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SOME TYPES OF FUZZY NORMAL SPACES IN CHANG'S SENSE

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Abstract

The notions of fuzzy normal spaces, fuzzy almost normal spaces and fuzzy mildly normal spaces have been defined and investigated in a Chang's fuzzy topological space.

1. Introduction

Chang introduced the notion of fuzzy topology in his classical paper [3]. Balasubramanian and Sundaram [2] introduced the concept of fuzzy generalized closed sets in Chang's fuzzy topology as an extension of generalized closed sets of Levine [7] in ordinary topology. More details about the generalized closed sets can be found in [4, 5, 9].

Here, we will define the concepts of fuzzy generalized α-closed sets and fuzzy generalized regular α-closed sets in Chang's fuzzy topological space. By using the above mentioned classes of generalized fuzzy closed

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sets we will introduce and study the concepts of fuzzy normal space, fuzzy almost normal space and fuzzy mildly normal space.

Throughout this paper, by (X, τ) or simply by X will be denoted fuzzy topological space (fts) due to Chang. The interior, the closure and the complement of a fuzzy set λ will be denoted by $\text{int}\,\lambda$, $\text{cl}\lambda$ and $1 - \lambda$, respectively.

A fuzzy set λ is quasi-coincident with a fuzzy set μ, denoted by $\lambda q \mu$, if there exists $x \in X$ such that $\lambda(x) + \mu(x) > 1$. Otherwise we denote $\lambda \overline{q} \mu$ [8].

Definition 1.1. Let λ be a fuzzy set of an fts (X, τ). Then λ is called

(1) a *fuzzy open set* if and only if $\lambda \in \tau$ [3];

(2) a *fuzzy regular open set* if and only if $\lambda = \text{int}(\text{cl}\lambda)$ [1];

(3) a *fuzzy* α-*open set* if and only if $\lambda \leq \text{int}(\text{cl}(\text{int}\,\lambda))$ [10].

Definition 1.2. Let λ be a fuzzy set of an fts (X, τ). Then λ is called

(1) a *fuzzy closed set* if and only if $1 - \lambda \in \tau$ [3];

(2) a *fuzzy regular closed set* if and only if $1 - \lambda$ is a fuzzy regular open set [1];

(3) a *fuzzy* α-*closed set* if and only if $1 - \lambda$ is a fuzzy α-open set [10].

Theorem 1.1. *Let X be an fts.*

(1) *Any union of fuzzy regular open* (*resp. fuzzy* α-*open*) *sets is a fuzzy open* (*resp. fuzzy regular open*, *fuzzy* α-*open*) *set* [1];

(2) *Any intersection of fuzzy regular closed* (*resp. fuzzy* α-*closed*) *sets is a fuzzy closed* (*resp. fuzzy regular closed*, *fuzzy* α-*closed*) *set* [10].

Theorem 1.2 [10]**.** *Let* λ *and* μ *be fuzzy sets of an fts X. Then the following statements are true*:

(1) *if* λ *is a fuzzy regular closed set*, *then* λ *is a fuzzy closed set*;

(2) *if* λ *is a fuzzy closed set*, *then* λ *is a fuzzy* α-*closed set*;

(3) *if* λ *is a fuzzy* α-*closed set and* $\mathrm{cl}(\mathrm{int}\lambda) \leq \mu \leq \lambda$, *then* μ *is a fuzzy* α-*closed set*.

Theorem 1.3 [10]. *Let* λ *and* μ *be fuzzy sets of an fts X. Then the following statements are true*:

(1) *if* λ *is a fuzzy regular open set*, *then* λ *is a fuzzy open set*;

(2) *if* λ *is a fuzzy open set*, *then* λ *is a fuzzy* α-*open set*;

(3) *if* λ *is a fuzzy* α-*open set and* $\lambda \leq \mu \leq \mathrm{int}(\mathrm{cl}\lambda)$, *then* μ *is a fuzzy* α-*open set*.

Theorem 1.4 [1]. *Let* λ *be a fuzzy set of an fts X. Then*

(1) $\mathrm{int}(1 - \lambda) = 1 - \mathrm{cl}\lambda$;

(2) $\mathrm{cl}(\mathrm{int}(\mathrm{cl}\lambda)) = \mathrm{cl}(\mathrm{int}(\mathrm{cl}(\mathrm{int}\,\lambda)))$;

(3) $\mathrm{int}(\mathrm{cl}\lambda) = \mathrm{int}(\mathrm{cl}(\mathrm{int}(\mathrm{cl}\lambda)))$.

Definition 1.3 [6]. Let λ be a fuzzy set of an fts X. Then λ is called

(1) a *generalized fuzzy closed set* if and only if $\mathrm{cl}\lambda \leq \mu$, for each fuzzy open set μ such that $\lambda \leq \mu$;

(2) a *generalized fuzzy regular closed set* if and only if $\mathrm{cl}\lambda \leq \mu$, for each fuzzy regular open set μ such that $\lambda \leq \mu$.

Definition 1.4 [6]. Let λ be a fuzzy set of an fts X. Then λ is called

(1) a *generalized fuzzy open set* if and only if $1 - \lambda$ is a generalized fuzzy closed set;

(2) a *generalized fuzzy regular open set* if and only if $1 - \lambda$ is a generalized fuzzy regular closed set.

