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A CONVERGENCE THEOREM FOR *I*-QUASI-NONEXPANSIVE MAPPINGS

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Abstract

In this paper, we are concerned with the study of the Ishikawa iterative scheme involving an I-quasi-nonexpansive mapping T. Under some suitable conditions, the scheme is shown to converge strongly to a common fixed point of T and I.

1. Introduction and Preliminaries

Throughout this paper, we denote the set of all fixed points of a mapping T by $F(T)$ and $T^0 = E$, where E denotes the mapping $E : C \to C$ defined by $Ex = x$, respectively.

Let C be a closed convex bounded subset of a real normed linear space X. Then T is called *nonexpansive* on C if

$$\| Tx - Ty \| \leq \| x - y \|, \tag{1.1}$$

for all $x, y \in K$.

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In 1941, Tricomi introduced the quasi-nonexpansive mapping for real functions. Diaz and Metcalf [1] and Dotson Jr. [2] studied quasi-nonexpansive mappings in Banach spaces. In 1997, Kirk [4] introduced the quasi-nonexpansive mapping in metric spaces. In 2006, Rhoades and Temir [7] generalized the concept to a normed space as follows: T is said to be a *quasi-nonexpansive mapping* if

$$\| Tx - f \| \leq \| x - f \|, \tag{1.2}$$

for all $x \in C$ and $f \in F(T)$.

Remark 1.1. From the above definitions it is easy to see that if $F(T)$ is nonempty, then a nonexpansive mapping must be quasi-nonexpansive. But the converse does not hold.

Definition 1.1. Let C be a subset of X, and T and I be self-mappings of C. Then T is said to be *I-quasi-nonexpansive* on C if $\| Tx - f \| \leq \| Ix - f \|$, for all $x \in C$ and $f \in F(T) \cap F(I)$.

For studying common fixed points of a iterative scheme involving an I-quasi-nonexpansive mapping T and the nonexpansive mapping I, we quoted the Ishikawa scheme as follows:

Definition 1.2. Let $T : C \to C$ be an I-quasi-nonexpansive mapping and I be a nonexpansive mapping on C, where C is nonempty closed and convex subset of a Banach space X. Then an *iterative scheme* is a pair of sequences $\{x_n\}$ and $\{y_n\}$ defined by, for a given $x_0 \in C$,

$$\begin{aligned} y_n &= a'_n T x_n + b'_n x_n, \\ x_{n+1} &= a_n T y_n + b_n x_n, \end{aligned} \quad n \geq 0, \tag{1.3}$$

where $\{a_n\}$, $\{b_n\}$, $\{a'_n\}$ and $\{b'_n\}$ are real sequences in $(0, 1)$ with $a_n + b_n = 1 = a'_n + b'_n$.

For approximating fixed points of deterministic nonexpansive mappings, Senter and Dotson [9] introduced a Condition (A). Later on, Maiti and Ghosh [5] and Tan and Xu [10] studied Condition (A) and pointed out that Condition (A) is weaker than the requirement of semi-compactness on mappings.

Definition 1.3. A mapping $T : C \to C$ is said to satisfy Condition (A) if there exists a nondecreasing function $f : [0, +\infty) \to [0, +\infty)$ with $f(0) = 0$, $f(r) > 0$, for all $r \in (0, +\infty)$ such that $\| x - Tx \| \geq f(d(x, F(T)))$ for all $x \in C$, where $d(x, F(T)) = \inf\{d(x, x^*) : x^* \in F(T)\}$, $F(T)$ is the fixed point set of T.

