

## ON IMPROVEMENTS OF THE BUTTERFLY THEOREM

ZVONKO CERIN and GIAN MARIO GIANELLA

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### Abstract

This paper explores the locus of butterfly points of a quadrangle  $ABCD$  in the plane. These are the common midpoints of three segments formed from intersections of a butterfly line with the lines  $AB$ ,  $CD$ ,  $AD$ ,  $BC$ ,  $AC$ , and  $BD$ . The locus is the nine-point-conic of  $ABCD$  that goes through the midpoints of the segments  $AB$ ,  $CD$ ,  $AD$ ,  $BC$ ,  $AC$ , and  $BD$ . We also consider the problem to determine when two quadrangles share the nine-point-conic. Our proofs use analytic geometry of the rectangular Cartesian coordinates.

### 1. Introduction

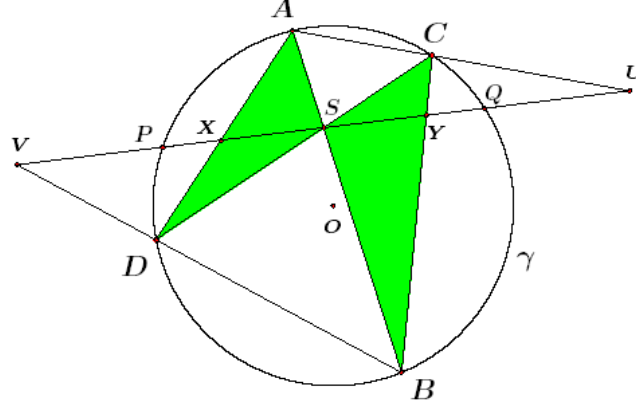
The classical butterfly theorem claims that whenever chords  $AB$  and  $CD$  of a circle  $\gamma$  intersect at the midpoint  $S$  of the third chord  $PQ$ , then  $S$  is also the midpoint of the segments  $XY$  and  $UV$  formed by the intersections  $X$ ,  $Y$ ,  $U$ , and  $V$  of the lines  $AD$ ,  $BC$ ,  $AC$ , and  $BD$  with the line  $PQ$  (see Figure 1).

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**Figure 1.** The point  $S$  is the body and the triangles  $ADS$  and  $BCS$  are the wings of the butterfly.

In recent years there has been a considerable activity in improving this interesting result. First it was observed in [23] that the line  $PQ$  could be replaced by any line  $\ell$  in the plane of the circle and take for the point  $S$  the projection of the center  $O$  of  $\gamma$  onto  $\ell$ . This was extended by the first author in [2] and [3], where the circle is replaced by any conic  $\gamma$ , the point  $S$  is a point on line of symmetry  $z$  of  $\gamma$  and the line  $\ell$  is the perpendicular to  $z$  at  $S$ . The reference [24] contains yet another improvement of this by replacing the line of symmetry  $z$  with any line  $\ell$  and taking for the point  $S$  the intersection of  $\ell$  with the diameter of the conic  $\gamma$  which is conjugate to the line  $\ell$ .

A further extension is accomplished in the first author's article [4] where he introduced the following technical definition in order to get shorter statements.

A pair  $(\ell, S)$  consisting of a line  $\ell$  and a point  $S$  on it is said to have the *butterfly property* with respect to the quadrangle  $ABCD$  provided  $S$  is a common midpoint of segments  $\ell_a\ell_c$ ,  $\ell_b\ell_d$  and  $\ell_e\ell_f$ , where  $\ell_a$ ,  $\ell_b$ ,  $\ell_c$ ,  $\ell_d$ ,  $\ell_e$ , and  $\ell_f$  are intersections of  $\ell$  with lines  $AB$ ,  $BC$ ,  $CD$ ,  $DA$ ,  $AC$

and  $BD$ . In this situation we shall write  $(\ell, S) \bowtie ABCD$  or  $\ell \bowtie_S ABCD$  and use also the phrase “the line  $\ell$  has the butterfly property with respect to  $ABCD$  at the point  $S$ ”. Of course, we consider lines  $\ell$  and quadrangles  $ABCD$  for which all six intersections  $\ell_a, \ell_b, \ell_c, \ell_d, \ell_e$ , and  $\ell_f$  are well defined points in the (finite) plane.

The main results in [4] show that for most points  $S$  in the plane of a conic  $\gamma$  there is a unique line  $\ell$  such that  $(\ell, S) \bowtie ABCD$  holds for every quadrangle  $ABCD$  inscribed to  $\gamma$ .

The article [22] explores for a given cyclic quadrangle  $ABCD$  what is the locus of all projections  $S$  of the circumcenter  $O$  of  $ABCD$  on lines  $\ell$  with the property that the relation  $(\ell, S) \bowtie ABCD$  holds.

