p. 40 of [15])} \\ &\subset \{[\nu_{23}]\eta_{26}, 2\iota_{27}, \eta_{27}\} \\ &= \{0, 2\iota_{27}, \eta_{27}\} && \text{by the fact } \pi_{27}(F_4/G_2 : 2) = 0 \\ &\equiv 0 \bmod \pi_{28}(F_4/G_2 : 2)\eta_{28}. \end{aligned}$$

Since $\pi_{28}(F_4/G_2 : 2) = 0$, we have

$$2([\nu_{23}]\nu_{26}) = 0.$$

Moreover we have

$$p_*([\nu_{23}]\nu_{26}) = (p_*[\nu_{23}])\nu_{26} = \nu_{23}^2.$$

This implies that the above sequence splits.

[1] For the case $i = 42$, we need some more consideration on the construction of generators.

Consider the case $i = 31$. By Lemma 4.2, we have an exact sequence

$$0 \to \mathbf{Z}_2 \oplus \mathbf{Z}_2 \xrightarrow{i_*} \pi_{31}(F_4/G_2 : 2) \xrightarrow{p_*} \mathbf{Z}_2 \oplus \mathbf{Z}_2 \to 0,$$

where the first $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ is generated by $\eta^{*\prime}$, ω_{15} and the second $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ is generated by $\overline{\nu}_{23}, \varepsilon_{23}$. By Lemma 3.1(3), we have $\{\sigma_{15}, \varepsilon_{22}, 2\iota_{30}\} \ni 0$. Hence, by Theorem 4.3, there exists an element $[\varepsilon_{23}] \in \pi_{31}(F_4/G_2 : 2)$ such that

$$p_*([\varepsilon_{23}]) = \varepsilon_{23} \quad \text{and} \quad 2[\varepsilon_{23}] = 0.$$

We can define a Toda bracket $\{[\nu_{23}], \eta_{26}, \nu_{27}\}$ using the fact that $[\nu_{23}]\eta_{26} \in \pi_{27}(F_4/G_2 : 2) = 0$. Since $\pi_{31}^{26} = \pi_{28}(F_4/G_2 : 2) = 0$, the indeterminacy of $\{[\nu_{23}], \eta_{26}, \nu_{27}\}$ is trivial. Then by Lemma 6.2 of [15], we have

$$p_*\{[\nu_{23}], \eta_{26}, \nu_{27}\} = \{\nu_{23}, \eta_{26}, \nu_{27}\} = \overline{\nu}_{23}.$$

Therefore, we can define $[\overline{\nu}_{23}] = \{[\nu_{23}], \eta_{26}, \nu_{27}\}$. Then we have

$$\begin{aligned} 2[\overline{\nu}_{23}] &= \{[\nu_{23}], \eta_{26}, \nu_{27}\} 2\iota_{31} \\ &\subset \{[\nu_{23}], 2\eta_{26}, \nu_{27}\} \\ &\equiv 0 \bmod [\nu_{23}]\pi_{31}^{26} + \pi_{28}(F_4/G_2 : 2)\nu_{28} = 0. \end{aligned}$$

This implies that the above sequence splits.

Consider the case $i = 32$. By Lemma 4.2, we have an exact sequence

$$0 \to \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \xrightarrow{i_*} \pi_{32}(F_4/G_2 : 2) \xrightarrow{p_*} \mathbf{Z}_2 \oplus \mathbf{Z}_2 \to 0,$$

where $\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$ is generated by $\eta^{*\prime}\eta_{31}, \varepsilon_{15}^*, \nu_{15}\kappa_{18}, \overline{\mu}_{15}$ and $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ is generated by $\nu_{23}^3, \varepsilon_{23}\eta_{31}$. We consider $[\nu_{23}]\nu_{26}^2$ and $[\varepsilon_{23}]\eta_{31}$. Then we have

$$p_*([\nu_{23}]\nu_{26}^2) = \nu_{23}^3,\ 2[\nu_{23}]\nu_{26}^2 = [\nu_{23}]2\nu_{26}^2 = 0,$$

$$p_*([\varepsilon_{23}]\eta_{31}) = \varepsilon_{23}\eta_{31},\ 2[\varepsilon_{23}]\eta_{31} = [\varepsilon_{23}]2\eta_{31} = 0.$$

Therefore, the above sequence splits.

Consider the case $i = 38$. By Lemma 4.2, we have an exact sequence

$$0 \to \mathbf{Z}_{16} \oplus \mathbf{Z}_8 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \xrightarrow{i_*} \pi_{38}(F_4/G_2 : 2) \xrightarrow{p_*} \mathbf{Z}_{16} \oplus \mathbf{Z}_2 \to 0,$$

where $\mathbf{Z}_{16} \oplus \mathbf{Z}_8 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$ is generated by $\bar{\rho}_{15}$, $\nu_{15}\bar{\kappa}_{18}$, ψ_{15}, $\bar{\varepsilon}^{*\prime}$, $\bar{\nu}^{*\prime}$ and $\mathbf{Z}_{16} \oplus \mathbf{Z}_2$ is generated by $2\rho_{23}$, $\bar{\varepsilon}_{23}$. By Lemma 3.1(6), we have $\{\sigma_{15}, \bar{\varepsilon}_{22}, 2\iota_{37}\} \ni 0$. Hence, by Theorem 4.3, there exists an element $[\bar{\varepsilon}_{23}] \in \pi_{38}(F_4/G_2 : 2)$ such that

$$p_*([\bar{\varepsilon}_{23}]) = \bar{\varepsilon}_{23} \ \text{ and } \ 2[\bar{\varepsilon}_{23}] = 0.$$