Theorem 1.5 [6]. *Let* λ *be a fuzzy set of an fts X. Then the following statements are true*:

(1) *if* λ *is a generalized fuzzy closed set*, *then* λ *is a generalized fuzzy regular closed set*.

(2) *if* λ *is a generalized fuzzy open set*, *then* λ *is a generalized fuzzy regular open set*.

Theorem 1.6 [6]. *Let* λ *and* μ *be fuzzy sets of an fts X. Then the following statements are true*:

(1) *if* λ *and* μ *are generalized fuzzy closed* (*resp. generalized fuzzy regular closed*) *sets, then* $\lambda \vee \mu$ *is a generalized fuzzy closed* (*resp. generalized fuzzy regular closed*) *set*;

(2) *if* λ *is a generalized fuzzy closed* (*resp. generalized fuzzy regular closed*) *set and* $\lambda \leq \mu \leq \mathrm{cl}\lambda$, *then* μ *is a generalized fuzzy closed* (*resp. generalized fuzzy regular closed*) *set*;

(3) *if* λ *is a fuzzy closed set, then* λ *is a generalized fuzzy closed set.*

Theorem 1.7 [6]. *Let* λ *and* μ *be fuzzy sets of an fts X. Then the following statements hold*:

(1) *if* λ *and* μ *are generalized fuzzy open* (*resp. generalized fuzzy regular open*) *sets, then* $\lambda \wedge \mu$ *is a generalized fuzzy open* (*resp. generalized fuzzy regular open*) *set*;

(2) *if* λ *is a generalized fuzzy open* (*resp. generalized fuzzy regular open*) *set and* $\mathrm{int}\,\lambda \leq \mu \leq \lambda$, *then* μ *is a generalized fuzzy open* (*resp. generalized fuzzy regular open*) *set*;

(3) *if* λ *is a fuzzy open set, then* λ *is a generalized fuzzy open set.*

2. Generalized Fuzzy α-closed Sets and Regular α-closed Sets

Definition 2.1. Let λ be a fuzzy set of an fts *X*. Then

$\alpha\mathrm{cl}\lambda = \wedge\{\mu \mid \lambda \leq \mu, \mu \text{ is a fuzzy } \alpha\text{-closed set}\}$ is called a *fuzzy α-closure* of λ;

$\alpha\,\mathrm{int}\,\lambda = \vee\{\mu \mid \mu \leq \lambda, \mu \text{ is a fuzzy } \alpha\text{-open set}\}$ is called a *fuzzy α-interior* of λ.

Theorem 2.1. *Let A be a fuzzy set of an fts X. Then the following statements are true*:

(1) $\alpha\,\mathrm{int}(1-\lambda) = 1 - \alpha\mathrm{cl}\lambda$; (2) $\alpha\mathrm{cl}\lambda = \lambda \vee \mathrm{cl}(\mathrm{int}(\mathrm{cl}\lambda))$;

(3) $\alpha\,\mathrm{int}\,\lambda = \lambda \wedge \mathrm{int}(\mathrm{cl}(\mathrm{int}\,\lambda))$; (4) $\lambda \leq \alpha\mathrm{cl}\lambda \leq \mathrm{cl}\lambda$;

(5) $\alpha\mathrm{cl}\lambda(\alpha\mathrm{cl}\lambda) = \alpha\mathrm{cl}\lambda$.

Proof. Let A be any fuzzy set of an fts X. Then

(1) We have $\alpha\,\mathrm{int}(1-\lambda) = \vee\{\mu \mid \mu \leq 1-\lambda,\ \mu$ is a fuzzy α-open set$\}$ $= 1 - \wedge\{1-\mu \mid 1-\mu \geq \lambda,\ 1-\mu$ is a fuzzy α-colsed set$\} = 1 - \alpha\mathrm{cl}\lambda$.

(2) We put $\mu = \lambda \vee \mathrm{cl}(\mathrm{int}(\mathrm{cl}\lambda))$. From $\mathrm{cl}\mu = \mathrm{cl}\lambda \vee \mathrm{cl}(\mathrm{cl}(\mathrm{int}(\mathrm{cl}\lambda)))$ $= \mathrm{cl}\lambda \vee \mathrm{cl}(\mathrm{int}(\mathrm{cl}\lambda)) = \mathrm{cl}\lambda$ follows that $\mathrm{cl}(\mathrm{int}(\mathrm{cl}\mu)) = \mathrm{cl}(\mathrm{int}(\mathrm{cl}\lambda)) \leq \mu$. Hence μ is a fuzzy α-closed set and $\lambda \leq \mu$, so $\alpha\mathrm{cl}\lambda \leq \lambda \vee \mathrm{cl}(\mathrm{int}(\mathrm{cl}\lambda))$.