This locus is shown to be the equilateral hyperbola that goes through the circumcenter  $O$  and the midpoints of segments  $AB, AC, AD, BC, BD$ , and  $CD$ . It also goes through the intersection of diagonals ( $AC$  and  $BD$ ) and the intersections  $(AB \cap CD)$  and  $(AD \cap BC)$  of opposite sides.

The goal of this paper is to lift the assumption that  $ABCD$  is a cyclic quadrangle from results in [22]. Our approach is through the analytic geometry. Perhaps some or all of our results could be proved synthetically (see the last sentence on p. 61 of [22]). However, with this miraculous method in [22] only a very special case of cyclic quadrangles was covered. We hope that one cannot impose methods of proofs and discovery in mathematics and that with computers our “heavy calculations” are in fact far easier to follow for an average person than to master projective or affine geometry.

## 2. Butterfly Points of Strong Quadrangles

In order to avoid repetitions of the phrase “without parallel diagonals or parallel opposite sides” we first introduce a broad class of quadrangles that will be subjects of our investigation.

We shall say that the quadrangle  $ABCD$  is *strong* provided the lines  $AB$ ,  $AC$ , and  $AD$  intersect in the points  $E$ ,  $F$ , and  $G$  with the lines  $CD$ ,  $BD$ , and  $BC$ . The triangle  $EFG$  is called the *diagonal triangle* of  $ABCD$ .

Let  $ABCD$  be a quadrangle in the plane. Then a point  $S$  from this plane is called a *butterfly point* of  $ABCD$  or a  $\beta_{ABCD}$ -point if there is a line  $\ell$  through  $S$  such that the relation  $(\ell, S) \bowtie ABCD$  is true.

Let us begin with a technical result which clarifies our definition of the relation  $(\ell, S) \bowtie ABCD$ . It shows that it suffices to require that only midpoints of two among segments  $\ell_a\ell_c$ ,  $\ell_b\ell_d$  and  $\ell_e\ell_f$  coincide.

**Lemma 1.** *Let  $ABCD$  be a strong quadrangle. Let a line  $\ell$  intersect the lines  $AB$ ,  $BC$ ,  $CD$ ,  $DA$ ,  $AC$ , and  $BD$  in the points  $\ell_a$ ,  $\ell_b$ ,  $\ell_c$ ,  $\ell_d$ ,  $\ell_e$  and  $\ell_f$ . Let  $M_{ac}$ ,  $M_{bd}$  and  $M_{ef}$  be midpoints of the segments  $\ell_a\ell_c$ ,  $\ell_b\ell_d$  and  $\ell_e\ell_f$ . If any two of these midpoints coincide, then they all coincide.*

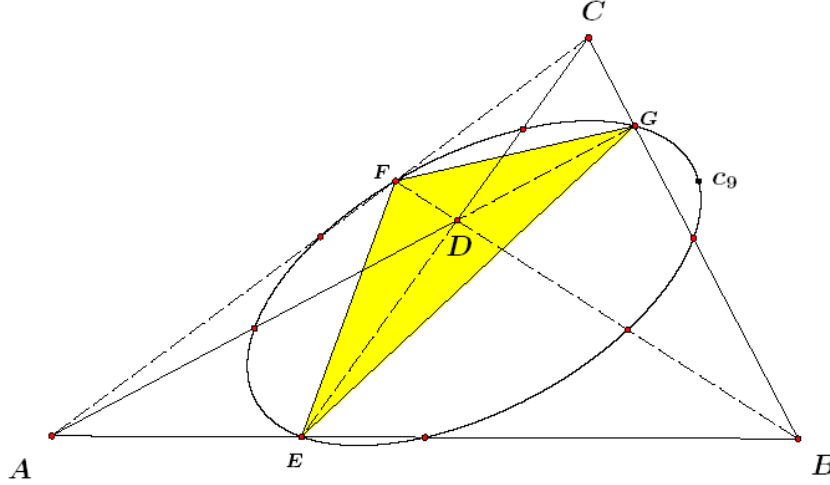
**Proof** (in Cartesian coordinates). We assume that the points  $A$ ,  $B$ ,  $C$  and  $D$  have the rectangular Cartesian coordinates  $(0, 0)$ ,  $(1, 0)$ ,  $(u, v)$  and  $(U, V)$ . Let  $u_0 = uU$ ,  $v_0 = vV$ ,  $u_1 = uV$ ,  $v_1 = vU$ ,  $u_2 = u - U$ ,  $v_2 = v - V$ ,  $w = v_1 - u_1$ ,  $W = v_1 + u_1$ . Let  $fx + gy + h = 0$  be the equation of the line  $\ell$ . Note that  $f^2 + g^2 \neq 0$ . Solving linear equations we easily find coordinates of all points and discover that the distances  $|M_{ac}M_{bd}|$ ,  $|M_{ac}M_{ef}|$  and  $|M_{bd}M_{ef}|$  are the absolute values of  $\frac{MK}{2\alpha\gamma\beta\delta}$ ,  $\frac{MK}{2\alpha\gamma\epsilon\varphi}$  and  $\frac{MK}{2\beta\delta\epsilon\varphi}$ , where  $\alpha = f$ ,  $\beta = (u - 1)f + vg$ ,  $\gamma = u_2f + v_2g$ ,  $\delta = Uf + Vg$ ,  $\epsilon = uf + vg$ ,  $\varphi = (U - 1)f + Vg$ ,  $K = \sqrt{f^2 + g^2}$ ,  $M = f(v_2 - w)(u_0f^2 + Wfg + v_0g^2) + h(Tf^2 + 2v_0u_2fg + v_0v_2g^2)$  and  $T = w - v_1U + uu_1$ . That the lemma holds is now obvious.