By Theorem 4.3 and Lemma 3.2(1) there exists an element $[2\rho_{23}] \in \pi_{38}(F_4/G_2 : 2)$ such that

$$p_*([2\rho_{23}]) = 2\rho_{23} \quad \text{and} \quad i_*(x\bar{\rho}_{15} + 4a\nu_{15}\bar{\kappa}_{18} + b\psi_{15} + c\phi_{15}) = 16[2\rho_{23}]$$

for some odd integer x and integers a, b, c. Therefore, we obtain

$$\pi_{38}(F_4/G_2 : 2) \cong \mathbf{Z}_{256} \oplus \mathbf{Z}_8 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2.$$

Consider the case $i = 39$. By Lemma 4.2, we have an exact sequence

$$0 \to \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \xrightarrow{i_*} \pi_{39}(F_4/G_2 : 2) \xrightarrow{p_*} \mathbf{Z}_2 \to 0,$$

where $\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$ is generated by $\bar{\sigma}'_{15}$, $\tilde{\varepsilon}_{15}$, $E\zeta^*$, $\mu^{*\prime}$ and $\mathbf{Z}_2$ is generated by $\sigma_{23}\mu_{30}$. By Theorem 4.3 and Lemma 3.2(2), there exists an element $[\sigma_{23}\mu_{30}] \in \pi_{39}(F_4/G_2 : 2)$ such that

$$p_*([\sigma_{23}\mu_{30}]) = \sigma_{23}\mu_{30} \quad \text{and} \quad 2[\sigma_{23}\mu_{30}] = 0.$$

Therefore, the above sequence splits.

Consider the case $i = 40$. By Lemma 4.2, we have an exact sequence

$$0 \to \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \xrightarrow{i_*} \pi_{40}(F_4/G_2 : 2) \xrightarrow{p_*} \mathbf{Z}_2 \oplus \mathbf{Z}_2 \to 0,$$

where $\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$ is generated by $\xi_{15}\sigma_{33}$, $\mu_{3,15}$, $D_1\mu_{31}$ and $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ is generated by $\sigma_{23}\mu_{30}\eta_{39}$, $\nu_{23}\kappa_{26}$. We consider $[\sigma_{23}\mu_{30}]\eta_{39}$ and $[\nu_{23}]\kappa_{26}$. Then we obtain the result by an argument similar to the case $i = 32$.

Consider the case $i = 43$. By Lemma 4.2, we have an exact sequence

$$0 \to \mathbf{Z}_2 \xrightarrow{i_*} \pi_{43}(F_4/G_2 : 2) \xrightarrow{p_*} \mathbf{Z}_8 \to 0,$$

where $\mathbf{Z}_2$ is generated by $\varepsilon_{15}\bar{\kappa}_{23}$ and $\mathbf{Z}_8$ is generated by $\bar{\kappa}_{23}$. By Theorem 4.3 and Lemma 3.2(5), there exists an element $[\bar{\kappa}_{23}] \in \pi_{43}(F_4/G_2 : 2)$ such that

$$p_*([\bar{\kappa}_{23}]) = \bar{\kappa}_{23} \quad \text{and} \quad 8[\bar{\kappa}_{23}] = 0.$$

Therefore, the above sequence splits.

Consider the case $i = 44$. By Lemma 4.2, we have an exact sequence

$$0 \to \mathbf{Z}_2 \xrightarrow{i_*} \pi_{44}(F_4/G_2 : 2) \xrightarrow{p_*} \mathbf{Z}_2 \oplus \mathbf{Z}_2 \to 0,$$

where $\mathbf{Z}_2$ is generated by L_1 and $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ is generated by $\bar{\kappa}_{23}\eta_{43}$, σ_{23}^3. We consider $[\bar{\kappa}_{23}]\eta_{43}$. Then we have

$$p_*([\bar{\kappa}_{23}]\eta_{43}) = \bar{\kappa}_{23}\eta_{43} \quad \text{and} \quad 2[\bar{\kappa}_{23}]\eta_{43} = [\bar{\kappa}_{23}]2\eta_{43} = 0.$$

By Theorem 4.3 and Lemma 3.2(6), there exists an element $[\sigma_{23}^3] \in \pi_{44}(F_4/G_2 : 2)$ such that

$$p_*([\sigma_{23}^3]) = \sigma_{23}^3 \quad \text{and} \quad 2[\sigma_{23}^3] = 0.$$

This implies that the above sequence splits.

