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A NEW CLASS OF NONLINEAR INTEGRATED MODELS

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Abstract

Trend-trend break stationarity versus difference-fractional difference stationarity around macroeconomic fluctuations has been a long standing debate. This paper proposes a new class of nonlinear autoregressive integrated models that covers a wide range of processes including two special cases of trend-trend break stationarity and difference stationarity. The new model is derived from a stochastic delay-differential equation based on the analysis of mechanics of a class of economic systems in which there are a restoring regulation with time delay, a resistance and exogenous disturbances. It is demonstrated that the mean of disturbances and the resistance coefficient determine the slope of time trend, thereby exogenous impact and shifts in the resistance can lead to trend breaks. The model can generate the fluctuations between stationary and nonstationary state, and the typical nonlinear dynamics such as self-sustained oscillation, limit cycle and chaos. The critical value to bifurcate these obviously different dynamics seems to depend on the relative strength of the restoring force to the resistance, which will be further studied. An indicator to measure the nonstationarity of economic time series is developed and used to evaluate three empirical series.

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1. Introduction

Economic theories can provide nothing about the structure of economic time series, so econometricians have to uncover the hidden structure by recognizing the important features of economic data. The data features recognized include trend stationarity (TS)-trend break stationarity, difference stationarity (DS)-fractional difference stationarity (i.e., integration-fractional integration), long memory, mean reversion, nonlinearity and chaos. Trend-trend break stationarity versus difference-fractional difference stationarity around macroeconomic fluctuations has been a long standing debate. However recent research suggests that there are certain connections among them (Granger and Teräsvirta [5]; Andersson and Teräsvirta [1]; Granger and Hyung [4]). An interesting problem is whether these data features result from a common data generating mechanism. This paper attempts to deal with the issue by modeling systematically a class of economic variables like prices. We develop a nonlinear differential equation whose discrete version encompasses trend-trend break stationarity and difference stationarity, as well as chaos and limit cycle as its special cases. We are required to incorporate as much available background theory as possible in modeling. The sense of the worth of applying physical approach for modeling economics can trace back to Hamburger [6], who modeled business cycles using the Van der Pol equation (also see Tong [12] and Mohammed [9]). In this paper we utilize the physical analogy to model.

Consider an object with mass m, position $y(t)$, velocity $\dot{y}(t)$ and acceleration $\ddot{y}(t)$ at time t. From Newton's second law, the acceleration $\ddot{y}(t)$ of the object is directly proportional to the net force F acting upon the object and inversely proportional to the mass m of the object, denoted by $\ddot{y}(t) = F/m$ or $F = m\ddot{y}(t)$. In actual modeling, F often includes random forces in stochastic systems (Tong [12]). Here we let $y(t)$ represent the price of a certain product at time t. Then $\dot{y}(t)$ is the price adjustment (the change in price), $\ddot{y}(t)$ is the rate of change of the price adjustment, F is the sum of the forces acting upon the price movement, and m is a scalar referring to market scale of the product.

Price adjustment relates costs but the costs resist price change. Some economic factors like fixed-price contracts that provide a disincentive to change induce the stickiness of price change. Hence there is always a resistance in price movement. Responses to price adjustment usually reflect the power of the resistance, thereby we assume that the resistance is a nonlinear function of the price adjustment $\dot{y}(t)$, denoted by $f[\dot{y}(t)]$.

Prices tend to revert to market normal level whenever they deviate from that normal level by market competition. Central banks also design various financial policies to maintain price stability. We call the force to revert prices to market normal level the restoring force and assume that it is a nonlinear function of the deviation $y(t) - \mu(t)$ from normal level $\mu(t)$, denoted by $g[y(t) - \mu(t)]$. There $\mu(t)$ is defined to be the unconditional mean of price by $\mu(t) = E(y_t)$. There is always a time delay in the regulatory effect of g, because g only responds after the deviation $y(t) - \mu(t)$ has occurred. A delay parameter τ is thus embedded in $g : g[y(t - \tau) - \mu(t - \tau)]$.

Price movement in an open system is disturbed by unexplained shocks (forces not explicitly taken into account), explained but unpredictable shocks (e.g., changes caused by an oil crisis or a war), and the impact of the related economic variables such as monetary shocks. To simplify matters, we only consider unexplained shocks, expressed by a Gaussian variable with a non-zero mean $e(t)$. The impact of other shocks is further discussed in Subsection 3.2.

2. Derivation of Models

If an economic variable is under the influence of the forces described above, its motion can be described as the following stochastic delay-differential equation:

$$m\ddot{y}(t) = f[\dot{y}(t)] + g[y(t - \tau) - \mu(t - \tau)] + e(t), \tag{1}$$

where f and g are two nonlinear continuous functions satisfying

Assumption A.

(a) $xH(x) < 0$ for $x \neq 0$ and $H(-x) = -H(x)$;

(b) $H(x) = O(|x|^{-(1+\delta)})$ for large $|x|$ and $\delta > 0$;

(c) $H(x)$ has second order continuous derivatives at $x = 0$,

where H denotes either f or g.