Our first theorem is a version of Theorem 3 in [22] that holds for all strong quadrangles and not only for the ones inscribed to a circle.

**Theorem 1.** *For any strong quadrangle  $ABCD$  the locus of all  $\beta_{ABCD}$ -points is the nine-point-conic  $c_9 = c_9(ABCD)$  that goes through the vertices of its diagonal triangle  $EFG$  and through the midpoints of segments  $AB$ ,  $BC$ ,  $CD$ ,  $DA$ ,  $AC$ , and  $BD$ .*

**Proof.** In this proof we shall use the same assumption and notation about the points  $A$ ,  $B$ ,  $C$  and  $D$  as we did in the proof of Lemma 1.

Let  $P(p, q)$  be a  $\beta_{ABCD}$ -point. Let  $fx + gy - fp - gq = 0$  be the equation of the line  $\ell$  with the property that  $P$  is the common midpoint of segments  $\ell_a\ell_c$ ,  $\ell_b\ell_d$ , and  $\ell_e\ell_f$ . (Note that the real number  $f$  cannot be zero because then the lines  $\ell$  and  $AB$  would be parallel.) This is true provided the following two conditions  $K_i$  hold



**Figure 2.** The quadrangle  $ABCD$  with its nine-point-conic.

$(a_i f - 2b_i g)p + (2A_i f - B_i g)q + C_i = 0$ , with indices  $i = 1, 2$  and with coefficients  $a_1 = B_1 = -v_2$ ,  $b_1 = 0$ ,  $a_2 = B_2 = V - W$ ,  $b_2 = v_0$ ,  $A_1 = u$ ,  $A_2 = u_0 - U$ ,  $C_1 = w$  and  $C_2 = fv_1 + gv_0$ . Since in each equation  $K_i$  the variable  $f$  appears linearly, we can solve it easily and get two quotients  $Q_i$  for values of  $f$ . Hence, it must be  $Q_1 - Q_2 = 0$ . The difference on the

left hand side has the following polynomial

$$c_9 = c_9(ABCD) = 2k_1p^2 - 4k_2pq + 2k_3q^2 - k_4p + k_5q + k_6$$

as the only factor that could be zero, where  $k_1 = v_0v_2$ ,  $k_2 = v_0u_2$ ,  $k_3 = w + uu_1 - Uv_1$ ,  $k_4 = v_0(v_2 + 2w)$ ,  $k_5 = v_1^2 - u_1^2 + u_1(V + 2v) - v_1(v + 2V)$  and  $k_6 = v_0w$ . We conclude that the locus of all  $\beta_{ABCD}$ -points is a conic whose equation is  $c_9 = 0$ . It is now easy to check that the midpoints  $A'$ ,  $B'$ ,  $C'$ ,  $D'$ ,  $E'$ , and  $F'$  of segments  $AB$ ,  $BC$ ,  $CD$ ,  $DA$ ,  $AC$ , and  $BD$  as well as the vertices  $E$ ,  $F$ , and  $G$  of the diagonal triangle lie on this conic.

**Theorem 2.** *Let  $ABCD$  be a strong quadrangle. Then the circumcircle  $O$  of the triangle  $ABC$  is a  $\beta_{ABCD}$ -point if and only if either  $ABC$  has a right angle or  $ABCD$  is a cyclic quadrangle.*

**Proof.** This follows immediately from the fact that the value of the polynomial  $c_9$  for  $p = \frac{1}{2}$  and  $q = \frac{u^2 + v^2 - u}{2v}$  (the coordinates of the circumcircle of the triangle  $ABC$ ) is the quotient

$$\frac{u(u^2 + v^2 - u)(1 - u)(v(U^2 + V^2 - U) - (u^2 + v^2 - u)V)}{2v^2}$$

and that  $v(p^2 + q^2 - p) - (u^2 + v^2 - u)q = 0$  is the equation of the circumcircle of the triangle  $ABC$ .