We suppose that $\alpha\mathrm{cl}\lambda \geq \lambda \vee \mathrm{cl}(\mathrm{int}(\mathrm{cl}\lambda))$. Then, there exist $x \in X$ and $t \in (0, 1)$ such that $\alpha\mathrm{cl}\lambda(x) < t < \lambda(x) \vee \mathrm{cl}(\mathrm{int}(\mathrm{cl}\lambda))(x)$. According to the definition of the fuzzy α-closure of a fuzzy set, there exists a fuzzy α-closed set ρ with $\lambda \leq \rho$ such that $\alpha\mathrm{cl}\lambda(x) \leq \rho(x) < t <$ $\lambda(x) \vee \mathrm{cl}(\mathrm{int}(\mathrm{cl}\lambda))(x)$. On the other hand, since ρ is a fuzzy α-closed set and $\lambda \leq \rho$, we have $\mathrm{cl}(\mathrm{int}(\mathrm{cl}\rho)) \leq \rho$ and $\lambda \vee \mathrm{cl}(\mathrm{int}(\mathrm{cl}\lambda)) \leq \rho \vee \mathrm{cl}(\mathrm{int}(\mathrm{cl}\rho)) = \rho$. It is a contradiction. Hence $\alpha\mathrm{cl}\lambda \geq \lambda \vee \mathrm{cl}(\mathrm{int}(\mathrm{cl}\lambda))$.

(3) It can be proved in a similar manner as (2).

(4) It follows from the fact that every fuzzy closed set is a fuzzy α-closed set.

(5) It can be proved by using Definition 2.1 and Theorem 1.1(2).

Theorem 2.2. *If* λ *is a fuzzy* α*-open set of an fts* X, *then* $\alpha\mathrm{cl}\lambda = \mathrm{cl}\lambda =$ $\mathrm{cl}(\mathrm{int}\,\lambda)$.

Proof. Let λ be any fuzzy α-open set. We have $\lambda \leq \mathrm{int}(\mathrm{cl}(\mathrm{int}\,\lambda))$, so $\mathrm{cl}\lambda \leq \mathrm{cl}(\mathrm{int}\,\lambda)$. Therefore $\mathrm{cl}\lambda = \mathrm{cl}(\mathrm{int}\,\lambda)$. Hence

$$\alpha\mathrm{cl}\lambda = \lambda \vee \mathrm{cl}(\mathrm{int}(\mathrm{cl}\lambda)) = \lambda \vee \mathrm{cl}(\mathrm{int}(\mathrm{cl}(\mathrm{int}\,\lambda))) = \lambda \vee \mathrm{cl}(\mathrm{int}\,\lambda) = \lambda \vee \mathrm{cl}\lambda = \mathrm{cl}\lambda.$$

Definition 2.2. A fuzzy set λ of an fts X is called

(1) a *generalized fuzzy* α*-closed set* if and only if $\alpha\mathrm{cl}\lambda \leq \mu$, for each fuzzy open set μ such that $\lambda \leq \mu$;

(2) a *generalized fuzzy regular* α*-closed set* if and only if $\alpha\mathrm{cl}\lambda \leq \mu$, for each fuzzy regular open set μ such that $\lambda \leq \mu$.

Definition 2.3. A fuzzy set λ of an fts X is called

(1) a *generalized fuzzy* α*-open set* if and only if $1 - \lambda$ is a generalized fuzzy regular α-closed set;

(2) a *generalized fuzzy regular* α*-open set* if and only if $1 - \lambda$ is generalized fuzzy regular α-closed set.

Remark 2.1. From the above definitions it is not difficult to conclude that the following diagram of implications is true.

$$\begin{array}{ccc} \text{generalized fuzzy closed} & \Rightarrow & \text{generalized fuzzy regular closed} \\ \Downarrow & & \Downarrow \\ \text{generalized fuzzy } \alpha\text{-closed} & \Rightarrow & \text{generalized fuzzy regular } \alpha\text{-closed} \end{array}$$

Example 2.1. Let $X = \{a, b\}$ and let λ, μ and ν are fuzzy sets defined by

$$\lambda(a) = 0,2;\ \lambda(b) = 0,4;\ \mu(a) = 0,9;\ \mu(b) = 0,4;\ \nu(a) = 0,1;\ \nu(b) = 0,4.$$

Let $\tau_1 = \{0, \lambda, \mu, 1\}$. By easy computation it can be shown that ν is a generalized fuzzy α-closed set, but it is not a generalized fuzzy closed set. Also, ν is a generalized fuzzy regular α-closed set, but it is not a generalized fuzzy regular closed set.

If we put $\tau_2 = \{0, \mu, 1\}$, then the fuzzy set ν is a generalized fuzzy regular α-closed set, but it is not a generalized fuzzy α-closed set. Also, ν is a generalized fuzzy regular closed set, but it is not a generalized fuzzy closed set.

Theorem 2.3. *Let* λ *and* μ *be fuzzy sets of an fts X. Then the following statements hold*:

(1) *if* λ *is a generalized fuzzy* α*-closed* (*resp. generalized fuzzy regular* α*-closed*) *set and* $\lambda \le \mu \le \alpha\text{cl}\lambda$, *then* μ *is a generalized fuzzy* α*-closed* (*resp. generalized fuzzy regular* α*-closed*) *set*;

(2) *if* λ *is a fuzzy closed set*, *then* λ *is both generalized fuzzy* α*-closed and generalized fuzzy regular* α*-closed*;

(3) *if* λ *is a fuzzy* α*-closed set*, *then* λ *is both generalized fuzzy* α*-closed and generalized fuzzy regular* α*-closed*.