Assumption A(a) is given for the following reasons. The function $f(g)$ is one in which an increase or decrease in the variable being regulated brings about responses that move the variable in the direction opposite to the direction of the change (the sign of the derivation) as the nature of a resistance (a restoring force). The functions f and g are thus required to satisfy $xH(x) < 0$ for $x \neq 0$. Further the response to the rising or falling price adjustment or the restoring regulation to a positive or negative deviation is usually symmetrical. Therefore, f and g are set to be odd functions by $H(-x) = -H(x)$. Assumption A(b) is an integrable condition to avoid an explosive solution due to f and g, because we wish to describe such a system where economic breakup could not occur (see Granger and Andersen [3]). To perform time series analysis, we need to obtain the discrete version of equation (1) by applying Taylor series expansion. Assumption A(c) is set to achieve this aim.

Let h be the interval of the time series. Then the regulatory delay is the integral part of τ/h, denoted by k, implying that $\tau \geq h$, $y_t = y(th)$, $\Delta y_t = y_t - y_{t-1}$, $\mu_{t-k} = \mu[(t-k)h]$, and $e_t = e(th)$. Let $\varepsilon_t = c - e_t \sim$ i.i.d. $N(0, \sigma^2)$, where $c = E(e_t)$. The initial values of $y_{-(k-1)}, \ldots, y_{-1}$ and y_0 are set to be constants with probability 1. The discrete version of equation (1) is given by

$$\frac{m\Delta^2 y_t}{h^2} = c + f\left(\frac{\Delta y_t}{h}\right) + g(y_{t-k} - \mu_{t-k}) + \varepsilon_t. \tag{2}$$

The equation can be regarded as a generalized equation in the following sense. From equation (2), we can derive two autoregressive processes as follows:

Theorem 1. *Let* $h = 1$. *Suppose that* $f(x)$ *and* $g(x)$ *satisfy Assumption A.*

(i) *Equation* (2) *approximates a nonlinear autoregressive integrated process* (*abbreviated as NLARI*):

$$y_t = a_0 + a_1 y_{t-1} + a_2 y_{t-2} + \gamma^* g(y_{t-k} - \mu_{t-k}) + v_t, \tag{3}$$

$$a_0 = \gamma^* c,\ a_1 = 1 + \zeta,\ a_2 = -\zeta,\ \zeta = \frac{1}{1 + \alpha/m},\ \gamma^* = \frac{1}{m + \alpha},$$

$$\alpha = -f'(0),\ \mu_t = y_0 + \frac{c}{\alpha} t = y_0 + \frac{a_0}{1 - \zeta} t,\ c = E(e_t),\ v_t = \gamma^* \varepsilon_t$$

for small $E|\Delta y_t|$, *where* $a_1 \in (1, 2)$, $a_2 \in (-1, 0)$, $\gamma^* > 0$ *and* $\alpha > 0$;

(ii) *Equation* (3) *approximates a stationary autoregressive process around a trend* (*abbreviated as TSAR*):

$$y_t = d_0 + d_1 t + a_1 y_{t-1} + a_2 y_{t-2} + a_k y_{t-k} + v_t,$$

$$d_1 = \gamma^* \frac{c\beta}{\alpha},\ a_k = -\gamma^* \beta,\ \beta = -g'(0) \tag{4}$$

for small $\mathrm{Var}(y_t)$, *where* $a_k < 0$, $k \geq 1$ *and* $\beta > 0$.

Proof. See Appendix A.

Remark 1. From Assumption A(a), $f(0) = g(0) = 0$, implying that $f(x) \approx -\alpha x$ and $g(x) \approx -\beta x$ around $x = 0$. Then we call α the *resistance coefficient*, β the *restoring force coefficient*, and $\gamma = (\beta/m)(1 + \alpha/m)^{-1}$ the *restoring force coefficient relative to the resistance*.

Remark 2. According to Wold's theorem, any stationary time series has an approximate stationary and invertible ARMA representation. Therefore, if the disturbance $\{v_t\}$ is serially correlated stationary pattern, equations (3) and (4) can be extended into a nonlinear autoregressive integrated process NLARI with the general order lag.

Remark 3. In most cases $E|\Delta y_t|$ for the logs of original series $\{x_t\}$ is less than $E|\Delta x_t|$ by $E|\Delta y_t| = E|\log(x_t / x_{t-1})|$, so the linearization condition "small $E|\Delta y_t|$" is easily satisfied. This suggests that equation (3) is likely a good practical approximation to the logs of economic series.

3. The Structure of NLARI

3.1. Encompassing DS and TS

From (2) to (4), we have

$$\begin{array}{l} \text{Eq. (2) for } f = g = 0 \\ \text{Eq. (3) for } g = 0, \text{ small } \alpha \\ \text{Eq. (3) for } g = 0, \text{ large } \alpha \\ \text{Eq. (3) for } g = 0 \\ \text{Eq. (3) for small Var } (y_t) \end{array} = \begin{cases} 2-\text{order integration } I(2) \\ \text{near } 2-\text{order integration } I(2) \\ \text{near random walk} \\ \text{ARI} : a_1 \in (1, 2), a_2 \in (-1, 0) \\ \text{TSAR} : a_1 \in (1, 2), a_2 \in (-1, 0), a_k < 0. \end{cases}$$

It is shown that equation (2) covers a wide range of processes including $I(2)$, near $I(2)$, near random walk, ARI and AR with special coefficient ranges.

