JP Journal of Fixed Point Theory and Applications
Volume 2, Number 2, 2007, Pages 139-148
Published online: January 31, 2008
This paper is available online at http://www.pphmj.com

TRIPLE POSITIVE PSEUDO-SYMMETRIC SOLUTIONS TO A THREE-POINT SECOND-ORDER INTEGRODIFFERENTIAL BOUNDARY VALUE PROBLEM WITH *p*-LAPLACIAN

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Abstract

In this paper, we use fixed point theorem due to Avery and Peterson to prove the existence of the triple positive pseudo-symmetric solutions to a three-point second-order p-Laplacian integrodifferential boundary value problem.

1. Introduction

The existence of solutions of second order multi-point boundary value problems with p-Laplacian has been studied by many authors using the nonlinear alternative of Leray-Schauder, coincidence degree theory, the upper and lower solution method and fixed point theorem in cones

2000 Mathematics Subject Classification: 34B10, 34B18.

Keywords and phrases: fixed point theorem, positive pseudo-symmetric solutions, p-Laplacian, cone.

Received August 18, 2007

(see [1-2, 4-7] and references therein). Very recently, by the monotone iterative technique, Ahmad and Nieto [1] studied the existence of triple positive pseudo-symmetric solutions for the following three-point boundary value problem with p-Laplacian

$$(\psi_p(x'(t)))' + a(t)\left\{f(t, x(t)) + \int_t^{\frac{1+\eta}{2}} K(t, \zeta, x(\zeta))d\zeta\right\} = 0, \quad t \in (0, 1),$$

$$x(0) = 0, \ \ x(\eta) = x(1), \ \ 0 < \eta < 1,$$

where $p > 1$, $\psi_p(s) = s|s|^{p-2}$.

In this paper, we consider the following three-point boundary value problems

$$(\phi_p(u'(t)))' + a(t)\left\{f(t, u(t)) + \int_{\frac{\eta}{2}}^{t} H(t, \xi, u(\xi))d\xi\right\} = 0, \quad t \in (0, 1), \tag{1.1}$$

$$u(0) = u(\eta), \ \ u(1) = \beta u'(\eta), \ \ 0 < \eta < 1, \tag{1.2}$$

where $\beta < 0$, $p > 1$, $\phi_p(s) = s|s|^{p-2}$, let ϕ_q be the inverse of ϕ_p. Here, we study the second-order three-point boundary value problems with p-Laplacian under the conditions of that f, a, u and H are pseudo-symmetric in t about $\frac{\eta}{2}$ on $(0, 1)$. To the best of our knowledge, this problem has not been studied before. Our main tool is the fixed point theorem due to Avery and Peterson.

2. Preliminaries and Lemmas

Definition 2.1. A functional $x \in E$ is said to be *pseudo-symmetric* about $\frac{\eta}{2}$ on $[0, 1]$, if x is symmetric over the interval $[0, \eta]$, that is, $x(t) = x(\eta - t)$ for $t \in [0, \eta]$.

Let γ and θ be nonnegative continuous convex functionals on K, α be a nonnegative continuous concave functional on K, and ψ be a nonnegative continuous functional on K. Then for positive real numbers a, b, c and d, we define the following convex sets:

$$K(\gamma, d) = \{x \in K \mid \gamma(x) < d\},$$

$$K(\gamma, a, b, d) = \{x \in K \mid b \le \alpha(x), \gamma(x) \le d\},$$

$$K(\gamma, \theta, \alpha, b, c, d) = \{x \in K \mid b \le \alpha(x), \theta(x) \le c, \gamma(x) \le d\},$$

$$R(\gamma, \psi, a, d) = \{x \in K \mid a \le \psi(x), \gamma(x) \le d\}.$$

Lemma 2.1 [3]. *Let K be a cone in a Banach space E. Let* γ *and* θ *be nonnegative continuous convex functionals on K,* α *be a nonnegative continuous concave functional on K, and* ψ *be a nonnegative continuous functional on K satisfying* $\psi(\lambda x) \le \lambda\psi(x)$ *for* $0 \le \lambda \le 1$, *such that for some positive numbers M and d,*

$$\alpha(x) \le \psi(x) \quad and \quad \| x \| \le M\gamma(x) \tag{2.1}$$

for all $x \in \overline{K(\gamma, d)}$. *Suppose* $T : \overline{K(\gamma, d)} \to \overline{K(\gamma, d)}$ *is completely continuous and there exist positive numbers* a, b *and c with* $a < b$ *such that*

(C_1) $\{x \in K(\gamma, \theta, \alpha, b, c, d) | \alpha(x) > b\} \ne \emptyset$ *and* $\alpha(Tx) > b$ *for* $x \in K(\gamma, \theta, \alpha, b, c, d)$,

(C_2) $\alpha(Tx) > b$ *for* $x \in K(\gamma, \alpha, b, d)$ *with* $\theta(Tx) > c$,

(C_3) $0 \notin R(\gamma, \psi, a, d)$ *and* $\psi(Tx) < a$ *for* $x \in R(\gamma, \psi, a, d)$, *with* $\psi(x) = a$.