Proof. (1) Let λ be any generalized fuzzy α-closed set and $\lambda \leq \mu \leq \alpha\text{cl}\lambda$. Then $\alpha\text{cl}\lambda \leq \rho$, for each fuzzy open set ρ such that $\lambda \leq \rho$. From $\mu \leq \alpha\text{cl}\lambda$, we have $\alpha\text{cl}\mu \leq \alpha\text{cl}(\alpha\text{cl}\lambda) = \alpha\text{cl}\lambda \leq \rho$.

Hence μ is a generalized fuzzy α-closed set.

The other case can be proved in a similar manner.

(2) It can be proved by using the relations $\alpha\text{cl}\lambda \leq \text{cl}\lambda = \lambda$.

(3) It follows immediately from the relation $\alpha\text{cl}\lambda = \lambda$.

Theorem 2.4. *Let λ and μ be fuzzy sets of an fts X. Then the following statements hold*:

(1) *if λ is a generalized fuzzy α-open* (*resp. generalized fuzzy regular α-open*) *set and* $\alpha\,\text{int}\,\lambda \leq \mu \leq \lambda$, *then μ is a generalized fuzzy α-open* (*resp. generalized fuzzy regular α-open*) *set*;

(2) *if λ is a fuzzy open set, then λ is both generalized fuzzy α-open and generalized fuzzy regular α-open*;

(3) *if λ is a fuzzy α-open set, then λ is both generalized fuzzy α-open and generalized fuzzy regular α-open*.

Proof. It follows from Theorem 2.1(1) and Theorem 2.3.

Definition 2.4. Let λ be a fuzzy set of an fts *X*. Then

$g\alpha\text{cl}\lambda = \wedge\{\mu \mid \lambda \leq \mu, \mu$ is a generalized fuzzy α-closed set$\}$ is called a *generalized fuzzy α-closure* of λ;

$gr\alpha\text{cl}\lambda = \wedge\{\mu \mid \lambda \leq \mu, \mu$ is a generalized fuzzy regular α-closed set$\}$ is called a *generalized fuzzy regular α-closure* of λ.

Theorem 2.5. *Let λ be a fuzzy set of an fts X. Then the following statements hold*:

(1) $\lambda \leq gr\alpha\text{cl}\lambda \leq g\alpha\text{cl}\lambda \leq \alpha\text{cl}\lambda$; (2) $g\alpha\text{cl}(g\alpha\text{cl}\lambda) = g\alpha\text{cl}\lambda$;

(3) $gr\alpha\text{cl}(gr\alpha\text{cl}\lambda) = gr\alpha\text{cl}\lambda$; (4) $\alpha\text{cl}(g\alpha\text{cl}\lambda) = \alpha\text{cl}\lambda = g\alpha\text{cl}(\alpha\text{cl}\lambda)$;

(5) $\alpha\text{cl}(gr\alpha\text{cl}\lambda) = \alpha\text{cl}\lambda = gr\alpha\text{cl}(\alpha\text{cl}\lambda)$.

Proof. (1) It follows immediately from Definition 2.2 and Remark 2.1.

(2) We suppose that $g\alpha\text{cl}\lambda \geq g\alpha\text{cl}(g\alpha\text{cl}\lambda)$. Then there exist $x \in X$ and $t \in (0, 1)$ such that $g\alpha\text{cl}\lambda(x) < t < g\alpha\text{cl}(g\alpha\text{cl}\lambda)(x)$. According to the definition of the generalized α-closure of a fuzzy set, there exists a generalized fuzzy α-closed set ρ with $\lambda \leq \rho$ such that $g\alpha\text{cl}\lambda(x) \leq \rho(x) < t$. On the other hand, since ρ is a generalized fuzzy α-closed set and $\lambda \leq \rho$, we have $g\alpha\text{cl}\lambda \leq \rho$. Then $g\alpha\text{cl}(g\alpha\text{cl}\lambda) \leq g\alpha\text{cl}\rho = \rho$, so we obtain that $g\alpha\text{cl}(g\alpha\text{cl}\lambda)(x) \leq \rho(x) < t$. This is a contradiction. Hence $g\alpha\text{cl}\lambda \geq g\alpha\text{cl}(g\alpha\text{cl}\lambda)$.

The equality follows from the last relation and the evident relation $g\alpha\text{cl}\lambda \leq g\alpha\text{cl}(g\alpha\text{cl}\lambda)$.

(3) It can be proved in a similar manner as (2).

(4) Since $\alpha\text{cl}\lambda$ is a fuzzy α-closed set, it follows from Theorem 2.3(3) that $\alpha\text{cl}\lambda$ is a generalized fuzzy α-closed set and a generalized fuzzy regular α-closed set. Hence $g\alpha\text{cl}(\alpha\text{cl}\lambda) = \alpha\text{cl}\lambda$.

We will prove the first equality of (4). From $\lambda \leq g\alpha\text{cl}\lambda$ it follows that $\alpha\text{cl}\lambda \leq \alpha\text{cl}(g\alpha\text{cl}\lambda)$.

We suppose that $\alpha\text{cl}\lambda \geq \alpha\text{cl}(g\alpha\text{cl}\lambda)$. Then there exist $x \in X$ and $t \in (0, 1)$ such that $\alpha\text{cl}(g\alpha\text{cl}\lambda)(x) > t > \alpha\text{cl}\lambda(x)$.