3.2. Exogenously impact

This subsection focuses on the impact of exogenous shocks to the dynamics of the NLARI (3). The drift in (3), $a_0 = c(m + \alpha)^{-1}$, where $c = E(e_t)$, reflects the mean disturbance relative to the resistance, thereby a_0 is a key parameter to reflect exogenous disturbances. Equation (3) can be expanded as follows:

$$y_t = y_0 + c\alpha^{-1}t + \gamma^* \sum_{i=k+1}^{t} \sum_{j=0}^{i-1} \zeta^j g(y_{i-k-j} - \mu_{i-k-j}) + \gamma^* \sum_{i=1}^{t} \sum_{j=0}^{i-1} \zeta^j \varepsilon_{i-j} \tag{5}$$

(for proof see Appendix A). From the expansion (5), the values of $E(e_t)$ and α determine the slope of trend $c\alpha^{-1}t$ that dominates the dynamics of the process $\{y_t\}$ if $\gamma^* \sum_{i=1}^{t} (\varepsilon_i^* + g_{i-k}^*)$ is of a probability order less than t. We thus see that trend breaks may result from the shift in $E(e_t)$ due to various exogenous factors including casual shocks like those induced by an oil price crisis or a war, as well as the shift in the value of α. In the following subsection, we show other factors to result in trend breaks.

3.3. Endogenously fluctuations

This subsection investigates how the deterministic terms of the NLARI (6) influence the dynamics of the process. To do this, we let $E(e_t) = 0$ $(a_0 = 0)$ and the value of $\sigma^2 = \mathrm{Var}(e_t)$ be unchanged in each case. The function $g(x) = -\beta x / \exp(x^2)$ satisfies Assumption A and so is chosen as the restoring force function. To simplify matters, let $k = m = 1.$ Equation (3) can be specified by

$$y_t = (1+\zeta)y_{t-1} - \zeta y_{t-2} - \gamma \frac{y_{t-1} - y_0}{\exp[(y_{t-1} - y_0)^2]} + v_t,$$

$$\zeta = \frac{1}{1+\alpha}, \ \gamma = \frac{\beta}{1+\alpha}, \ v_t = \frac{\varepsilon_t}{1+\alpha}. \tag{6}$$

We can demonstrate that $\beta < 4 + 2\alpha$ if the NLARI (6) approximates a stationary TSAR (for proof see Appendix B). This suggests that the NLARI (6) may possess more complex dynamics if $\beta \geq 4 + 2\alpha$. From Figure 1 we see that when $\beta \geq 4 + 2\alpha = 6$, the NLARI (6) exhibits serious oscillations that are very different from the case $\beta < 4 + 2\alpha = 6$ (in each case $\alpha = 1, \sigma = 0.01$, and the initial values are zero). When $\beta < 6$, the NLARI process fluctuated as if Gaussian white noise for $\beta = 1$, and oscillates between unit root process $(\beta = 0)$ and Gaussian white noise for $\beta = 0.04$. When $\beta \geq 6$, the dynamics of the NLARI process is closely to random self-sustained oscillations for $\beta = 6.06$, random limit cycle for $\beta = 8$, and random chaos for $\beta = 12$. Interestingly, it is shown that a larger value of β corresponds to the smaller amplitudes of the fluctuations plotted by the NLARI process for $\beta < 4 + 2\alpha$, while a larger value of β is accompanied by the larger amplitudes for $\beta \geq 4 + 2\alpha$. The latter suggests that a larger restoring force can induce the overshoot and overcompensate resulting in serious oscillations.

4. A Measure of Nonstationarity

This section develops an indicator to measure the nonstationarity. We first introduce the series $\{Y_{t,s}\}_{t=1}^{n}$ and $\{Y_{t,s}\}_{s=1}^{m}$, where $Y_{t,s} =$

$s^{-1}\sum_{i=1}^{s} y_{(t-1)s+i}$ and $\{y_j\}_{j=1}^{N}$ is generated by the NLARI $(N = sn = tm)$. For the two special cases of the NLARI, ARI and TSAR, we have the following.

Theorem 2. *Suppose* $E(e_t) = 0$. *Let* λ *be an integer larger than one.*

(i°) *For ARI,*

$$\frac{1}{\sqrt{(t-1)s}} Y_{t,\lambda s} \Rightarrow \sqrt{\frac{\lambda}{1-\zeta^2}} X \text{ as } t \to \infty$$

for a given s, where $X \sim N(0, \gamma^* \sigma^2)$.

(ii°) *For TSAR,*

$$\sqrt{s} Y_{t,\lambda s} \Rightarrow \frac{1}{\sqrt{\lambda}} X \text{ as } s \to \infty$$

for a given t.