Consider the case $i = 45$. By Lemma 4.2, we have an exact sequence

$$0 \to \mathbf{Z}_8 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \xrightarrow{i_*} \pi_{44}(F_4/G_2 : 2) \xrightarrow{p_*} \mathbf{Z}_2 \oplus \mathbf{Z}_2 \to 0,$$

where $\mathbf{Z}_8 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$ is generated by $\theta^{\vee}$, $\omega_{15}\kappa_{31}$, $\psi_{15}\sigma_{38}$ and $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ is generated by $\varepsilon_{23}\kappa_{31}$, $\nu_{23}\bar{\sigma}_{26}$. We consider $[\varepsilon_{23}]\kappa_{31}$ and $[\nu_{23}]\bar{\sigma}_{26}$. Then we obtain the result by an argument similar to the case $i = 32$.

5. The Homotopy Groups $\pi_i(\Omega\prod : 2)$ for $i \leq 37$

We consider the fibration

$$S^7 \xrightarrow{i} \Omega\prod \xrightarrow{p} \Omega S^{23},$$

which is given by Davis and Mahowald [2]. Then we have a long homotopy exact sequence

$$\cdots \to \pi_{i+1}(\Omega S^{23}) \xrightarrow{\Delta_{i+1}} \pi_i(S^7) \xrightarrow{i_*} \pi_i(\Omega\prod) \xrightarrow{p_*} \pi_i(\Omega S^{23}) \xrightarrow{\Delta_i} \cdots \tag{2}$$

associated with the above fibration. By making use of this exact sequence, we calculate homotopy groups $\pi_i(\Omega\prod : 2)$. Let $\mathrm{ad} : \pi_{i+1}(S^{23}) \to \pi_i(\Omega S^{23})$ be the adjoint isomorphism. Then for an element α of $\pi_k(S^{22})$, we have the formula

$$\mathrm{ad}(E\alpha) = \mathrm{ad}(\iota_{23})\alpha.$$

We define the homomorphism

$$\Delta'_i : \pi_i(S^{23}) \to \pi_{i-2}(S^7)$$

by

$$\Delta'_i = \Delta_{i-1}\mathrm{ad}.$$

Then we have the following.

Lemma 5.1. *We have the formulas*

$$\Delta'_{23}(\iota_{23}) = -\sigma'\sigma_{14} + E^{-1}[[\iota_8, \iota_8], \iota_8],$$

$$\Delta'_i(E^2\alpha) = (-\sigma'\sigma_{14} + E^{-1}[[\iota_8, \iota_8], \iota_8])\alpha$$

for $\alpha \in \pi_{i-2}(S^{21})$, *where* $[\ ,\ ]$ *is the Whitehead product and* E *is the suspension homomorphism.*

Proof. By Davis-Mahowald [2] and Mimura [7], we have

$$\Omega\prod \simeq S^7 \underset{\beta}{\cup} e^{22} \cup e^{29} \cup \cdots,$$

where $\beta = -\sigma'\sigma_{14} + E^{-1}[[\iota_8, \iota_8], \iota_8]$. Hence we have

$$\Delta'_{23}(\iota_{23}) = -\sigma'\sigma_{14} + E^{-1}[[\iota_8, \iota_8], \iota_8].$$

Then we obtain the second formula from the familiar property of the boundary homomorphism of a fibration.

From now on, we restrict our attention to the 2-primary component. For $\Delta'_i : \pi_i^{23} \to \pi_{i-2}^7$, by Lemma 5.1, we have the following formulas

$$\Delta'_{23}(\iota_{23}) = -\sigma'\sigma_{14} \quad \text{and} \quad \Delta'_i(E^2\alpha) = -\sigma'\sigma_{14}\alpha, \tag{3}$$

where α is an arbitrary element of π_{i-2}^{21}.

Using Tables 2.2, 2.3, 2.4, Lemma 2.6 and formula (3), we calculate the boundary homomorphism $\Delta'_i : \pi_i^{23} \to \pi_{i-2}^7$ for $i \leq 39$. The results are stated below.

Lemma 5.2. (1) *We have the following table of the cokernels of* $\Delta'_{i+2} : \pi_{i+2}^{23} \to \pi_i^7$ $(22 \leq i \leq 37)$

i	22	23
Coker Δ'_{i+2}	$\mathbf{Z}_8 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$	$\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$
generator	ρ'', $\sigma'\bar{\nu}_{14}$, $\bar{\varepsilon}_7$	$\sigma'\mu_{14}$, $E\zeta'$, $\mu_7\sigma_{16}$

24	25	26	27
$\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$	$\mathbf{Z}_8 \oplus \mathbf{Z}_2$	$\mathbf{Z}_8 \oplus \mathbf{Z}_2$	$\mathbf{Z}_8$
$\sigma'\eta_{14}\mu_{15}$, $\nu_7\kappa_{10}$, $\bar{\mu}_7$, $\eta_7\mu_8\sigma_{17}$	$\zeta_7\sigma_{18}$, $\eta_7\bar{\mu}_8$	$\bar{\zeta}_7$, $\bar{\sigma}_7$	$\bar{\kappa}_7$