Then T has at least three fixed points x_1, x_2 *and* $x_3 \in \overline{K(\gamma, d)}$ *such that*

$$\gamma(x_i) \le d \quad for\ i = 1, 2, 3,$$

$$b < \alpha(x_1), \ a < \psi(x_2) \ with\ \alpha(x_2) < b \ and\ \psi(x_3) < a.$$

Throughout the paper, we assume the following conditions hold:

(H_1) $f : [0, 1] \times [0, \infty) \to [0, \infty)$ is continuous nondecreasing in u, and for any fixed $u \in [0, \infty)$, $f(t, u)$ is pseudo-symmetric in t about $\frac{\eta}{2}$ on $(0, 1)$;

(H_2) $H : [0, 1] \times [0, 1] \times [0, \infty) \to [0, \infty)$ is continuous nondecreasing in u, and for any fixed $(\xi, u) \in [0,1] \times [0, \infty)$, $H(t, \xi, u)$ is pseudo-symmetric in t about $\frac{\eta}{2}$ on $(0, 1)$;

(H_3) $a(t) \in L(0, 1)$ is nonnegative on $(0, 1)$ and pseudo-symmetric in t about $\frac{\eta}{2}$ on $(0, 1)$. Further, $a(t) \not\equiv 0$ on any nontrivial compact subinterval of $(0, 1)$.

Lemma 2.2. *For any* $u \in K$, *we have* $u(t) \geq \sigma \| u \|$, $t \in \left[\frac{\eta}{2}, \eta\right]$, *where* $\sigma = (1 - \eta)\left(1 - \frac{\eta}{2}\right)^{-1}$.

Proof. For any $u \in K$, we define

$$u_\eta(t) = \begin{cases} u(\eta - t), & t \in [\eta - 1, 0], \\ u(t), & t \in [0, 1], \end{cases}$$

and note that u_η is nonnegative, concave and symmetric on $[\eta - 1, 1]$ with $\| u_\eta \| = \| u \|$. It follows from the concavity and symmetry of u_η that

$$u_\eta(t) \geq \begin{cases} \| u \| \left(1 - \frac{\eta}{2}\right)^{-1} (1 - \eta + t), & t \in \left[\eta - 1, \frac{\eta}{2}\right], \\ \| u \| \left(1 - \frac{\eta}{2}\right)^{-1} (1 - t), & t \in \left[\frac{\eta}{2}, 1\right], \end{cases}$$

which, in view of $u_\eta(t) = u(t)$ on $[0, 1]$, yields

$$u(t) \geq \| u \| \left(1 - \frac{\eta}{2}\right)^{-1} \min\{1 - \eta + t, 1 - t\}, \quad t \in [0, 1].$$

So, for $t \in \left[\frac{\eta}{2}, \eta\right]$, $u(t) \geq \| u \| (1 - \eta)\left(1 - \frac{\eta}{2}\right)^{-1}$.

Let $E = C[0,1]$ be Banach space equipped with norm $\| u \| = \max\limits_{0 \leq t \leq 1} | u(t) |$

and K be a cone in E defined by $K = \{u \in E : u$ is nonnegative, concave on $[0, 1]$, pseudo-symmetric about $\frac{\eta}{2}$ on $[0, 1]$ and $\min_{\frac{\eta}{2} \le t \le \eta} u(t) \ge \sigma \| u \|\}$.