Otherwise the results (i°) *and* (ii°) *hold for* $y_t^* = y_t - E(y_t)$.

Proof. See Appendix B.

Corollary 1. *Suppose* $E(e_t) = 0$ *and* λ *is an integer larger than one. The ratio of the sample standard errors of* $\{Y_{t,\lambda s}\}$ *and* $\{Y_{t,s}\}$ *is given by*

(i*) *For ARI*

$$r_s(\lambda) = \frac{se\left(\frac{Y_{t,\lambda s}}{\sqrt{(t-1)s}}\right)}{se\left(\frac{Y_{t,s}}{\sqrt{(t-1)s}}\right)} \approx \sqrt{\lambda}$$

for large m and a given s;

(ii*) *For TSAR*

$$r_t(\lambda) = \frac{se(\sqrt{s} Y_{t,\lambda s})}{se(\sqrt{s} Y_{t,s})} \approx \frac{1}{\sqrt{\lambda}}$$

for large n and a given t.

From Corollary 1, the data supports ARI if $r_s(\lambda) = \sqrt{\lambda},$ the data supports TSAR if $r_t(\lambda) = 1/\sqrt{\lambda},$ and the data is closely tied to neither ARI nor TSAR but it may tend toward ARI or TSAR if $1/\sqrt{\lambda}, < r_t(\lambda),\ r_s(\lambda) < \sqrt{\lambda}.$ Thus $r_s(\lambda)$ and $r_t(\lambda)$ reflect the nonstationarity and stationarity of the NLARI, called the *nonstationarity ratio* and the *stationarity ratio*, respectively.

As an example, we applied the nonstationarity ratio $r_s(\lambda)$ to evaluate the Canada/US foreign exchange rate (EXCAUS), the effective federal funds rate (FEDFUNDS) and the unemployment rate (UNRATE) over 1971:1 to 2005:12. They were three monthly series without time trend implying that $E(e_t) = 0.$ Table 1 reports the values of $r_s(\lambda)$ for the common logarithms of the three series. It is shown that the three series were between the integrated nonstationary and stationary state. UNRATE was closest to stationary and the farthest from the integrated state, but EXCAUS was close relatively to the integrated state. Our result is consistent with other empirical results.

5. Concluding Remarks

This paper proposes a new class of nonlinear autoregressive integrated NLARI process that can make transition from DS to TS as the relative restoring force coefficient γ changes. The existence of the NLARI explains why it is difficult to distinguish between DS and TS in empirical research. We have discussed how exogenous impact leads to trend breaks under the endogenous structure. It has been demonstrated that both the mean of disturbances and the resistance coefficient determine the slope of time trend, thereby the changes in them can cause trend breaks if $\gamma^* \sum_{i=1}^{t} (\varepsilon_i^* + g_{i-k}^*)$ is of a probability order less than t. Simultaneously, we have investigated how the deterministic part of the NLARI dominates the dynamics of the process. The bifurcation of the NLARI from stationary state to serious nonlinear oscillations seems to depend on whether β is smaller than $4 + 2\alpha.$ The issue refers to complex bifurcation theory in random dynamic systems and so remains in future.

Our results appear to be close to previous relevant work. For example, the NLARI with the p-order lag appears to be very similar to smooth transition autoregressive model (STAR)

$$y_t - \mu = \sum_{j=1}^{p} \alpha_j (y_{t-j} - \mu) + \left[\sum_{j=1}^{p} \beta_j (y_{t-j} - \mu) \right] \Phi(\delta;\ y_{t-d} - \mu) + \varepsilon_t \tag{7}$$

if the transition function Φ is specified as $\Phi(\delta;\ y_{t-d} - \mu) = 1 - \exp[-\delta^2 (y_{t-d} - \mu)^2]$ (see Michael et al. [8]; Peel and Taylor [10]; Taylor et al. [11]). However, there is not any relationship in economic meaning or innumeracy connection among the parameters α_j, β_j, μ and ε_t in equation (7), particularly μ is not time-dependent. Therefore, the NLARI is essentially different from the STAR. The latter cannot be expected to detect how economic endogenous structure and exogenous impact influence the dynamics of economic time series.

An indicator to measure the nonstationarity, $r_s(\lambda)$, has been introduced and used to evaluate three empirical series. However, the method was not developed into a test-of-significance approach. It seems to be very difficult to obtain the sampling distribution of $r_s(\lambda)$ or the analytic limit distribution of the NLARI process that can fluctuate between stationary and nonstationary state.

The next work is to estimate the NLARI including the delay parameter k and provide evidence that the NLARI can capture the long-memory and mean reverting properties in economic time series. But the NLARI is not suitable for analyzing the samples with very high- or low- frequency. A very high-frequency series usually corresponds to a long-delayed regulation, while a very low-frequency series easily involves a large interval that violates the discrete requirement $\tau \geq h$. Both cases likely lead to poor results.