There are a number of recent results on fixed points of nonexpansive and quasi-nonexpansive mappings in Banach spaces and metric spaces. For example, the strong and weak convergences of the sequence of certain iterates to a fixed point of quasi-nonexpansive maps were studied by Petryshyn and Williamson Jr. [6]. Their analysis was connected with the convergence of Mann iterates studied by Dotson Jr. [2]. Subsequently, the convergence of Ishikawa iterates of quasi-nonexpansive mappings in Banach spaces was discussed by Ghosh and Debnath [3]. In [11], the weakly convergence theorem for I-asymptotically quasi-nonexpansive mapping defined in Hilbert space was proved. And in 2006, Rhoades and Temir [7] proved the weak convergence of the sequence of Mann iterates to a common fixed point of an I-nonexpansive mapping T and a nonexpansive mapping I. But the strongly convergence theorem for I-nonexpansive mapping has not been studied, yet.

The aim of our study is to prove a strongly convergence theorem for an I-quasi-nonexpansive mapping in a uniformly convex Banach space. We establish the strongly convergence of the sequence of iterates (1.3) to a common fixed point of T and I if and only if $\liminf_{n\to\infty} d(x_n, F(T) \cap F(I))$. Our theorems improve and generalize some previous results.

We restate the following lemma which plays important roles in our proofs.

Lemma 1.1 [8]. *Suppose that X is a uniformly convex Banach space and $0 < p \leq t_n \leq q < 1$ for all positive integers n. Also suppose that $\{x_n\}$ and $\{y_n\}$ are two sequences in X such that $\limsup_{n\to\infty} \| x_n \| \leq r$, $\limsup_{n\to\infty} \| y_n \| \leq r$ and $\limsup_{n\to\infty} \| t_n x_n + (1 - t_n) y_n \| = r$ hold for some $r \geq 0$. Then $\lim_{n\to\infty} \| x_n - y_n \| = 0$.*

2. Main Results

Lemma 2.1. *Let C be a nonempty closed convex subset of a normed space X. Let $T : C \to C$ be an I-quasi-nonexpansive mapping and I be a nonexpansive mapping on C. Suppose the sequence $\{x_n\}$ is generated by* (1.3). *If $F_1 \doteq F(T) \cap F(I) \neq \varnothing$, then $\lim_{n\to\infty} \| x_n - x^* \|$ exists for any $x^* \in F_1$.*

Proof. For any $x^* \in F_1$,

$$\begin{aligned}
0 &\leq \| x_{n+1} - x^* \| \\
&= \| a_n T y_n + (1 - a_n) x_n - x^* \| \\
&= \| a_n (T y_n - x^*) + (1 - a_n)(x_n - x^*) \| \\
&\leq a_n \| I y_n - x^* \| + (1 - a_n) \| x_n - x^* \| \\
&\leq a_n \| y_n - x^* \| + (1 - a_n) \| x_n - x^* \| \\
&= a_n \| a'_n (T x_n - x^*) + (1 - a'_n)(x_n - x^*) \| + (1 - a_n) \| x_n - x^* \| \\
&\leq a_n a'_n \| I x_n - x^* \| + a_n (1 - a'_n) \| x_n - x^* \| + (1 - a_n) \| x_n - x^* \| \\
&\leq a_n a'_n \| x_n - x^* \| + a_n (1 - a'_n) \| x_n - x^* \| + (1 - a_n) \| x_n - x^* \| \\
&\leq \| x_n - x^* \|.
\end{aligned}$$

Thus the sequence $\{\| x_n - x^* \|\}$ is decreasing, so $\lim_{n\to\infty} \| x_n - x^* \|$ exists for any $x^* \in F_1$. It also implies that $\{x_n\}$ is bounded. The proof is completed.