Define an operator $T : K \to E$ by

$$(Tu)(t) = \begin{cases} \beta\phi_q\left(\int_\eta^{\frac{\eta}{2}} a(s)\left\{f(s, u(s)) + \int_{\frac{\eta}{2}}^{s} H(s, \xi, u(\xi))d\xi\right\}ds\right) \\ + \int_\eta^1 \phi_q\left(\int_{\frac{\eta}{2}}^{s} a(\tau)\left\{f(\tau, u(\tau)) + \int_{\frac{\eta}{2}}^{\tau} H(\tau, \xi, u(\xi))d\xi\right\}d\tau\right)ds \\ + \int_0^t \phi_q\left(\int_s^{\frac{\eta}{2}} a(\tau)\left\{f(\tau, u(\tau)) + \int_\tau^{\frac{\eta}{2}} H(\tau, \xi, u(\xi))d\xi\right\}d\tau\right)ds, & 0 \le t \le \frac{\eta}{2}, \\ \beta\phi_q\left(\int_\eta^{\frac{\eta}{2}} a(s)\left\{f(s, u(s)) + \int_{\frac{\eta}{2}}^{s} H(s, \xi, u(\xi))d\xi\right\}ds\right) \\ + \int_t^1 \phi_q\left(\int_{\frac{\eta}{2}}^{s} a(\tau)\left\{f(\tau, u(\tau)) + \int_{\frac{\eta}{2}}^{\tau} H(\tau, \xi, u(\xi))d\xi\right\}d\tau\right)ds, & \frac{\eta}{2} \le t \le 1. \end{cases}$$

It is obvious that u is a solution of problems (1.1) and (1.2) if and only if $Tu = u$.

Lemma 2.3. *Suppose* (H_1), (H_2) *and* (H_3) *hold. Then* $T : K \to K$ *is completely continuous.*

Proof. For each $u \in K$, let $v = Tu$. Then,

$$v'(t) = \phi_q\left(\int_t^{\frac{\eta}{2}} a(\tau)\left\{f(\tau, u(\tau)) + \int_\tau^{\frac{\eta}{2}} H(\tau, \xi, u(\xi))d\xi\right\}d\tau\right), \tag{2.2}$$

$$(\phi_p(v'(t)))' = -a(t)\left\{f(t, u(t)) + \int_t^{\frac{\eta}{2}} H(t, \xi, u(\xi))d\xi\right\} \le 0. \tag{2.3}$$

So $v = Tu$ is concave.

Next, we show Tu is pseudo-symmetric about $\frac{\eta}{2}$ on $[0, 1]$. For $t \in \left[\frac{\eta}{2}, \eta\right]$, we have

$$\int_0^{\eta-t} \phi_q\left(\int_s^{\frac{\eta}{2}} a(\tau)\left\{f(\tau, u(\tau)) + \int_\tau^{\frac{\eta}{2}} H(\tau, \xi, u(\xi))d\xi\right\}d\tau\right)ds$$

$$= -\int_\eta^t \phi_q\left(\int_{\eta-s}^{\frac{\eta}{2}} a(\tau)\left\{f(\tau, u(\tau)) + \int_\tau^{\frac{\eta}{2}} H(\tau, \xi, u(\xi))d\xi\right\}d\tau\right)ds$$

$$= \int_t^\eta \phi_q\left(\int_{\frac{\eta}{2}}^s a(\tau)\left\{f(\tau, u(\tau)) + \int_{\eta-\tau}^{\frac{\eta}{2}} H(\tau, \xi, u(\xi))d\xi\right\}d\tau\right)ds$$

$$= \int_t^\eta \phi_q\left(\int_{\frac{\eta}{2}}^s a(\tau)\left\{f(\tau, u(\tau)) + \int_{\frac{\eta}{2}}^\tau H(\tau, \xi, u(\xi))d\xi\right\}d\tau\right)ds.$$

Thus, for any $t \in \left[\frac{\eta}{2}, \eta\right] \left(\eta - t \in \left[0, \frac{\eta}{2}\right]\right)$, we obtain

$$(Tu)(\eta - t) = \beta\phi_q\left(\int_\eta^{\frac{\eta}{2}} a(s)\left\{f(s, u(s)) + \int_{\frac{\eta}{2}}^s H(s, \xi, u(\xi))d\xi\right\}ds\right)$$

$$+ \int_t^1 \phi_q\left(\int_{\frac{\eta}{2}}^s a(\tau)\left\{f(\tau, u(\tau)) + \int_{\frac{\eta}{2}}^\tau H(\tau, \xi, u(\xi))d\xi\right\}d\tau\right)ds$$

$$= (Tu)(t).$$

We can get $(Tu)'\left(\frac{\eta}{2}\right) = 0$ by the symmetry of Tu on $[0, \eta]$. And for $t \in \left[\frac{\eta}{2}, 1\right]$, the concavity of Tu implies that $(Tu)'(t) \le 0$. Therefore,

$$(Tu)(0) = (Tu)(\eta) \ge (Tu)(1) = \beta(Tu)'(\eta) \ge 0.$$

Consequently, we have $(Tu)(t) \ge 0$ as (Tu) is concave. And it is obvious that $\min(Tu)(t) \ge \sigma\| Tu \|$, for $\frac{\eta}{2} \le t \le \eta$. Hence, we obtain that $TK \subseteq K$.