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Appendix A

Proofs of Theorem 1 and Equation (5). Assumption A implies that f is bounded and $f(0) = 0$. We can expand $f(x)$ in a Taylor series as follows:

$$f(\Delta y_t) = f'(0)(\Delta y_t) + \frac{f''[\theta_2(\Delta y_t)]}{2}(\Delta y_t)^2.$$

Suppose $\Delta y_t = O_p(r_t)$. Then $\Delta y_t = O_p(r_t)$. If $E|\Delta y_t| = o(1)$, then $r_t = o(1)$ by Corollary 5.1.1.1 in Fuller [2]. Therefore,

$$f(\Delta y_t) = f'(0)(\Delta y_t) + O_p(r_t^2) \approx f'(0)(\Delta y_t),$$

provided by Corollary 5.1.5 in Fuller [2]. Then equation (2) approximates

$$\Delta^2 y_t = \frac{1}{m}c + \frac{1}{m}f'(0)(\Delta y_t) + \frac{1}{m}g(y_{t-k} - \mu_{t-k}) + \frac{1}{m}\varepsilon_t.$$

Let $\alpha = -f'(0)$. The condition $xf(x) < 0$ leads to $f'(0) < 0$ and $\alpha > 0$. Therefore, the first order approximation of f for equation (2) is given by

$$y_t = \frac{c}{m+\alpha} + \left(1 + \frac{m}{m+\alpha}\right)y_{t-1} - \frac{m}{m+\alpha}y_{t-2}$$

$$+ \frac{1}{m+\alpha}g(y_{t-k} - \mu_{t-k}) + \frac{1}{m+\alpha}\varepsilon_t.$$

Let

$$a_0 = \gamma^* c,\ \gamma^* = \frac{1}{m+\alpha},\ \zeta = \frac{m}{m+\alpha},\ a_1 = 1+\zeta,\ a_2 = -\zeta.$$

We obtain equation (3)

$$y_t = a_0 + a_1 y_{t-1} + a_2 y_{t-2} + \gamma^* g(y_{t-k} - \mu_{t-k}) + \gamma^*\varepsilon_t,$$

which can be rewritten as

$$y_t - (1+\zeta)y_{t-1} + \zeta y_{t-2} = a_0 + \gamma^* g(y_{t-k} - \mu_{t-k}) + \gamma^*\varepsilon_t.$$

Letting $\Delta = 1 - L$ (L is the lag operator), the above equation is given by

$$(1 - \zeta L)(1 - L) y_t = a_0 + \gamma^* g(y_{t-k} - \mu_{t-k}) + \gamma^* \varepsilon_t,$$

which yields

$$\begin{aligned}(1 - L) y_t &= \frac{a_0}{1 - \zeta} + \gamma^* \frac{g(y_{t-k} - \mu_{t-k})}{1 - \zeta L} + \gamma^* \frac{\varepsilon_t}{1 - \zeta L} \\ &= \frac{c}{\alpha} + \gamma^* \sum_{j=0}^{t-1} \zeta^j g(y_{t-k-j} - \mu_{t-k}) + \gamma^* \sum_{j=0}^{t-1} \zeta^j \varepsilon_{t-j}.\end{aligned}$$

Let

$$\varepsilon_t^* = \sum\nolimits_{j=0}^{t-1} \zeta^j \varepsilon_{t-j}$$

and

$$g_{t-k}^* = g^*(y_{t-k} - \mu_{t-k}) = \sum\nolimits_{j=0}^{t-1} \zeta^j g(y_{t-k-j} - \mu_{t-k-j}).$$

$$\begin{aligned} y_t &= y_{t-1} + \frac{c}{\alpha} + \gamma^* g_{t-k}^* + \gamma^* \varepsilon_t^* \\ &= y_{t-2} + 2\frac{c}{\alpha} + \gamma^*(g_{t-k}^* + g_{t-k-1}^*) + \gamma^*(\varepsilon_t^* + \varepsilon_{t-1}^*) \\ &\;\;\vdots \\ &= y_0 + t\frac{c}{\alpha} + \gamma^* \sum_{i=1}^{t} g_{i-k}^* + \gamma^* \sum_{i=1}^{t} \varepsilon_i^*. \end{aligned}$$

Define $\xi_t = \sum_{i=1}^{t} \varepsilon_i^*$. Obviously, $\mu_1 = E(y_1) = y_0 + c\alpha^{-1}$ and

$$y_1 - \mu_1 = \xi_1,\; E[g^*(y_1 - \mu_1)] = E[g^*(\xi_1)].$$

Since $\varepsilon_t \sim \text{i.i.d.}\, N(0, \sigma^2)$, $-\varepsilon_t$ and ε_t have identical distributions and it follows that $-\xi_1$ and ξ_1 have identical distributions. Hence, $g^*(\xi_1)$ and

$g^*(-\xi_1)$ have identical distributions. From the condition $g(-x) = -g(x)$, we have

$$E[g^*(\xi_1)] = E[g^*(-\xi_1)] = -E[g^*(\xi_1)],$$

implying that $E[g^*(\xi_1)] = 0$ and $E[g^*(y_1 - \mu_1)] = 0$. Then,

$$E(y_2) = \mu_2 = y_0 + 2\frac{c}{\alpha}, \; y_2 - \mu_2 = \xi_2 + g^*(\xi_1).$$