Lemma 2.2. *Let X be a uniformly convex Banach space and C be a nonempty closed convex subset of X. Let T, I and $\{x_n\}$ be same as in Lemma* 2.1. *If $F_1 \neq \varnothing$, then $\lim_{n\to\infty} \| T x_n - x_n \| = 0$.*

Proof. By Lemma 2.1, for any $x^* \in F_1$, $\lim_{n\to\infty}\| x_n - x^* \|$ exists. Assume $\lim_{n\to\infty}\| x_n - x^* \| = c \geq 0$. Since

$$\begin{aligned} 0 &\leq \| y_n - x^* \| \\ &= \| \alpha'_n Tx_n + (1 - \alpha'_n) x_n - x^* \| \\ &\leq \alpha'_n \| Tx_n - x^* \| + (1 - \alpha'_n) \| x_n - x^* \| \\ &\leq \alpha'_n \| Ix_n - x^* \| + (1 - \alpha'_n) \| x_n - x^* \| \\ &\leq \alpha'_n \| x_n - x^* \| + (1 - \alpha'_n) \| x_n - x^* \| \\ &\leq \| x_n - x^* \|, \end{aligned}$$

taking lim sup on both sides in above inequality, we have

$$\limsup_{n\to\infty} \| y_n - x^* \| \leq c. \tag{2.1}$$

Since T is I-quasi-nonexpansive,

$$\| Ty_n - x^* \| \leq \| Iy_n - x^* \| \leq \| y_n - x^* \|,$$

which implies that $\limsup_{n\to\infty} \| Ty_n - x^* \| \leq c$.

Further, $\lim_{n\to\infty} \| x_{n+1} - x^* \| = c$ means that

$$\lim_{n\to\infty} \| a_n (Ty_n - x^*) + b_n (x_n - x^*) \| = c.$$

Applying Lemma 1.1, we obtain

$$\lim_{n\to\infty} \| Ty_n - x_n \| = 0. \tag{2.2}$$

Next,

$$\begin{aligned} \| x_n - x^* \| &\leq \| x_n - Ty_n \| + \| Ty_n - x^* \| \\ &\leq \| x_n - Ty_n \| + \| Iy_n - x^* \| \\ &\leq \| x_n - Ty_n \| + \| y_n - x^* \| \end{aligned}$$

gives

$$c = \lim_{n\to\infty} \| x_n - x^* \| \le \liminf_{n\to\infty} \| y_n - x^* \|. \tag{2.3}$$

By (2.1) and (2.3),

$$\lim_{n\to\infty} \| y_n - x^* \| = c. \tag{2.4}$$

Now, $\lim_{n\to\infty} \| y_n - x^* \| = c$ is expressible as

$$\lim_{n\to\infty} \| a'_n (Tx_n - x^*) + b'_n (x_n - x^*) \| = c.$$

In addition, we have

$$\limsup_{n\to\infty} \| Tx_n - x^* \| \le \limsup_{n\to\infty} \| Ix_n - x^* \|$$
$$\le \lim_{n\to\infty} \| x_n - x^* \| = c.$$

Hence, it follows from Lemma 1.1 that

$$\lim_{n\to\infty} \| Tx_n - x_n \| = 0. \tag{2.5}$$

The proof is completed.

Theorem 2.3. *Let X be a uniformly convex Banach space and C, T, I and $\{x_n\}$ be same as in Lemma* 2.2.

(1) *If $F_1 = F(T) \cap F(I) \neq \emptyset$, then F_1 is a closed set.*

(2) *$\{x_n\}$ converges strongly to a common fixed point of T and I if and only if* $\liminf_{n\to\infty} d(x_n, F_1) = 0$.

Proof. (1) Let $\{\xi_n\} \subset F_1$ be such that $\xi_n \to x$ as $n \to \infty$. Then

$$\| Tx - x \| = \| Tx - \xi_n + \xi_n - x \| \le \| Ix - \xi_n \| + \| \xi_n - x \| \le 2\| \xi_n - x \|.$$

This implies that $Tx = x$. Also,

$$\| Ix - x \| = \| Ix - \xi_n + \xi_n - x \| \le 2\| \xi_n - x \|.$$

So, x is a common fixed point of T and I. Thus F_1 is a closed set.