An easy consequence of Theorem 2 is the following corollary which includes Theorem 3 in [22]. We also describe precisely what are the lines with the butterfly property in this situation.

**Corollary 1.** *The nine-point-conic  $c_9$  of a strong cyclic quadrangle  $ABCD$  is an equilateral hyperbola which goes through the center  $O$  of its circumcircle.*

For  $P = O$  let  $\ell = \ell_P$  be the normal of the equilateral hyperbola  $c_9$  in  $O$  and for every point  $P \in c_9 \setminus \{O\}$  let  $\ell = \ell_P$  be the perpendicular at  $P$  to the segment  $OP$ . Then the line  $\ell$  has the property that  $P$  is a common midpoint of segments  $\ell_a\ell_c$ ,  $\ell_b\ell_d$ , and  $\ell_e\ell_f$ , where  $\ell_a$ ,  $\ell_b$ ,  $\ell_c$ ,  $\ell_d$ ,  $\ell_e$ , and  $\ell_f$  are intersections of  $\ell$  with lines  $AB$ ,  $BC$ ,  $CD$ ,  $DA$ ,  $AC$ , and  $BD$ .

**Proof.** The first part is an easy consequence of Theorem 2 and the following two well-known theorems. A strong quadrangle  $ABCD$  is cyclic if and only if the circumcenter  $O$  of the triangle  $ABC$  is the orthocenter of its diagonal triangle  $EFG$ . A conic which goes through the vertices and the orthocenter of a triangle is an equilateral hyperbola.

In order to prove the second part, which describes precisely the position of the line with the butterfly property, we must repeat the proof of Theorem 1 under the assumption that  $A = T(0)$ ,  $B = T(u)$ ,  $C = T(v)$ ,

and  $D = T(w)$  are points on the unit circle, where  $T(x) = \left( \frac{1-x^2}{1+x^2}, \frac{2x}{1+x^2} \right)$ .

Let  $s_1 = u + v + w$ ,  $s_2 = uv + vw + wu$  and  $s_3 = uvw$ . Then the equation of  $c_9$  is

$$(s_1 - s_3)p^2 + 2(s_2 - s_3)pq + (s_3 - s_1)q^2 - (s_3 + s_1)p + 2q = 0$$

and the parameter  $u$  is  $\frac{\beta(2(1-vw)p + (s_1 - s_3)q - 2)}{(s_1 - s_3)p + 2u(v+w)q - s_1 - s_3}$ . Since the

perpendicular at  $P$  to the line  $OP$  is  $px + qy - p^2 - q^2 = 0$ , we conclude that this line will agree with the butterfly line  $fx + gy - fp - gq = 0$  if and only if  $P$  satisfies the above equation of the equilateral hyperbola  $c_9$ .

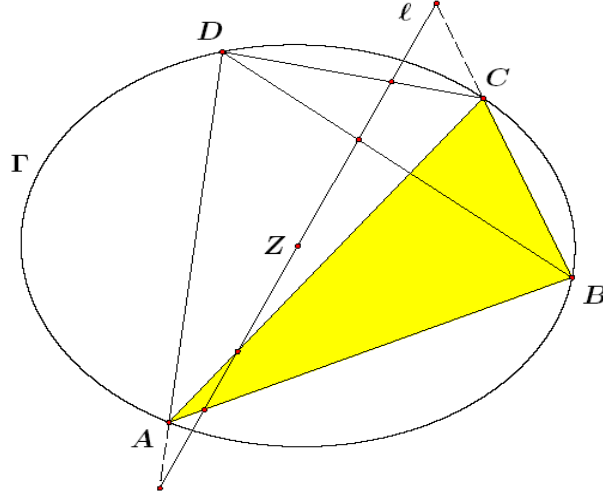
When  $P = O$ , then the normal to  $c_9$  at  $P$  and the butterfly line of the same point of course both have the equation  $2x + (s_1 + s_3)y = 0$ .

### 3. Centers as Butterfly Points

Our next theorem shows that the center of a conic through the vertices of a triangle  $ABC$  will be the butterfly point of a strong quadrangle  $ABCD$  if and only if the point  $D$  is on this conic. It could therefore be regarded as a converse of Theorem 3 in [4].

**Theorem 3.** *Let  $ABCD$  be a strong quadrangle. Then the center  $S$  of a nondegenerate conic  $\Gamma$  through the vertices of the triangle  $ABC$  is a  $\beta_{ABCD}$ -point if and only if  $D$  lies on  $\Gamma$ .*

**Proof.** In this proof we shall use the same assumption about the points  $A, B, C$  and  $D$  as we did in proof of Lemma 1 and Theorem 1.