28	29
$\mathbf{Z}_2 \oplus \mathbf{Z}_2$	$\mathbf{Z}_8 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$
$\eta_7\bar{\kappa}_8$, $\sigma'\kappa_{14}$	$\sigma'\rho_{14}$, $\bar{\nu}_7\kappa_{15}$, $\varepsilon_7\kappa_{15}$, $\nu_7\bar{\sigma}_{10}$

30	31
$\mathbf{Z}_8 \oplus \mathbf{Z}_8 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$	$\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$
$\nu_7\bar{\kappa}_{10}$, $\bar{\rho}'$, ϕ_7, $\bar{\kappa}_7\nu_{27} - \nu_7\bar{\kappa}_{10}$, $\sigma'\omega_{14}$	δ_7, $\bar{\mu}_7\sigma_{24}$, $\nu_7\sigma_{10}\kappa_{17}$, $\bar{\sigma}'_7$, $\sigma'\bar{\mu}_{14}$, $\sigma'\omega_{14}\eta_{30}$

32
$(\mathbf{Z}_4 \oplus \mathbf{Z}_8) \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$
$\{\phi''', E^2\phi''\}$, $\sigma'\eta_{14}\bar{\mu}_{15}$, $\mu_{3,7}$, $\eta_7\bar{\mu}_8\sigma_{25}$

33	34
$\mathbf{Z}_8 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$	$\mathbf{Z}_8 \oplus \mathbf{Z}_8 \oplus \mathbf{Z}_2$
$\bar{\zeta}_7\sigma_{26}$, $\sigma'\omega_{14}\nu_{30}$, $\bar{\kappa}_7\nu_{27}^2$, $\bar{\sigma}_7\sigma_{26}$, $\phi_7\nu_{30}$, $\nu_7^2\bar{\kappa}_{13}$, $\eta_7\mu_{3,8}$	$\sigma'\bar{\kappa}_{14}$, $\zeta_{3,7}$, $\bar{\nu}_7\bar{\sigma}_{15}$

35	36
$\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$	$\mathbf{Z}_8 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$
$\phi'''\nu_{32}$, $\sigma'\eta_{14}\bar{\kappa}_{15}$, $\bar{\nu}_7\bar{\kappa}_{15}$, $\varepsilon_7\bar{\kappa}_{15}$	δ'', $\sigma'\varepsilon_{14}\kappa_{22}$, $\sigma'\omega_{14}\nu_{30}^2$, $\phi_7\nu_{30}^2$, $\eta_7\varepsilon_8\bar{\kappa}_{16}$

37
$\mathbf{Z}_8 \oplus \mathbf{Z}_8 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$
$\sigma'\overline{\rho}_{14},\ \nu_7\sigma_{10}\overline{\kappa}_{17},\ \sigma'\psi_{14},\ \phi_7\sigma_{30}$

(2) *The homomorphisms* $\Delta'_{i+1} : \pi^{23}_{i+1} \to \pi^{7}_{i-1}$ *are monomorphisms for* $i = 23, 24, 26, 27, 32, 34, 35$. *For other values of* $i\,(22 \le i \le 37)$ *we have the following table of the kernel of* Δ'_{i+1}.

i	22	25	28	29	30	31	33	36	37
Ker Δ'_{i+1}	$\mathbf{Z}$	$\mathbf{Z}_8$	$\mathbf{Z}_2$	$\mathbf{Z}_{16}$	$\mathbf{Z}_2 \oplus \mathbf{Z}_2$	$\mathbf{Z}_2 \oplus \mathbf{Z}_2$	$\mathbf{Z}_8$	$\mathbf{Z}_2 \oplus \mathbf{Z}_2$	$\mathbf{Z}_{32} \oplus \mathbf{Z}_2$
generator	$8\iota_{23}$	ν_{23}	ν^2_{23}	σ_{23}	$\overline{\nu}_{23},\ \varepsilon_{23}$	$\nu^3_{23},\ \varepsilon_{23}\eta_{31}$	ζ_{23}	$\sigma^2_{23},\ \kappa_{23}$	$\rho_{23},\ \overline{\varepsilon}_{23}$

From the exact sequence (2), we have the following short exact sequence

$$0 \to \operatorname{Coker} \Delta_{i+1} \xrightarrow{i_*} \pi_i(\Omega \textstyle\prod : 2) \xrightarrow{p_*} \operatorname{Ker} \Delta_i \to 0,$$

from which we will calculate $\pi_i(\Omega \prod : 2)$. We remark that

$$\operatorname{Ker} \Delta_i = \operatorname{ad}(\iota_{23})E^{-1}(\operatorname{Ker} \Delta'_{i+1}) \quad \text{and} \quad \operatorname{Coker} \Delta_{i+1} = \operatorname{Coker} \Delta'_{i+2}$$

for dimensions under consideration.