(2) Suppose that $\{x_n\}$ converges strongly to a fixed point q of F_1. Then $\lim_{n\to\infty} \| x_n - q \| = 0$. Since $0 \le d(x_n, F_1) \le \| x_n - q \|$, we have $\liminf_{n\to\infty} d(x_n, F_1) = 0$.

Conversely, suppose that $\liminf_{n\to\infty} d(x_n, F_1) = 0$. For any $x^* \in F_1$, by Lemma 2.1, we have $0 \leq \| x_{n+1} - x^* \| \leq \| x_n - x^* \|$. Thus,

$$0 \leq d(x_{n+1}, F_1) \leq d(x_n, F_1).$$

So, $\lim_{n\to\infty} d(x_n, F_1)$ exists. Furthermore, since $\liminf_{n\to\infty} d(x_n, F_1) = 0$, we have

$$\lim_{n\to\infty} d(x_n, F_1) = 0. \tag{2.6}$$

We now prove that $\{x_n\}$ is a Cauchy sequence.

For any $\varepsilon > 0$, since $\lim_{n\to\infty} d(x_n, F_1) = 0$, there exists natural number N_1 such that when $n \geq N_1$, $d(x_n, F_1) < \frac{\varepsilon}{3}$. Thus, there exists $x^* \in F_1$ such that for above ε there exists positive integer $N_2 \geq N_1$ such that as $n \geq N_2$

$$\| x_n - x^* \| < \frac{\varepsilon}{2}.$$

Now for arbitrary $n, m \geq N_2$, consider

$$\| x_n - x_m \| \leq \| x_n - x^* \| + \| x_m - x^* \| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This implies $\{x_n\}$ is a Cauchy sequence. Therefore, there exists $p \in C$ such that $\{x_n\}$ converges strongly to p. In addition, $\lim_{n\to\infty} x_n = p$ and $\lim_{n\to\infty} d(x_n, F_1) = 0$ give that $d(p, F_1) = 0$, F_1 is closed as indicated in Theorem 2.3(1), therefore $p \in F_1$. The proof is completed.

Corollary 2.4. *Let X be a uniformly convex Banach space and* C, T, I *and* $\{x_n\}$ *be same as in Lemma* 2.1. *If* $F_1 \neq \emptyset$, T *satisfies Condition* (A) *and* $\liminf_{n\to\infty} d(x_n, F(I)) = 0$, *then* $\{x_n\}$ *converges to a common random fixed point of T and I.*

Proof. For any $x^* \in F_1$, by Lemma 2.1, for each $\omega \in \Omega$, $\lim_{n\to\infty} \| x_n(\omega) - x^*(\omega) \|$ exists. Let it be c for some $c \geq 0$. If $c = 0$, then there is nothing to prove.

If $c > 0$, then by Lemma 2.2, $\lim_{n\to\infty} \| Tx_n - x_n \| = 0$. Moreover,

$$\| x_{n+1} - x^* \| \leq \| x_n - x^* \|$$

gives that

$$\inf_{x^* \in F_1} \| x_{n+1} - x^* \| \leq \inf_{x^* \in F_1} \| x_n - x^* \|.$$

That implies that $0 \leq d(x_{n+1}, F_1) \leq d(x_n, F_1)$. Thus $\lim_{n\to\infty} d(x_n, F_1)$ exists.

Since T satisfies Condition (A), we have

$$\| x_n - Tx_n \| \geq f(d(x_n, F(T))).$$

It follows from (2.5) that we have $\lim_{n\to\infty} f(d(x_n, F(T))) \leq 0$. Since f is a nondecreasing function with $f(0) = 0$, therefore, $\lim_{n\to\infty} d(x_n, F(T)) = 0$.

Further, by $\liminf_{n\to\infty} d(x_n, F(I)) = 0$ and $F_1 \neq \emptyset$, we get

$$\liminf_{n\to\infty} d(x_n, F_1) = 0.$$

It follows from Theorem 2.3 that $\{x_n\}$ converges to the common random fixed point of T and I. This completes the proof.

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