3. Main Result

Let the nonnegative continuous concave functional α on K, the nonnegative continuous convex functionals θ, γ and the nonnegative continuous functional ψ be defined on the cone K by

$$\alpha(u) = \min_{\frac{\eta}{2} \le t \le \eta} |u(t)|, \quad \gamma(u) = \theta(u) = \psi(u) = \max_{0 \le t \le 1} |u(t)|$$

for $u \in K$. Clearly,

$$\sigma\theta(u) \le \alpha(u) \le \theta(u) = \gamma(u) = \psi(u). \tag{3.1}$$

By Lemma 2.2, we have

$$\alpha(u) \ge \sigma \| u \|. \tag{3.2}$$

From (3.1) and (3.2), we know (2.1) hold.

Let

$$L = \beta\phi_q\left(\int_{\eta}^{\frac{\eta}{2}} a(s)ds\right) + \int_{\frac{\eta}{2}}^{1} \phi_q\left(\int_{\frac{\eta}{2}}^{s} a(\tau)d\tau\right)ds, \quad N = \int_{\frac{\eta}{2}}^{1} \phi_q\left(\int_{\frac{\eta}{2}}^{s} a(\tau)d\tau\right)ds.$$

Theorem 3.1. *Suppose* (H_1), (H_2) *and* (H_3) *hold. Let* $0 < a < b \le \sigma d$, *and the following conditions hold,*

(A_1) $f(t, u(t)) + \int_{\frac{\eta}{2}}^{t} H(t, \xi, u(\xi))d\xi \le \phi_p\left(\frac{d}{L}\right)$ *for* $t \in [0, 1]$, $u \in [0, d]$,

(A_2) $f(t, u(t)) + \int_{\frac{\eta}{2}}^{t} H(t, \xi, u(\xi))d\xi < \phi_p\left(\frac{a}{L}\right)$ *for* $t \in [0, 1]$, $u \in [0, a]$,

(A_3) $f(t, u(t)) + \int_{\frac{\eta}{2}}^{t} H(t, \xi, u(\xi))d\xi > \phi_p\left(\frac{b}{\sigma N}\right)$ *for* $t \in \left[\frac{\eta}{2}, \eta\right]$, $u \in \left[b, \frac{b}{\sigma}\right]$.

Then BVP (1.1) *and* (1.2) *have at least three pseudo-symmetric positive solutions* u_1, u_2 *and* u_3 *such that*

$$\max_{0 \le t \le 1} |u_i(t)| \le d \quad \text{for } i = 1, 2, 3, \ b < \min_{\frac{\eta}{2} \le t \le \eta} |u_1(t)|, \ \max_{0 \le t \le 1} |u_1(t)| \le d,$$

$$a < \max_{0 \le t \le 1} |u_2(t)| < \frac{b}{\sigma} \text{ with } \min_{\frac{\eta}{2} \le t \le \eta} |u_2(t)| < b \text{ and } \max_{0 \le t \le 1} |u_3(t)| < a.$$

Proof. Firstly, we check $T:\overline{K(\gamma, d)} \to \overline{K(\gamma, d)}$ is completely continuous operator.

If $u \in \overline{K(\gamma, d)}$, then $\gamma(u) \le d$. From (A_1), we have

$$\gamma(Tu) = \max_{0\le t\le 1}|Tu(t)|$$

$$= \beta\phi_q\left(\int_\eta^{\frac{\eta}{2}} a(s)\left\{f(s, u(s)) + \int_{\frac{\eta}{2}}^s H(s, \xi, u(\xi))d\xi\right\}ds\right)$$

$$+ \int_{\frac{\eta}{2}}^1 \phi_q\left(\int_{\frac{\eta}{2}}^s a(\tau)\left\{f(\tau, u(\tau)) + \int_{\frac{\eta}{2}}^\tau H(\tau, \xi, u(\xi))d\xi\right\}d\tau\right)ds$$

$$\le \frac{d}{L}\beta\phi_q\left(\int_\eta^{\frac{\eta}{2}} a(s)ds\right) + \frac{d}{L}\int_{\frac{\eta}{2}}^1 \phi_q\left(\int_{\frac{\eta}{2}}^s a(\tau)d\tau\right)ds = \frac{d}{L}L = d.$$

Therefore, $T : \overline{K(\gamma, d)} \to \overline{K(\gamma, d)}$. Standard applications of Arzela-Ascoli Theorem imply that T is completely continuous operator.