Since $\alpha\text{cl}\lambda(x) < t$, from the definition of the fuzzy α-closure of a fuzzy set, it follows that there exists a fuzzy α-closed set ρ with $\lambda \leq \rho$ such that $\alpha\text{cl}(g\alpha\text{cl}\lambda)(x) > t > \rho(x) \geq \alpha\text{cl}\lambda(x)$.

From Theorem 2.3(3), it follows that ρ is a generalized fuzzy α-closed set, so $\rho = g\alpha\text{cl}\rho$. Hence $g\alpha\text{cl}\lambda \leq g\alpha\text{cl}\rho = g\alpha\text{cl}(\alpha\text{cl}\rho) = \alpha\text{cl}\rho = \rho$. Thus $\alpha\text{cl}(g\alpha\text{cl}\lambda) \leq \rho$. It is a contradiction, so $\alpha\text{cl}(g\alpha\text{cl}\lambda) \leq \alpha\text{cl}\lambda$.

(5) It can be proved in a similar manner as (4).

3. Fuzzy Normal, Fuzzy Almost and Fuzzy Mildly Normal Spaces

Definition 3.1. An fts (X, τ) is called:

(1) *fuzzy normal* if and only if for each pair of fuzzy closed sets λ_1 and λ_2 such that $\lambda_1 \overline{q} \lambda_2$, there exist fuzzy open sets μ_1 and μ_2, such that $\lambda_1 \leq \mu_1, \lambda_2 \leq \mu_2$ and $\mu_1 \overline{q} \mu_2$;

(2) *fuzzy almost normal* if and only if for each fuzzy closed set λ_1 and each fuzzy regular closed set λ_2, such that $\lambda_1 \overline{q} \lambda_2$, there exist fuzzy open sets μ_1 and μ_2, such that $\lambda_1 \leq \mu_1, \lambda_2 \leq \mu_2$ and $\mu_1 \overline{q} \mu_2$;

(3) *fuzzy mildly normal* if and only if for each fuzzy regular closed sets λ_1 and λ_2 such that $\lambda_1 \overline{q} \lambda_2$, there exist fuzzy open sets μ_1 and μ_2, such that $\lambda_1 \leq \mu_1, \lambda_2 \leq \mu_2$ and $\mu_1 \overline{q} \mu_2$.

Remark 3.1. Since each fuzzy regular closed set is a fuzzy closed set, it is not difficult to prove that the following diagram of implications is true.

fuzzy normal space $\Rightarrow$ fuzzy almost normal space $\Rightarrow$ fuzzy mildly normal space

The following example shows that the converse may not be true.

Example 3.1. Let $X = I$ and let λ, μ, ν, η and ω are fuzzy sets defined by

$\lambda(x) = 0, 4,$ for each $x \in I$, $\mu(x) = 0, 7,$ for each $x \in I$, $\nu(x) = 0, 8,$ for each $x \in I$,

$\eta(x) = 0, 6,$ for each $x \in I$, $\omega(x) = 0, 2,$ for each $x \in I$.

Let $\tau_1 = \{0, \lambda, \mu, 1\}$ and $\tau_2 = \{0, \mu, \nu, \eta, \omega, 1\}$. By easy computation it can be shown that (X, τ_1) is a fuzzy mildly normal space which is not a fuzzy almost normal space. The fts (X, τ_2) is a fuzzy almost normal space, but it is not a fuzzy normal space.

Lemma 3.1. *Let X be an fts. Then the following statements hold*:

(1) $\operatorname{int}(\operatorname{cl}(\operatorname{int}\lambda)) \leq \operatorname{cl}(\operatorname{int}(\operatorname{cl}\lambda))$, *for each fuzzy set* λ *of X*;

(2) *if* $\lambda \overline{q} \mu$, *then* $\operatorname{int}(\operatorname{cl}(\operatorname{int}\lambda)) \overline{q} \operatorname{int}(\operatorname{cl}(\operatorname{int}\mu))$, *for each fuzzy set* λ *and* μ *of X.*

Proof. (1) From $\mathrm{cl}(\mathrm{int}\,\lambda) \le \mathrm{cl}\lambda$, it follows that $\mathrm{int}(\mathrm{cl}(\mathrm{int}\,\lambda)) \le \mathrm{int}(\mathrm{cl}\lambda) \le \mathrm{cl}(\mathrm{int}(\mathrm{cl}\lambda))$;

(2) if $\lambda \overline{q} \mu$, then $\lambda \le 1 - \mu$, so

$$\mathrm{int}(\mathrm{cl}(\mathrm{int}\,\lambda)) \le \mathrm{int}(\mathrm{cl}(\mathrm{int}(1-\mu))) = 1 - \mathrm{cl}(\mathrm{int}(\mathrm{cl}\mu)) \le 1 - \mathrm{int}(\mathrm{cl}(\mathrm{int}\,\mu)).$$

Theorem 3.2. *Let* (X, τ) *be an fts. Then the following statements are equivalent*:

(1) (X, τ) *is a fuzzy mildly normal space*;