Similarly, ξ_2 and $-\xi_2$ have identical distributions and it follows that $\xi_2 + g^*(\xi_1)$ and $-\xi_2 + g^*(-\xi_1)$ have identical distributions, leading to

$$E\{g^*[\xi_2 + g^*(\xi_1)]\} = 0$$

and

$$E\{g^*(\xi_1) + g^*[\xi_2 + g^*(\xi_1)]\} = 0,$$

which yields $\mu_3 = y_0 + 3c\alpha^{-1}$. By the inductive method, we easily prove that

$$\mu_t = y_0 + \frac{c}{\alpha}t = y_0 + \frac{a_0}{1-\zeta}t.$$

We have thus completed the proofs of Theorem 1(i) and equation (5).

For small $\mathrm{Var}(y_t)$, one can ignore everything except the linear terms of the Taylor series expansion for $g(y_{t-k} - \mu_{t-k})$. Then, the first-order approximation of g in equation (3) is given by

$$\Delta^2 y_t = \frac{c}{m} - \frac{1}{m}\cdot\alpha(y_t - y_{t-1}) + \frac{g'(0)}{m}(y_{t-k} - \mu_{t-k}) + \frac{\varepsilon_t}{m}.$$

Let $\beta = -g'(0)$. Since $xg(x) < 0$, we have $\beta > 0$. Therefore,

$$y_t = \gamma^* c + \gamma^*\beta\mu_{t-k} + a_1 y_{t-1} + a_2 y_{t-2} + a_k y_{t-k} + e_t,$$

where $a_1 = 1 + \zeta$, $a_2 = -\zeta$ and $a_k = -\gamma^*\beta$. Since y_0 and y_{-1} are constants with probability 1, μ_0 is a constant. Taking expectations of both sides of the model results in

$$\mu_t = \gamma^* c + (1 + \zeta)\,\mu_{t-1} - \zeta\mu_{t-2}$$

and it follows that $(1 - \zeta L)(1 - L)\mu_t = \gamma^* c$ implying that

$$(1 - L)\mu_t = \frac{\gamma^* c}{1 - \zeta} = \frac{c}{\alpha}.$$

Therefore, $\mu_t = y_0 + c\alpha^{-1}t.$ We can write

$$\begin{aligned} \gamma^* c + \gamma^* \beta\mu_{t-k} &= \gamma^* c + \gamma^* \beta[y_0 + (t - k)c\alpha^{-1}] \\ &= \gamma^*(c + \beta y_0 - k\beta c\alpha^{-1}) + \gamma^* \beta c\alpha^{-1} t \\ &\equiv d_0 + d_1 t, \text{ say.} \end{aligned}$$

This derives equation (4).

Next, we will demonstrate that equation (4) describes a strictly stationary process. Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ denote the roots of equation

$$1 - a_1 L - a_2 L^2 - a_k L^k = 0.$$

Equation (4) has the expansion

$$y_t = b_0 + \sum_{j=0}^{\infty} \varphi_j v_{t-j},\ \varphi_j = \sum_{s=1}^{k} b_s \lambda_s^j,$$

where $y_0, y_{-1}, \ldots, y_{-k}$ are constants with probability 1, implying that $v_0, v_{-1}, \ldots$ are zero. The expansion can be rewritten as

$$y_t = b_0 + \sum_{j=0}^{t-1} \varphi_j v_{t-j},\ \varphi_j = \sum_{s=1}^{k} b_s \lambda_s^j.$$

If $|\lambda_{s_0}| \geq 1$ for a certain s_0, then $\text{Var}(y_t)$ diverges, which is contradicted by the assumption that $\text{Var}(y_t)$ is small. Hence $|\lambda_{s_0}| < 1$ and the process $\{y_t\}$ is second-order stationary. Since y_t is a linear combination of Gaussian white noise v_t, the joint distribution of $y_1, y_2, \ldots, y_t$ is multivariate normal for $1, 2, \ldots, t$. Equation (4) thus describes a strictly stationary process.

Appendix B

Proof of "$\beta < 4 + 2\alpha$". For the stationary process TSAR(4) with $k = 1$, two roots λ_1 and λ_2 of the equation $1 - (a_1 + a_k)L - a_2 L^2 = 0$ or must satisfy the conditions $|\lambda_1| < 1$ and $|\lambda_2| < 1$. From the relationship between the equation and the roots, we have

$$a_1 + a_k = \lambda_1 + \lambda_2 = \frac{2 + \alpha - \beta}{1 + \alpha}$$

$$-a_2 = \lambda_1 \lambda_2 = \frac{1}{1 + \alpha},$$

implying that

$$\begin{aligned} a_1 + a_k + a_2 &= \frac{1 + \alpha - \beta}{1 + \alpha} \\ &= \lambda_2 + \frac{1}{\lambda_2(1 + \alpha)} - \frac{1}{1 + \alpha} \\ &\equiv f(\lambda_2) \text{ say.} \end{aligned}$$