**Figure 3.** The center  $Z$  of a conic  $\Gamma$  through the vertices of  $ABC$  is a  $\beta_{ABCD}$ -point if and only if  $D$  is on  $\Gamma$ .

A conic has the equation

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0.$$

When we substitute coordinates of points  $A$ ,  $B$  and  $C$  for  $x$  and  $y$  and solve these linear equations in  $d$ ,  $e$  and  $f$ , we conclude that the equation of our conic  $\Gamma$  that goes through the vertices of  $ABC$  is

$$\Gamma(x, y) = vQ(x, y) - Q(u, v)y = 0,$$

where  $Q$  is a function that takes  $(x, y)$  into  $ax^2 + 2bxy + cy^2 - ax$ . Let  $T = Q(u, v)$ .

The above conic will have a center (i.e., it will be either an ellipse or a hyperbola) provided  $\Delta = ac - b^2 \neq 0$ . Then the center  $S$  is the point  $\left( \frac{acv - bT}{2v\Delta}, \frac{a(T - bv)}{2v\Delta} \right)$ .

Let  $S_x$  and  $S_y$  denote the coordinates of  $S$ . Then the equation of a line  $\ell$  through  $S$  is  $fx + gy - fS_x - gS_y = 0$ , for some real numbers  $f$  and  $g$  with  $f^2 + g^2 \neq 0$ .

We can evaluate  $|M_{ac}S|$  and  $|M_{bd}S|$  to find that they are absolute values of  $\frac{(fP_j + gQ_j)K}{4v\Delta S_j T_j}$  for  $j = 1, 2$  with

$$S_1 = \alpha, S_2 = \beta, T_1 = \gamma, T_2 = \delta, P_1 = 2vw\Delta + \varrho(a, b)T - av\varrho(b, c),$$

$$Q_1 = av_2(T - bv), P_2 = 2vv_1\Delta + \sigma(a, b)T - av\sigma(b, c),$$

$$Q_2 = 2vv_0\Delta + \tau(a, b)T - av\tau(b, c), \varrho(a, b) = 2au_2 + bv_2,$$

$$\sigma(a, b) = 2aU(u - 1) + b(W - V),$$

and

$$\tau(a, b) = a(W - V) + 2bv_0$$

(for our notation see the proof of Lemma 1).

Let us assume for the moment that  $Q_1 \neq 0$ . Then the center  $S$  is the midpoint  $M_{ac}$  if and only if  $g = -\frac{fP_1}{Q_1}$ . Substituted into  $fP_2 + gQ_2$  this value gives  $\frac{2k\Gamma(U, V)}{av_2}$ , where  $k = (a(u - 1) + bv)(au + bv)$ .

When the point  $D$  lies on the conic  $\Gamma$ , then  $\Gamma(U, V) = 0$  so that  $S$  is also the midpoint  $M_{bd}$ . Hence,  $S$  is the  $\beta_{ABCD}$ -point by Lemma 1.

Conversely, if  $S$  is the  $\beta_{ABCD}$ -point, then  $k\Gamma(U, V) = 0$ . In other words, either  $\Gamma(U, V) = 0$  (when the point  $D$  lies on the conic  $\Gamma$ ), or  $au + bv = 0$ , or  $a(u - 1) + bv = 0$ . When  $au + bv = 0$ , then  $b = -\frac{au}{v}$  so that the center of  $\Gamma$  is the midpoint  $A_g$  of the segment  $BC$  and the line  $\ell$  agrees with the line  $BC$ . In this situation the point  $\ell_b$  is not determined which implies that  $au + bv = 0$  cannot happen. Similarly, when  $a(u - 1) + bv = 0$ , then  $b = \frac{a(1 - u)}{v}$  so that the center of  $\Gamma$  is the midpoint  $B_g$  of the segment  $AC$  and the line  $\ell$  agrees with the line  $AC$  and we again conclude that  $a(u - 1) + bv = 0$  cannot happen.

Note that  $Q_1$  is equal to zero provided either  $\alpha = 0$  or  $v - V = 0$  or  $T - bv = 0$ . When  $T - bv = 0$ , then  $c = \frac{au(1-u) + bv(1-2u)}{v^2}$  and  $S = C_g$ , the midpoint of the segment  $AB$ . Also,  $fP_j + gQ_j = \frac{fkF_j}{v}$ , where  $F_1$  and  $F_2$  are  $v_2 - 2w$  and  $V + w$ . It follows that the center  $C_g$  is the  $\beta_{ABCD}$ -point if and only if either  $f = 0$  or  $F_1 = 0$  and  $F_2 = 0$ . If  $f = 0$ , then the line  $\ell$  agrees with the line  $AB$  which cannot happen for the similar reason which prevents  $au + bv = 0$  and  $a(u - 1) + bv = 0$  to hold. On the other hand,  $F_1 = 0$  and  $F_2 = 0$  only for  $U = 1 - u$  and  $V = -v$  when  $ABCD$  is a parallelogram which is ruled out by our assumption that  $ABCD$  is a strong quadrangle.