We choose $u(t) = \frac{4bt}{\eta}$, $0 \le t \le 1$. It is easy to check that

$$u(t) \in K\left(\gamma, \theta, \alpha, b, \frac{b}{\sigma}, d\right)$$

and

$$\alpha(u) = \alpha\left(\frac{4bt}{\eta}\right) > b$$

for

$$\frac{\eta}{2} \le t \le \eta.$$

So

$$\left\{u \in K\left(\gamma, \theta, \alpha, b, \frac{b}{\sigma}, d\right) | \alpha(u) > b\right\} \ne \emptyset.$$

Thus, for $u \in K\left(\gamma, \theta, \alpha, b, \frac{b}{\sigma}, d\right)$, one has $b \le u(t) \le \frac{b}{\sigma}$. We obtain

$$\alpha(Tu) = \min_{\frac{\eta}{2} \le t \le \eta} |Tu(t)| \ge \sigma \max_{0 \le t \le 1} |Tu(t)|$$

$$= \sigma\left\{\beta\phi_q\left(\int_{\eta}^{\frac{\eta}{2}} a(s)\left\{f(s, u(s)) + \int_{\frac{\eta}{2}}^{s} H(s, \xi, u(\xi))d\xi\right\}ds\right)\right.$$

$$\left. + \int_{\frac{\eta}{2}}^{1} \phi_q\left(\int_{\frac{\eta}{2}}^{s} a(\tau)\left\{f(\tau, u(\tau)) + \int_{\frac{\eta}{2}}^{\tau} H(\tau, \xi, u(\xi))d\xi\right\}d\tau\right)ds\right\}$$

$$\ge \sigma\int_{\frac{\eta}{2}}^{1} \phi_q\left(\int_{\frac{\eta}{2}}^{s} a(\tau)\left\{f(\tau, u(\tau)) + \int_{\frac{\eta}{2}}^{\tau} H(\tau, \xi, u(\xi))d\xi\right\}d\tau\right)ds$$

$$\ge \sigma\frac{b}{\sigma N}\int_{\frac{\eta}{2}}^{1} \phi_q\left(\int_{\frac{\eta}{2}}^{s} a(\tau)d\tau\right)ds = \frac{b}{N} N = b.$$

Therefore, condition (C_1) of Lemma 2.1 is satisfied.

Secondly, we show (C_2) of Lemma 2.1 is satisfied. From (3.1), we have $\alpha(Tu) = \min_{\frac{\eta}{2} \le t \le \eta} |Tu(t)| \ge \sigma\theta(Tu) > b$, for all $u \in K(\gamma, \alpha, b, d)$ with $\theta(Tu) > \frac{b}{\sigma}$.

Finally, we show condition (C_3) of Lemma 2.1 is also satisfied. Obviously, as $\psi(0) = 0 < a$, there holds $0 \notin R(\gamma, \psi, a, d)$. Suppose $u \in R(\gamma, \psi, a, d)$ with $\psi(u) = a$. Then, by condition (A_2), we get

$$\psi(Tu) = \max_{0 \le t \le 1} |(Tu)(t)|$$

$$= \beta\phi_q\left(\int_{\eta}^{\frac{\eta}{2}} a(s)\left\{f(s, u(s)) + \int_{\frac{\eta}{2}}^{s} H(s, \xi, u(\xi))d\xi\right\}ds\right)$$

$$+ \int_{\frac{\eta}{2}}^{1} \phi_q\left(\int_{\frac{\eta}{2}}^{s} a(\tau)\left\{f(\tau, u(\tau)) + \int_{\frac{\eta}{2}}^{\tau} H(\tau, \xi, u(\xi))d\xi\right\}d\tau\right)ds$$

$$\le \frac{a}{L}\beta\phi_q\left(\int_{\eta}^{\frac{\eta}{2}} a(s)ds\right) + \frac{a}{L}\int_{\frac{\eta}{2}}^{1} \phi_q\left(\int_{\frac{\eta}{2}}^{s} a(\tau)d\tau\right)ds = \frac{a}{L} L = a.$$

According to Lemma 2.1, there exist three positive pseudo-symmetric solutions u_1, u_2 and u_3 for BVP (1.1) and (1.2) such that

$$\max_{0 \le t \le 1} |u_i(t)| \le d \quad \text{for } i = 1, 2, 3, \quad b < \min_{\frac{\eta}{2} \le t \le \eta} |u_1(t)|, \ \max_{0 \le t \le 1} |u_1(t)| \le d,$$

$$a < \max_{0 \le t \le 1} |u_2(t)| < \frac{b}{\sigma} \text{ with } \min_{\frac{\eta}{2} \le t \le \eta} |u_2(t)| < b \text{ and } \max_{0 \le t \le 1} |u_3(t)| < a.$$

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