Since

$$|\lambda_1| = (1 + \alpha)^{-1} / |\lambda_2| < 1, \ \lambda_2^2 > (1 + \alpha)^{-2}.$$

It can be proved that

$$|f(\lambda_2)|_{\max} = \left| f\left(\lambda_2 = -\frac{1}{1 + \alpha}\right) \right| = 1 + \frac{2}{1 + \alpha}$$

for $(1 + \alpha)^{-2} < \lambda_2^2 \leq (1 + \alpha)^{-1}$ and

$$|f(\lambda_2)|_{\max} = |f(\lambda_2 = -1)| = 1 + \frac{2}{1 + \alpha}$$

for $(1 + \alpha)^{-1} < \lambda_2^2 < 1$. Therefore, $|f(\lambda_2)|_{\max} < 1 + 2(1 + \alpha)^{-1}$, so that

$$|a_1 + a_k + a_2| = \frac{|1 + \alpha - \beta|}{1 + \alpha} < 1 + \frac{2}{1 + \alpha}$$

and then $\beta < 4 + 2\alpha$.

Appendix C

Proof of Theorem 2. Since $E(e_t) = 0$ means $\mu_t = y_0$, equation (3) can be written as

$$y_t = y_0 + \gamma^* \sum_{i=1}^{t} \varepsilon_i^* + \gamma^* \sum_{i=1}^{t} g^*(y_{i-k} - y_0^*),$$

where $y_0^* = y_{i-k}$ for $i < k$ otherwise $y_0^* = y_0$. Therefore, we can write

$$\begin{aligned}
Y_{t,s} &= \frac{1}{s} \sum_{i=1}^{s} y_{(t-1)s+i} \\
&= \frac{y_0}{s} + \gamma^* \frac{1}{s} \sum_{i=1}^{s} \sum_{j=1}^{(t-1)s+i} \varepsilon_j^* + \gamma^* \frac{1}{s} \sum_{i=1}^{s} \sum_{j=1}^{(t-1)s+i} g^*(y_{j-k} - y_0^*) \\
&= \frac{y_0}{s} + \gamma^* \sum_{i=1}^{(t-1)s} \varepsilon_i^* + \gamma^* \sum_{i=1}^{(t-1)s} g^*(y_{i-k} - y_0^*) \\
&\quad + \gamma^* \sum_{i=1}^{s} \left(1 - \frac{i-1}{s}\right) [\varepsilon_{(t-1)s+i}^* + g^*(y_{(t-1)s+i-k} - y_0^*)] \\
&= \gamma^* \sum_{i=1}^{(t-1)s} \varepsilon_i^* + o_p(\sqrt{(t-1)s}) + O_p(s).
\end{aligned}$$

In the last line, the term $o_p(\sqrt{(t-1)s})$ is obtained by

$$\sum_{i=1}^{(t-1)s} g^*(y_{i-k} - y_0^*) = o_p\left(\sum_{i=1}^{(t-1)s} \varepsilon_i^*\right) = o_p(\sqrt{(t-1)s})$$

due to $\sum_{i=1}^{(t-1)s} \varepsilon_i^* = O_p(\sqrt{(t-1)s})$. The term $O_p(s)$ is given by

$$\sum_{i=1}^{s} \left(1 - \frac{i-1}{s}\right) [\varepsilon_{(t-1)s+i}^* + g^*(y_{(t-1)s+i-k} - y_0^*)] = O_p(s)$$

because g is bounded and $0 < \zeta < 1$ implying that g^* is bounded. Similarly,

$$Y_{t,\lambda s} = \gamma^* \sum_{i=1}^{(t-1)\lambda s} \varepsilon_i^* + o_p(\sqrt{(t-1)\lambda s}) + O_p(\lambda s).$$

Using the above two formulas yields result (i$^\circ$).

For a stationary process, TSAR (4) can be written as $y_t = y_0 + \gamma^* \sum_{j=0}^{\infty} \phi_j^* \varepsilon_{t-j}$, where $\sum_{j=0}^{\infty} j|\phi_j^*| < \infty$. It is shown that

$$sY_{t,s} = \sum_{i=1}^{s} y_{(t-1)s+i} = y_0 + \gamma^* \sum_{i=1}^{s} \sum_{j=0}^{\infty} \phi_j^* \varepsilon_{(t-1)s+i-j}$$

$$= y_0 + \gamma^* \sum_{i=(t-1)s+1}^{ts} \sum_{j=0}^{\infty} \phi_j^* \varepsilon_{i-j}.$$