It remains to consider the case  $a(v - V) = 0$ . Of course, there are two subcases  $\alpha = 0$  and  $V = v$ . Since the ordinate of the vertex  $C$  is also  $v$ , we infer that the second subcase is impossible because the lines  $AB$  and  $CD$  would be parallel.

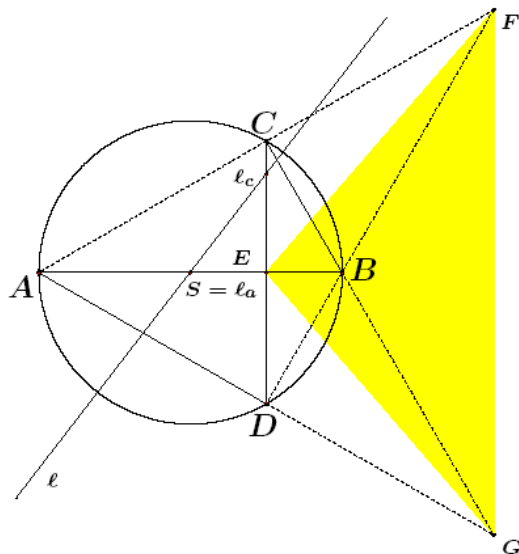
When  $\alpha = 0$ , then the center  $S$  of  $\Gamma$  is the point  $\left(\frac{2bu + cv}{2b}, 0\right)$  on the line  $AB$  and the conic  $\Gamma$  degenerates into two lines ( $AB$  and  $CS$ ) which we prohibited with our assumptions.

**Remark 1.** One can wonder if the statement of Theorem 3 is completely true. Like in the particular case formulated in Theorem 2, there should be an exception, when  $S$  is located on a side of the triangle  $ABC$ . Then  $D$  can be anywhere provided the quadrangle  $ABCD$  is still strong.

The following Figure 4 shows that the last claim is wrong.

On this figure  $ABCD$  is a strong quadrangle (its diagonal triangle  $EFG$  is well defined), the point  $S$  is on the side  $AB$  of the triangle  $ABC$  and the center of the circumcircle of  $ABC$  but it is not the  $\beta_{ABCD}$ -point because for any line through  $S$  the point  $S$  cannot be the midpoint of the segment  $\ell_a\ell_c$ , where  $\ell_a = S$  and  $\ell_c$  are the intersections of the line  $\ell$  with the lines  $AB$  and  $CD$ .

*ABC.* In the statement we use central points from [15].



**Figure 4.** The counterexample for the above statement.

triangle  $ABC$ .

of  $ABC$ , respectively.

desired conclusion.

Another version of Theorem 3 is the following statement which was formulated for cyclic quadrangles in [22] as Theorem 4.

**Theorem 4.** *Let  $ABCD$  be a strong quadrangle. Then the locus of centers  $S$  of all nondegenerate conics  $\Gamma$  through the vertices of  $ABCD$  is the nine-point-conic  $c_9$  of the quadrangle  $ABCD$ .*

**Proof.** We shall make the same assumptions about points  $A, B, C$ , and  $D$  as in the proof of Theorem 3. Replacing  $x$  and  $y$  with  $U$  and  $V$  in the polynomial  $\Gamma(x, y)$  we can solve for  $c$  to obtain that the equation of a nondegenerate conic through the vertices of  $ABCD$  is

$$\begin{aligned} &v_0v_2x(ax + 2by - a) + [(v_1U - u_1u - w)a - 2v_0u_2b]y^2 \\ &+ [(v_1v - u_1V - wW)a - 2v_0wb]y = 0. \end{aligned}$$

Its center  $S$  is  $\left( \frac{u_1\alpha(u, v)\beta(V) - v_1\alpha(U, V)\beta(v)}{2(\varrho a^2 + 2\sigma ab + \tau b^2)}, \frac{a(V^2\gamma(u, v) - v^2\gamma(U, V))}{2(\varrho a^2 + 2\sigma ab + \tau b^2)} \right)$ ,

where

$$\alpha(u, v) = a(u - 1) + 2bv, \quad \beta(V) = a - bV,$$

$$\varrho = u_1u - v_1U + w, \quad \sigma = u_1v - v_1V, \quad \tau = v_0v_2$$

and

$$\gamma(u, v) = au(u - 1) + bv(2u - 1).$$

In order to obtain the locus of these centers we will eliminate the real variable  $b$ . This could be done as follows.