(2) *for each pair of fuzzy regular closed sets* λ_1 *and* λ_2 *such that* $\lambda_1 \overline{q} \lambda_2$, *there exist fuzzy regular open sets* μ_1 *and* μ_2, *such that* $\lambda_1 \le \mu_1$, $\lambda_2 \le \mu_2$ *and* $\mu_1 \overline{q} \mu_2$;

(3) *for each pair of fuzzy regular closed sets* λ_1 *and* λ_2 *such that* $\lambda_1 \overline{q} \lambda_2$, *there exist generalized fuzzy open sets* μ_1 *and* μ_2, *such that* $\lambda_1 \le \mu_1$, $\lambda_2 \le \mu_2$ *and* $\mu_1 \overline{q} \mu_2$;

(4) *for each pair of fuzzy regular closed sets* λ_1 *and* λ_2 *such that* $\lambda_1 \overline{q} \lambda_2$, *there exist generalized fuzzy* α*-open sets* μ_1 *and* μ_2, *such that* $\lambda_1 \le \mu_1$, $\lambda_2 \le \mu_2$ *and* $\mu_1 \overline{q} \mu_2$;

(5) *for each fuzzy regular closed set* λ *and each fuzzy regular open set* μ *such that* $\lambda \le \mu$, *there exists a fuzzy open set* ρ *such that* $\lambda \le \rho \le \mathrm{cl}\rho \le \mu$;

(6) *for each fuzzy regular closed set* λ *and each fuzzy regular open set* μ *such that* $\lambda \le \mu$, *there exists a fuzzy regular open set* ρ *such that* $\lambda \le \rho \le \mathrm{cl}\rho \le \mu$;

(7) *for each fuzzy regular closed set* λ *and each fuzzy regular open set* μ *such that* $\lambda \le \mu$, *there exists a generalized fuzzy open set* ρ *such that* $\lambda \le \rho \le \mathrm{cl}\rho \le \mu$;

(8) *for each fuzzy regular closed set* λ *and each fuzzy regular open set* μ *such that* $\lambda \le \mu$, *there exists a generalized fuzzy* α*-open set* ρ *such that* $\lambda \le \rho \le \alpha\mathrm{cl}\rho \le \mu$;

(9) *for each fuzzy regular closed set* λ *and each fuzzy regular open set* μ *such that* $\lambda \leq \mu$, *there exists a fuzzy* α*-open set* ρ *such that* $\lambda \leq \rho \leq \alpha\mathrm{cl}\rho \leq \mu$;

(10) *for each pair of fuzzy regular closed sets* λ_1 *and* λ_2 *such that* $\lambda_1 \overline{q} \lambda_2$, *there exist fuzzy* α*-open sets* μ_1 *and* μ_2, *such that* $\lambda_1 \leq \mu_1$, $\lambda_2 \leq \mu_2$ *and* $\mu_1 \overline{q} \mu_2$.

Proof. (1) $\Rightarrow$ (5) Let λ be any fuzzy regular closed set and let μ be fuzzy regular open set such that $\lambda \leq \mu$. Then $\lambda \overline{q} 1 - \mu$. According to the assumption there exist fuzzy regular open sets ρ and ν, such that $\lambda \leq \rho$, $1 - \mu \leq \nu$ and $\rho \overline{q} \nu$. It follows that $\lambda \leq \rho \leq \mathrm{cl}\rho \leq 1 - \nu \leq \mu$.

(5) $\Rightarrow$ (2) Let λ_1 and λ_2 be any fuzzy regular closed sets such that $\lambda_1 \overline{q} \lambda_2$. Then $\lambda_1 \leq 1 - \lambda_2$. According to the assumption there exists a fuzzy open set ρ such that $\lambda_1 \leq \rho \leq \mathrm{cl}\rho \leq 1 - \lambda_2$. It follows that $\lambda_1 \leq \mathrm{int}(\mathrm{cl}\rho) \leq 1 - \lambda_2$ and $\lambda_2 \leq 1 - \mathrm{cl}\rho = \mathrm{int}(1 - \rho)$. Then $\mu_1 = \mathrm{int}(\mathrm{cl}\rho)$ and $\mu_2 = \mathrm{int}(1 - \rho)$ are fuzzy regular open sets such that $\lambda_1 \leq \mu_1$, $\lambda_2 \leq \mu_2$ and $\mu_1 \overline{q} \mu_2$.

(2) $\Rightarrow$ (6) Let λ be any fuzzy regular closed set and let μ be a fuzzy regular open set such that $\lambda \leq \mu$. Then $\lambda \overline{q} 1 - \mu$. According to the assumption there exist fuzzy regular open sets ω and ν such that $\lambda \leq \omega$, $1 - \mu \leq \nu$ and $\omega \overline{q} \nu$. We put $\rho = \mathrm{int}(\mathrm{cl}\omega)$. Then ρ is a fuzzy regular open set such that $\lambda \leq \rho \leq \mathrm{cl}\rho \leq 1 - \nu \leq \mu$.