From the Beveridge-Nelson composition (Hamilton [7, p. 504]), we have

$$sY_{t,s} = \gamma^* \left(\sum_{i=1}^{ts} \sum_{j=0}^{\infty} \phi_j^* \varepsilon_{i-j} - \sum_{i=1}^{(t-1)s} \sum_{j=0}^{\infty} \phi_j^* \varepsilon_{i-j} \right) + y_0$$

$$= \gamma^* \left(\sum_{i=1}^{ts} \varepsilon_i - \sum_{i=1}^{(t-1)s} \varepsilon_i \right) + \gamma^* \left(\sum_{j=0}^{\infty} \phi_j \varepsilon_{ts-j} - \sum_{j=0}^{\infty} \phi_j \varepsilon_{(t-1)s-j} \right) + y_0,$$

where $\phi_j = -(\phi_{j+1}^* + \phi_{j+2}^* + \ldots)$ and $\sum_{j=0}^{\infty} |\phi_j| < \infty$. We have

$$\sum_{j=0}^{\infty} \phi_j \varepsilon_{ts-j} - \sum_{j=0}^{\infty} \phi_j \varepsilon_{(t-1)s-j} = O_p(1).$$

Therefore,

$$\sqrt{s} Y_{t,s} = \gamma^* \left[\frac{1}{\sqrt{s}} \sum_{i=(t-1)s+1}^{ts} \varepsilon_i + O_p\left(\frac{1}{\sqrt{s}} \right) \right]$$

and similarly

$$\sqrt{s}Y_{t,\lambda s} = \frac{\gamma^*}{\sqrt{\lambda}}\left[\frac{1}{\sqrt{\lambda s}}\sum_{i=(t-1)\lambda s+1}^{t\lambda s}\varepsilon_i + O_p\left(\frac{1}{\sqrt{\lambda s}}\right)\right],$$

leading to Theorem 2 (ii°). We have completed the proof of Theorem 2.

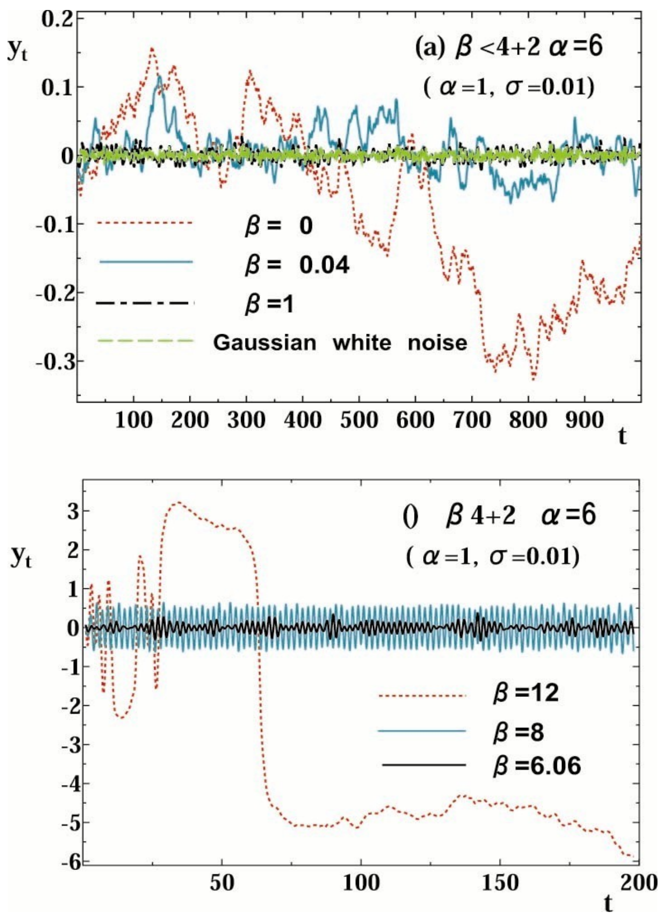

Figure 1. The obviously different dynamics of the NLARI (6) in the cases $\beta < 4 + 2\alpha = 6$ and $\beta \geq 4 + 2\alpha = 6$. (a) $\beta < 6$: the NLARI (6) is as if Gaussian white noise for $\beta = 1$, unit root process for $\beta = 0$, and between Gaussian white noise and unit root process for $\beta = 0.04$. (b) $\beta \geq 6$: the NLARI (6) is closely to random self-sustained oscillations for $\beta = 6.06$, random limit cycle for $\beta = 8$, and random chaos for $\beta = 12$.

Table 1. The values of the nonstationarity ratio $r_s(\lambda)$

Series	$s = 1$			$s = 2$		
	$\lambda = 2$	$\lambda = 3$	$\lambda = 4$	$\lambda = 2$	$\lambda = 3$	$\lambda = 4$
UNRATE	1.2710	1.4491	1.5881	1.2496	1.4118	1.5316
FEDFUNDS	1.3180	1.6065	1.8125	1.3486	1.5692	1.7247
EXCAUS	1.4440	1.7407	2.0169	1.4223	1.7543	2.0474
$\sqrt{\lambda}$ for ARI	1.4142	1.7321	2.0000	1.4142	1.7321	2.0000

UNRATE (unemployment rate): seasonally adjusted/percent/monthly/January 1971 to December 2005;

FEDFUNDS (effective federal funds rate): percent/monthly/January 1971 to December 2005;

EXCAUS (Canada/US foreign exchange rate): Can$ to one US$/monthly/January 1971 to December 2005.

Source of Data: Federal Reserve Economic Data (http://research.stlouisfed.org/fred2/).