Let the coordinates of  $S$  be  $x$  and  $y$ . Let  $H = v_2x - w$ ,  $G = -\varrho$ ,  $F = v_2 - 2w$ ,  $E = vv_1 - Vu_1 - wW$ . After the multiplication by the denominator of  $x$  and transfer of terms on the left hand side we get the following equations  $(e_1)$  and  $(e_2)$ .

$$(2x - 1)Ga^2 - 2v_0Hb^2 + [4v_0u_2x + E + 2(Uv_1 - vu_1)]ab = 0,$$

$$[E + 2Gy]a^2 - 2v_0v_2yb^2 - v_0[4u_2y - F]ab = 0.$$

We multiply  $(e_1)$  by  $v_2y$  and  $(e_2)$  by  $H$ , make their difference, and solve for  $b$ . We get  $b = \frac{a(EH + FGy)}{M}$ , where  $M = Ky - v_0FH$  and  $K = u_1^2(V + 3v) + v_1^2(3V + v) - 4u_0v_0(V + v) + v_0(u_1(V + 2v) - v_1(2V + v))$ .

Substituting this value back into  $(e_1)$  and  $(e_2)$  we obtain  $\frac{LH\Theta}{M^2} = 0$  and  $\frac{aLy\Theta}{M^2} = 0$ , where  $L = a^2v_2w(v - w)(V - w)(v_2 - w)$  and  $\Theta = v_0x(2H - 4u_2y - v_2) + 2(uu_1 - Uv_1 + w)y^2 + (2vu_1 - 2Vv_1 - E)y + v_0w$ .

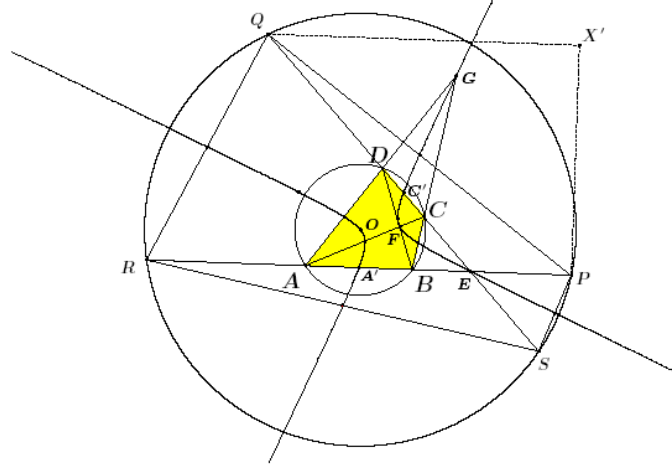
It is easy to check that  $\Theta = 0$  is in fact the equation of the nine-point-conic  $c_9$  of  $ABCD$  because the coordinates of the midpoints of segments  $AB$ ,  $AC$ ,  $AD$ ,  $BC$ ,  $BD$ , and  $CD$  satisfy it.

On the other hand, if  $a = 0$ , then the conic degenerates into two lines and if either  $v_2 = 0$ ,  $w = 0$ ,  $v - w = 0$ ,  $V - w = 0$ , or  $v_2 - w = 0$ , then the quadrangle  $ABCD$  is not strong which happens also when  $H = 0$  and  $y = 0$  (i.e., when  $U = 1 - u$  and  $V = -v$ ).

#### 4. Quadrangles Sharing the Nine-point-conics

The last Theorem 5 in [22] considers the question if different quadrangles can share the same nine-point-conic. It shows that for any cyclic quadrangle  $ABCD$  with the circumcenter  $O$  and for any circle with center at  $O$  which intersects the lines  $AB$  and  $CD$  in points  $P$ ,  $R$  and  $Q$ ,  $S$  the quadrangles  $ABCD$  and  $PQRS$  have the same nine-point-hyperbola (see Figure 4, i.e., Figures 6-8 in [22] without honeycombs).

We shall now prove an analogous result for an arbitrary strong quadrangle  $ABCD$ . We discover that there is a conic  $\omega$  with the property that for any of its points there is a simple construction  $\sigma$  that gives a quadrangle  $PQRS$  that shares the nine-point-conic with  $ABCD$ .

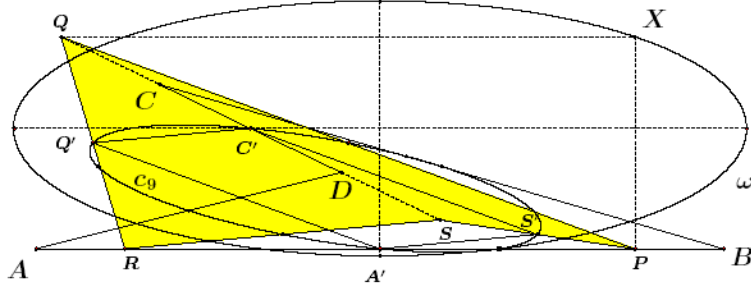


**Figure 5.** Cyclic quadrangles  $ABCD$  and  $PQRS$  with the concentric circumcircles and the identical nine-point-hyperbolas.