Theorem 5.3. *We have the following table of homotopy groups* $\pi_i(\Omega \prod : 2)$ *for* $i \le 37$.

i	$i \le 6$	7	8	9	10	11, 12	13
$\pi_i(\Omega\prod : 2)$	0	$\mathbf{Z}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}_8$	0	$\mathbf{Z}_2$
generator		$i_*(\iota_7)$	$i_*(\eta_7)$	$i_*(\eta_7^2)$	$i_*(\nu_7)$		$i_*(\nu_7^2)$

14	15	16
$\mathbf{Z}_8$	$\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$	$\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$
$i_*(\sigma')$	$i_*(\sigma'\eta_{14})$, $i_*(\bar{\nu}_7)$, $i_*(\varepsilon_7)$	$i_*(\sigma'\eta_{14}^2)$, $i_*(\nu_7^3)$, $i_*(\mu_7)$, $i_*(\eta_7\varepsilon_8)$

17	18	19	20	21
$\mathbf{Z}_8 \oplus \mathbf{Z}_2$	$\mathbf{Z}_8 \oplus \mathbf{Z}_2$	0	$\mathbf{Z}_2$	$\mathbf{Z}_4$
$i_*(\nu_7\sigma_{10})$, $i_*(\eta_7\mu_8)$	$i_*(\zeta_7)$, $i_*(\bar{\nu}_7\nu_{15})$		$i_*(\nu_7\sigma_{10}\nu_{17})$	$i_*(\kappa_7)$

22	23
$\mathbf{Z} \oplus \mathbf{Z}_8 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$	$\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$
$[8\iota_{23}]$, $i_*(\rho'')$, $i_*(\sigma'\bar{\nu}_{14})$, $i_*(\bar{\varepsilon}_7)$	$i_*(\sigma'\mu_{14})$, $i_*(E\zeta')$, $i_*(\mu_7\sigma_{16})$

24	25
$\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$	$\mathbf{Z}_{64} \oplus \mathbf{Z}_2$
$i_*(\sigma'\eta_{14}\mu_{15})$, $i_*(\nu_7\kappa_{10})$, $i_*(\bar{\mu}_7)$, $i_*(\eta_7\mu_8\sigma_{17})$	$[\nu_{23}]$, $i_*(\eta_7\bar{\mu}_8)$

26	27	28
$\mathbf{Z}_8 \oplus \mathbf{Z}_2$	$\mathbf{Z}_8$	$\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$
$i_*(\bar{\zeta}_7)$, $i_*(\bar{\sigma}_7)$	$i_*(\bar{\kappa}_7)$	$[\nu_{23}]\nu_{25}$, $i_*(\eta_7\bar{\kappa}_8)$, $i_*(\sigma'\kappa_{14})$

29
$\mathbf{Z}_{128} \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$
$[\sigma_{23}]$, $i_*(\bar{\nu}_7\kappa_{15})$, $i_*(\varepsilon_7\kappa_{15})$, $i_*(\nu_7\bar{\sigma}_{10})$

30
$\mathbf{Z}_8 \oplus \mathbf{Z}_8 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$
$i_*(\nu_7\bar{\kappa}_{10})$, $i_*(\bar{\rho}')$, $[\bar{\nu}_{23}]$, $[\varepsilon_{23}]$, $i_*(\phi_7)$, $i_*(\bar{\kappa}_7\nu_{27} - \nu_7\bar{\kappa}_{10})$, $i_*(\sigma'\omega_{14})$

31
$\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$
$[\nu_{23}]\nu_{25}^2$, $[\varepsilon_{23}]\eta_{31}$, $i_*(\delta_7)$, $i_*(\bar{\mu}_7\sigma_{24})$, $i_*(\nu_7\sigma_{10}\kappa_{17})$, $i_*(\bar{\sigma}'_7)$, $i_*(\sigma'\bar{\mu}_{14})$, $i_*(\sigma'\omega_{14}\eta_{30})$

32
$(\mathbf{Z}_4 \oplus \mathbf{Z}_8) \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$
$\{i_*(\phi'''), i_*(E^2\phi'')\}$, $i_*(\sigma'\eta_{14}\bar{\mu}_{15})$, $i_*(\mu_{3,7})$, $i_*(\eta_7\bar{\mu}_8\sigma_{25})$

33
$\mathbf{Z}_{64} \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$
$[\zeta_{23}]$, $i_*(\sigma'\omega_{14}\nu_{30})$, $i_*(\bar{\kappa}_7\nu_{27}^2)$, $i_*(\bar{\sigma}_7\sigma_{26})$, $i_*(\phi_7\nu_{30})$, $i_*(\nu_7^2\bar{\kappa}_{13})$, $i_*(\eta_7\mu_{3,8})$

34	35
$\mathbf{Z}_8 \oplus \mathbf{Z}_8 \oplus \mathbf{Z}_2$	$\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$
$i_*(\sigma'\bar{\kappa}_{14})$, $i_*(\zeta_{3,7})$, $i_*(\bar{\nu}_7\bar{\sigma}_{15})$	$i_*(\phi'''\nu_{32})$, $i_*(\sigma'\eta_{14}\bar{\kappa}_{15})$, $i_*(\bar{\nu}_7\bar{\kappa}_{15})$, $i_*(\varepsilon_7\bar{\kappa}_{15})$

36
$\mathbf{Z}_{16} \oplus \mathbf{Z}_4 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$
$[\sigma_{23}^2]$, $[\kappa_{23}]$, $i_*(\sigma'\varepsilon_{14}\kappa_{22})$, $i_*(\sigma'\omega_{14}\nu_{30}^2)$, $i_*(\eta_7\varepsilon_8\overline{\kappa}_{16})$