(6) $\Rightarrow$ (2) Let λ_1 and λ_2 be any fuzzy regular closed sets such that $\lambda_1 \overline{q} \lambda_2$. Then $\lambda_1 \leq 1 - \lambda_2$. According to the assumption there exists fuzzy regular open set ρ such that $\lambda_1 \leq \rho \leq \mathrm{cl}\rho \leq 1 - \lambda_2$. For $\mathrm{cl}\rho \leq 1 - \lambda_2$, there exists fuzzy regular open set ω such that $\mathrm{cl}\rho \leq \omega \leq \mathrm{cl}\omega \leq 1 - \lambda_2$. Then $\lambda_2 \leq 1 - \mathrm{cl}\omega = \mathrm{int}(1 - \omega)$ and $\mathrm{int}(1 - \omega)$ is a fuzzy regular open set. Finally, $\rho \leq \mathrm{cl}\omega$ implies $\rho \overline{q}\, \mathrm{int}(1 - \omega)$.

It is not difficult to prove the implications (1) $\Rightarrow$ (3) $\Rightarrow$ (4), (3) $\Rightarrow$ (7) $\Rightarrow$ (8) and (2) $\Rightarrow$ (1).

(4) $\Rightarrow$ (8) Let λ be any fuzzy regular closed set and let μ be a fuzzy regular open set such that $\lambda \leq \mu$. Then $\lambda \overline{q} 1 - \mu$. According to the assumption there exist generalized fuzzy α-open sets ρ and ν such that $\lambda \leq \rho, 1 - \mu \leq \nu$ and $\rho \overline{q} \nu$. Since ν is a generalized fuzzy α-open set and $1 - \mu$ is a fuzzy regular closed set, from $1 - \mu \leq \nu$ follows that $1 - \mu \leq \alpha \operatorname{int} \nu$. Thus $1 - \mu \leq \alpha \operatorname{int} \nu \leq \nu \leq 1 - \rho$.

Since $1 - \alpha \operatorname{int} \nu$ is a generalized fuzzy α-closed set, from $\rho \leq 1 - \alpha \operatorname{int} \nu$ we obtain that $\alpha - \operatorname{cl} \rho \leq 1 - \alpha \operatorname{int} \nu$. Hence $\lambda \leq \rho \leq \alpha \operatorname{cl} \rho \leq \mu$.

(8) $\Rightarrow$ (9) Let λ be any fuzzy regular closed set and let μ be a fuzzy regular open set such that $\lambda \leq \mu$. Then $\lambda \overline{q} 1 - \mu$. According to the assumption there exists a generalized fuzzy α-open set ω such that $\lambda \leq \omega \leq \alpha \operatorname{cl} \omega \leq \mu$. Since ω is a generalized fuzzy α-open set, from $\lambda \leq \omega$, it follows that $\lambda \leq \alpha \operatorname{int} \omega$. Then, $\rho = \alpha \operatorname{int} \omega$ is fuzzy α-open set and $\lambda \leq \rho \leq \alpha \operatorname{cl} \rho \leq \alpha \operatorname{cl} \omega \leq \mu$.

(9) $\Rightarrow$ (10) Let λ_1 and λ_2 be fuzzy regular closed sets such that $\lambda_1 \overline{q} \lambda_2$. Then $\lambda_1 \leq \overline{1} - \lambda_2$. According to the assumption there exists fuzzy α-open set μ_1 such that $\lambda_1 \leq \mu_1 \leq \alpha \operatorname{cl} \mu_1 \leq 1 - \lambda_2$. Then $\mu_2 = 1 - \alpha \operatorname{cl} \mu$ is a fuzzy α-open set and $\mu_1 \overline{q} \mu_2$.

(10) $\Rightarrow$ (1) Let λ_1 and λ_2 be fuzzy regular closed sets such that $\lambda_1 \overline{q} \lambda_2$. Then $\lambda_1 \leq 1 - \lambda_2$. According to the assumption there exist fuzzy α-open sets ν_1 and ν_2, such that $\lambda_1 \leq \mu_1, \lambda_2 \leq \mu_2$ and $\mu_1 \overline{q} \mu_2$. We put $\rho_1 = \operatorname{int}(\operatorname{cl}(\operatorname{int} \mu_1))$ and $\rho_2 = \operatorname{int}(\operatorname{cl}(\operatorname{int} \mu_2))$. Then ρ_1 and ρ_2 are fuzzy open sets. From Lemma 3.1(2) follows that $\mu_1 \overline{q} \mu_2$ implies $\rho_1 \overline{q} \rho_2$. Hence (X, τ) is a fuzzy normal space.

Theorem 3.3. *Let* (X, τ) *be an fts. Then the following statements are equivalent*:

(1) (X, τ) *is a fuzzy normal space* (*resp. fuzzy almost normal space*);

(2) *for each pair of fuzzy closed sets* λ_1 *and* λ_2 *such that* $\lambda_1 \overline{q} \lambda_2$, *there exist fuzzy regular open sets* μ_1 *and* μ_2, *such that* $\lambda_1 \leq \mu_1, \lambda_2 \leq \mu_2$ *and* $\mu_1 \overline{q} \mu_2$;

(3) *for each pair of fuzzy closed sets* λ_1 *and* λ_2 *such that* $\lambda_1 \overline{q} \lambda_2$, *there exist generalized fuzzy open sets* μ_1 *and* μ_2, *such that* $\lambda_1 \leq \mu_1$, $\lambda_2 \leq \mu_2$ *and* $\mu_1 \overline{q} \mu_2$;