The steps of the construction  $\sigma$  go as follows. Let  $ABCD$  be a strong quadrangle. Let  $A'$  and  $C'$  be midpoints of segments  $AB$  and  $CD$ . Let  $X$  be a point different from  $A'$  and  $C'$ . Let  $P$  be the orthogonal projection of  $X$  on the line  $AB$  and let  $Q$  denote the intersection of the line  $CD$  and the parallel through  $X$  to the line  $AB$ . Let  $R$  and  $S$  denote the reflections of points  $P$  and  $Q$  at the points  $A'$  and  $C'$ , respectively. Then we shall say that  $PQRS$  is obtained from  $X$  and  $ABCD$  by the construction  $\sigma$  and write  $PQRS = \sigma(X, ABCD)$ .

Of course, when  $ABCD$  is a cyclic quadrangle with the circumcenter  $O$  and  $k$  is any circle with center at  $O$  which intersects the lines  $AB$  and  $CD$  in points  $P, R$  and  $Q, S$ , then for the intersection  $X$  of the perpendicular at  $P$  and the parallel at  $Q$  to the line  $AB$  we have  $PQRS = \sigma(X, ABCD)$  so that our construction  $\sigma$  includes the one from [22] as a special case (see Figure 4).

**Theorem 5.** *Let  $ABCD$  be a strong quadrangle. Then the locus of all points  $X$  with the property that the quadrangles  $PQRS = \sigma(X, ABCD)$  and  $ABCD$  share the nine-point-conic is a conic  $\omega$ . The conics  $c_9$  and  $\omega$  are of the same type. The lines of symmetry of the conic  $\omega$  are the perpendicular bisector of the segment  $AB$  and the parallel to  $AB$  at the midpoint of the segment  $CD$ .*



**Figure 6.** The quadrangles  $PQRS = \sigma(X, ABCD)$  and  $ABCD$  share the nine-point-conic if and only if  $X$  is on the conic  $\omega$ .

**Proof.** We shall retain notation from the proofs of Theorems 3 and 4. Let the coordinates of the point  $X$  be  $s$  and  $t$ . Then the vertices of the quadrangle  $PQRS$  have coordinates  $P(s, 0)$ ,  $R(1-s, 0)$ ,  $Q\left(\frac{w+tu_2}{v_2}, t\right)$ , and  $S\left(\frac{uv-UV-tu_2}{v_2}, v+V-t\right)$ . It is clear that the nine-point-conics of  $ABCD$  and  $PQRS$  coincide if and only if the midpoints  $Q'$  and  $S'$  of the segments  $QR$  and  $SP$  are on  $c_9 = c_9(ABCD)$ . Note that  $Q'A'S'C'$  is a parallelogram (see Figure 5).

Recall that the equation of the conic  $c_9$  is  $\Theta = 0$ . If we substitute the coordinates of either  $Q'$  or  $S'$  for  $x$  and  $y$  in  $\Theta$ , then we shall get  $\frac{\Psi}{2v_2}$ , where

$$\Psi = v_0 v_2^2 s(s-1) + w(v_2 - w)(t^2 - (v+V)t + v_0).$$

We conclude that if the coordinates of the point  $X$  satisfy the condition  $\Psi = 0$ , then the quadrangles  $PQRS$  and  $ABCD$  will have the same nine-point-conic. Hence, the locus of points  $X$  is indeed a conic  $\omega$ . Since

the  $D$ -invariants (the expression  $ac - b^2$  whose sign determines the type of the conic) of  $c_9$  and  $\omega$  are  $4v_0w(v_2 - w)$  and  $v_2^2v_0w(v_2 - w)$ , it follows that  $c_9$  and  $\omega$  are of the same type. The possibility that  $D(\omega) = 0$  and  $D(c_9) \neq 0$  (for  $V = v$ ) is ruled out by the assumption that  $ABCD$  is a strong quadrangle.

The statement about the lines of symmetry of the conic  $\omega$  is easily checked by substitution. More precisely, if  $X$  is a point on  $\omega$ , then its reflection  $(1 - s, t)$  at the perpendicular bisector of the segment  $AB$  and its reflection  $(s, v + V - t)$  at the parallel to  $AB$  through the midpoint of  $CD$  are also on  $\omega$ .

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Kopernikova 7  
10010 Zagreb, Croatia  
e-mail: cerin@math.hr

Dipartimento di Matematica  
Universita di Torino  
Torino, Italy  
e-mail: gianella@dm.unito.it