37
$\mathbf{Z}_{256} \oplus \mathbf{Z}_8 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$
$[\rho_{23}]$, $i_*(\nu_7\sigma_{10}\overline{\kappa}_{17})$, $[\overline{\varepsilon}_{23}]$, $i_*(\sigma'\psi_{14})$, $i_*(\phi_7\sigma_{30})$

Here we denote by $[\alpha]$ *an element of* $\pi_i(\Omega\prod : 2)$ *such that* $p_*([\alpha]) = \mathrm{ad}(\alpha) \in \pi_i(\Omega S^{23} : 2)$. *The following relations hold*:

$$8[\nu_{23}] = i_*(a_1\zeta_7\sigma_{18}),$$

$$16[\sigma_{23}] = i_*(a_2\sigma'\rho_{14}),$$

$$8[\zeta_{23}] = i_*(a_3\overline{\zeta}_7\sigma_{26}),$$

$$2[\sigma_{23}^2] = i_*(\delta''),$$

$$2[\kappa_{23}] = i_*(\phi_7\nu_{30}^2 + b_1\sigma'\omega_{14}\nu_{30}^2),$$

$$32[\rho_{23}] = i_*(a_4\sigma'\overline{\rho}_{14} + 4b_2\nu_7\sigma_{10}\overline{\kappa}_{17} + b_3\sigma'\psi_{14} + b_4\sigma'\phi_{14}),$$

where a_i *is an odd integer and* $b_j = 0$ *or* 1.

Proof. The homomorphisms $i_* : \mathrm{Coker}\,\Delta_{i+1} \to \pi_i(\Omega\prod : 2)$ are isomorphisms for $i \leq 21$ and $i = 23, 24, 26, 27, 32, 34, 35$.

Consider the case $i = 22$. By Lemma 5.2, we have an exact sequence

$$0 \to \mathbf{Z}_8 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \xrightarrow{i_*} \pi_{22}(\Omega\prod : 2) \xrightarrow{p_*} \mathbf{Z} \to 0.$$

Since $\mathbf{Z}$ is free, this exact sequence splits.

Consider the case $i = 25$. By Lemma 5.2, we have an exact sequence

$$0 \to \mathbf{Z}_8 \oplus \mathbf{Z}_2 \xrightarrow{i_*} \pi_{25}(\Omega \prod : 2) \xrightarrow{p_*} \mathbf{Z}_8 \to 0,$$

where $\mathbf{Z}_8 \oplus \mathbf{Z}_2$ is generated by $\zeta_7\sigma_{18}$, $\eta_7\overline{\mu}_8$ and $\mathbf{Z}_8$ is generated by $\mathrm{ad}(\nu_{23})$. We consider the Toda bracket $\{\sigma'\sigma_{14}, \nu_{21}, 8\iota_{24}\}$. Then, by Lemma 3.1(1), we have

$$\{\sigma'\sigma_{14}, \nu_{21}, 8\iota_{24}\} \supset \sigma'\{\sigma_{14}, \nu_{21}, 8\iota_{24}\} \ni x\sigma'\zeta_{14}$$

for some odd integer x. By Lemma 12.12 of [15], we have $\sigma'\zeta_{14} = x'\zeta_7\sigma_{18}$ for some odd integer x'. So we have

$$\{\sigma'\sigma_{14}, \nu_{21}, 8\iota_{24}\} \ni x''\zeta_7\sigma_{18}$$

for some odd integer x''. Therefore, by Theorem 4.3, there exists an element $[\nu_{23}] \in \pi_{25}(\Omega \prod : 2)$ such that

$$p_*([\nu_{23}]) = \mathrm{ad}(\nu_{23}) \quad \text{and} \quad i_*(x''\zeta_7\sigma_{18}) = 8[\nu_{23}].$$

Therefore, we have $\pi_{25}(\Omega \prod : 2) \cong \mathbf{Z}_{64} \oplus \mathbf{Z}_2$.

By an argument similar to the case of $\pi_{25}(\Omega \prod : 2)$, we obtain $\pi_{29}(\Omega \prod : 2)$ from Lemma 3.3(2).

Consider the case $i = 28$. By Lemma 5.2, we have an exact sequence

$$0 \to \mathbf{Z}_2 \oplus \mathbf{Z}_2 \xrightarrow{i_*} \pi_{28}(\Omega \prod : 2) \xrightarrow{p_*} \mathbf{Z}_2 \to 0,$$

where $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ is generated by $\eta_7\overline{\kappa}_8$, $\sigma'\kappa_{14}$ and $\mathbf{Z}_2$ is generated by $\mathrm{ad}(\nu_{23}^2)$. By Theorem 4.3 and Lemma 3.3(1), there exists an element $\varepsilon \in \pi_{28}(\Omega \prod : 2)$ such that $p_*(\varepsilon) = \mathrm{ad}(\nu_{23}^2)$ and $2\varepsilon = 0$. Since $[\nu_{23}]\nu_{25} - \varepsilon \in \mathrm{Im}\, i_*$, we have $2[\nu_{23}]\nu_{25} = 0$. Therefore, we can choose $[\nu_{23}]\nu_{25}$ as a generator. Then we obtain the required result.