(4) *for each pair of fuzzy closed sets* λ_1 *and* λ_2 *such that* $\lambda_1 \overline{q} \lambda_2$, *there exist generalized fuzzy* α*-open sets* μ_1 *and* μ_2, *such that* $\lambda_1 \leq \mu_1$, $\lambda_2 \leq \mu_2$ *and* $\mu_1 \overline{q} \mu_2$;

(5) *for each fuzzy closed set* λ *and each fuzzy open set* μ *such that* $\lambda \leq \mu$, *there exists a fuzzy open set* ρ *such that* $\lambda \leq \rho \leq \mathrm{cl}\rho \leq \mu$;

(6) *for each fuzzy closed set* λ *and each fuzzy open set* μ *such that* $\lambda \leq \mu$, *there exists a fuzzy regular open set* ρ *such that* $\lambda \leq \rho \leq \mathrm{cl}\rho \leq \mu$;

(7) *for each fuzzy closed set* λ *and each fuzzy open set* μ *such that* $\lambda \leq \mu$, *there exists a generalized fuzzy open set* ρ *such that* $\lambda \leq \rho \leq \mathrm{cl}\rho \leq \mu$;

(8) *for each fuzzy closed set* λ *and each fuzzy open set* μ *such that* $\lambda \leq \mu$, *there exists a generalized fuzzy open* α*-open set* ρ *such that* $\lambda \leq \rho \leq \alpha\mathrm{cl}\rho \leq \mu$;

(9) *for each fuzzy closed set* λ *and each fuzzy open set* μ *such that* $\lambda \leq \mu$, *there exists a fuzzy* α*-open set* ρ *such that* $\lambda \leq \rho \leq \alpha\mathrm{cl}\rho \leq \mu$;

(10) *for each pair of fuzzy closed sets* λ_1 *and* λ_2 *such that* $\lambda_1 \overline{q} \lambda_2$, *there exist fuzzy* α*-open sets* μ_1 *and* μ_2, *such that* $\lambda_1 \leq \mu_1$, $\lambda_2 \leq \mu_2$ *and* $\mu_1 \overline{q} \mu_2$.

Proof. It can be proved in a similar manner as Theorem 3.2.

Theorem 3.4. *Let* (X, τ) *be an fts. Then the following statements are equivalent*:

(1) (X, τ) *is a fuzzy almost normal space*;

(2) *for each fuzzy closed set* λ_1 *and each fuzzy regular closed set* λ_2 *such that* $\lambda_1 \overline{q} \lambda_2$, *there exist fuzzy regular open sets* μ_1 *and* μ_2, *such that* $\lambda_1 \leq \mu_1$, $\lambda_2 \leq \mu_2$ *and* $\mu_1 \overline{q} \mu_2$;

(3) *for each fuzzy closed set* λ_1 *and each fuzzy regular closed set* λ_2 *such that* $\lambda_1 \overline{q} \lambda_2$, *there exist generalized fuzzy open sets* μ_1 *and* μ_2, *such that* $\lambda_1 \leq \mu_1$, $\lambda_2 \leq \mu_2$ *and* $\mu_1 \overline{q} \mu_2$;

(4) *for each fuzzy closed set* λ_1 *and each fuzzy regular closed set* λ_2 *such that* $\lambda_1 \overline{q} \lambda_2$, *there exist generalized fuzzy* α*-open sets* μ_1 *and* μ_2, *such that* $\lambda_1 \leq \mu_1$, $\lambda_2 \leq \mu_2$ *and* $\mu_1 \overline{q} \mu_2$;

(5) *for each fuzzy closed set* λ *and each fuzzy regular open set* μ *such that* $\lambda \leq \mu$, *there exists a fuzzy open set* ρ *such that* $\lambda \leq \rho \leq \mathrm{cl}\rho \leq \mu$;

(6) *for each fuzzy closed set* λ *and each fuzzy regular open set* μ *such that* $\lambda \leq \mu$, *there exists a fuzzy regular open set* ρ *such that* $\lambda \leq \rho \leq \mathrm{cl}\rho \leq \mu$;

(7) *for each fuzzy closed set* λ *and each fuzzy regular open set* μ *such that* $\lambda \leq \mu$, *there exists a generalized fuzzy open set* ρ *such that* $\lambda \leq \rho \leq \mathrm{cl}\rho \leq \mu$;

(8) *for each fuzzy closed set* λ *and each fuzzy regular open set* μ *such that* $\lambda \leq \mu$, *there exists a generalised fuzzy* α*-open set* ρ *such that* $\lambda \leq \rho \leq \alpha\mathrm{cl}\rho \leq \mu$;

(9) *for each fuzzy closed set* λ *and each fuzzy regular open set* μ *such that* $\lambda \leq \mu$, *there exists a fuzzy* α*-open set* ρ *such that* $\lambda \leq \rho \leq \alpha\mathrm{cl}\rho \leq \mu$;

(10) *for each fuzzy closed set* λ_1 *and each fuzzy regular closed set* λ_2 *such that* $\lambda_1 \overline{q} \lambda_2$, *there exist fuzzy* α*-open sets* μ_1 *and* μ_2 *such that* $\lambda_1 \leq \mu_1$, $\lambda_2 \leq \mu_2$ *and* $\mu_1 \overline{q} \mu_2$.

Proof. It can be proved in a similar manner as Theorem 3.2.

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