Consider the case $i = 30$. By Lemma 5.2, we have an exact sequence

$$0 \to \mathbf{Z}_8 \oplus \mathbf{Z}_8 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \xrightarrow{i_*} \pi_{30}(\Omega \prod : 2) \xrightarrow{p_*} \mathbf{Z}_2 \oplus \mathbf{Z}_2 \to 0,$$

where $\mathbf{Z}_8 \oplus \mathbf{Z}_8 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$ is generated by $\nu_7\bar{\kappa}_{10}$, $\bar{\rho}'$, ϕ_7, $\bar{\kappa}_7\nu_{27} - \nu_7\bar{\kappa}_{10}$, $\sigma'\omega_{14}$ and $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ is generated by $\mathrm{ad}(\bar{\nu}_{23})$, $\mathrm{ad}(\varepsilon_{23})$. By Lemma 3.3(3), we have

$$\{\sigma'\sigma_{14}, \bar{\nu}_{21}, 2\iota_{29}\} \ni 0.$$

Therefore, by Theorem 4.3, there exists an element $[\bar{\nu}_{23}] \in \pi_{30}(\Omega\prod : 2)$ such that

$$p_*([\bar{\nu}_{23}]) = \mathrm{ad}(\bar{\nu}_{23}) \quad \text{and} \quad 2[\bar{\nu}_{23}] = 0.$$

By Lemma 3.1(3), we have

$$\{\sigma'\sigma_{14}, \varepsilon_{21}, 2\iota_{29}\} \supset \sigma'\{\sigma_{14}, \varepsilon_{21}, 2\iota_{29}\} \ni 0.$$

Therefore, by Theorem 4.3, there exists an element $[\varepsilon_{23}] \in \pi_{30}(\Omega\prod : 2)$ such that

$$p_*([\varepsilon_{23}]) = \mathrm{ad}(\varepsilon_{23}) \quad \text{and} \quad 2[\varepsilon_{23}] = 0.$$

Therefore, the above sequence splits.

Consider the case $i = 31$. By Lemma 5.2, we have an exact sequence

$$0 \to \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \xrightarrow{i_*} \pi_{31}(\Omega\prod : 2) \xrightarrow{p_*} \mathbf{Z}_2 \oplus \mathbf{Z}_2 \to 0,$$

where $\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$ is generated by δ_7, $\bar{\mu}_7\sigma_{24}$, $\nu_7\sigma_{10}\kappa_{17}$, $\bar{\sigma}'_7$, $\sigma'\bar{\mu}_{14}$, $\sigma'\omega_{14}\eta_{30}$ and $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ is generated by $\mathrm{ad}(\nu_{23}^3)$, $\mathrm{ad}(\varepsilon_{23}\eta_{31})$. We consider $[\nu_{23}]\nu_{25}^2$ and $[\varepsilon_{23}]\eta_{30}$. Then we have

$$p_*([\nu_{23}]\nu_{25}^2) = \mathrm{ad}(\nu_{23}^3),\ 2[\nu_{23}]\nu_{25}^2 = 0,$$

$$p_*([\varepsilon_{23}]\eta_{30}) = \mathrm{ad}(\varepsilon_{23}\eta_{31}),\ 2[\varepsilon_{23}]\eta_{30} = 0.$$

Therefore, the above sequence splits.

Consider the case $i = 33$. By Lemma 5.2, we have an exact sequence

$$0 \to \mathbf{Z}_8 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \xrightarrow{i_*} \pi_{33}(\Omega\Pi : 2) \xrightarrow{p_*} \mathbf{Z}_8 \to 0,$$

where $\mathbf{Z}_8 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$ is generated by $\overline{\zeta}_7\sigma_{26}$, $\sigma'\omega_{14}\nu_{30}$, $\overline{\kappa}_7\nu_{27}^2$, $\overline{\sigma}_7\sigma_{26}$, $\phi_7\nu_{30}$, $\nu_7^2\overline{\kappa}_{13}$, $\eta_7\mu_{3,8}$ and $\mathbf{Z}_8$ is generated by $\mathrm{ad}(\zeta_{23})$. We consider $\{\sigma'\sigma_{14}, \zeta_{21}, 8\iota_{32}\}$. Since $\pi_{33}^{21} = 0$, we have $\sigma'\sigma_{14}\pi_{33}^{21} + 8\pi_{33}^{7} = 0$, which is the indeterminacy of $\{\sigma'\sigma_{14}, \zeta_{21}, 8\iota_{32}\}$. Therefore, this Toda bracket consists of only one element. We have

$$
\begin{aligned}
\{\sigma'\sigma_{14}, \zeta_{21}, 8\iota_{32}\} &= \sigma'\{\sigma_{14}, \zeta_{21}, 8\iota_{32}\} \\
&= x\sigma'\overline{\zeta}_{14} + y\sigma'\omega_{14}\nu_{30} && \text{by Lemma 3.1(4)} \\
&= x\sigma'\overline{\zeta}_{14} && \text{since } \sigma'\omega_{14}\nu_{30} \text{ is of order two} \\
&= x'\overline{\zeta}_7\sigma_{26} && \text{by Part I (8.14) of [13],}
\end{aligned}
$$

where x, x' are odd integers and y is an even integer. Therefore, by Theorem 4.3, there exists an element $[\zeta_{23}] \in \pi_{33}(\Omega\prod : 2)$ such that

$$p_*([\zeta_{23}]) = \mathrm{ad}(\zeta_{23}) \quad \text{and} \quad i_*\left(x'\overline{\zeta}_7\sigma_{26}\right) = 8[\zeta_{23}].$$

Therefore, we obtain the required result.

Consider the case $i = 36$. By Lemma 5.2, we have an exact sequence

$$0 \to \mathbf{Z}_8 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \xrightarrow{i_*} \pi_{36}(\Omega\prod : 2) \xrightarrow{p_*} \mathbf{Z}_2 \oplus \mathbf{Z}_2 \to 0,$$

where $\mathbf{Z}_8 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$ is generated by δ'', $\sigma'\varepsilon_{14}\kappa_{22}$, $\sigma'\omega_{14}\nu_{30}^2$, $\phi_7\nu_{30}^2$, $\eta_7\varepsilon_8\overline{\kappa}_{16}$ and $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ is generated by $\mathrm{ad}(\sigma_{23}^2)$, $\mathrm{ad}(\kappa_{23})$. By Lemma 3.3(4), we have

$$\{\sigma'\sigma_{14}, \kappa_{21}, 2\iota_{35}\} \ni \phi_7\nu_{30}^2 + x\sigma'\omega_{14}\nu_{30}^2$$

for some integer x. Therefore, by Theorem 4.3, there exists an element $[\kappa_{23}] \in \pi_{36}(\Omega\prod : 2)$ such that

$$p_*([\kappa_{23}]) = \mathrm{ad}(\kappa_{23}) \quad \text{and} \quad 2[\kappa_{23}] = i_*(\phi_7\nu_{30}^{2+x}\sigma'\omega_{14}\nu_{30}^2).$$

By Lemma 3.3(5), we have

$$\{\sigma'\sigma_{14},\ \sigma_{21}^2,\ 2\iota_{35}\} \ni \delta''.$$

Therefore, by Theorem 4.3, there exists an element $[\sigma_{23}^2] \in \pi_{36}(\Omega\prod : 2)$ such that

$$p_*([\sigma_{23}^2]) = \mathrm{ad}(\sigma_{23}^2) \quad \text{and} \quad 2[\sigma_{23}^2] = i_*(\delta'').$$

Therefore, we have $\pi_{36}(\Omega\prod : 2) \cong \mathbf{Z}_{16} \oplus \mathbf{Z}_4 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$.

Consider the case $i = 37$. By Lemma 5.2, we have an exact sequence

$$0 \to \mathbf{Z}_8 \oplus \mathbf{Z}_8 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \xrightarrow{i_*} \pi_{37}(\Omega\prod : 2) \xrightarrow{p_*} \mathbf{Z}_{32} \oplus \mathbf{Z}_2 \to 0,$$

where $\mathbf{Z}_8 \oplus \mathbf{Z}_8 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$ is generated by $\sigma'\overline{\rho}_{14}$, $\nu_7\sigma_{10}\overline{\kappa}_{17}$, $\sigma'\psi_{14}$, $\phi_7\sigma_{30}$ and $\mathbf{Z}_{32} \oplus \mathbf{Z}_2$ is generated by $\mathrm{ad}(\rho_{23}), \mathrm{ad}(\overline{\varepsilon}_{23})$. By Lemma 3.1(6), we have

$$\{\sigma'\sigma_{14},\ \overline{\varepsilon}_{21},\ 2\iota_{36}\} \supset \sigma'\{\sigma_{14},\ \overline{\varepsilon}_{21},\ 2\iota_{36}\} \ni 0.$$

Therefore, by Theorem 4.3, there exists an element $[\overline{\varepsilon}_{23}] \in \pi_{37}(\Omega\prod : 2)$ such that

$$p_*([\overline{\varepsilon}_{23}]) = \mathrm{ad}(\overline{\varepsilon}_{23}) \quad \text{and} \quad 2[\overline{\varepsilon}_{23}] = 0.$$

By Theorem 4.3 and Lemma 3.3(6), there exists an element $[\rho_{23}] \in \pi_{37}(\Omega\prod : 2)$ such that

$$p_*([\rho_{23}]) = \mathrm{ad}(\rho_{23})$$

and

$$i_*(x\sigma'\overline{\rho}_{14} + 4a\nu_7\sigma_{10}\overline{\kappa}_{17} + b\sigma'\psi_{14} + c\sigma'\phi_{14}) = 32[\rho_{23}]$$

for some odd integer x and some integers a, b, c. Therefore, we have $\pi_{37}(\Omega\prod : 2) \cong \mathbf{Z}_{256} \oplus \mathbf{Z}_8 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$.

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