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ON THE RELIABILITY OF THE WEIBULL LENGTH-BIASED DISTRIBUTION

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Abstract

In this paper the Weibull length-biased distribution is studied from reliability point of view. The hazard rate function is derived and its behavior is examined; it is shown that the hazard rate function is upside down bathtub shaped for values of the shape parameter that are less than unity and increasing otherwise. Bayes estimates of the reliability function are obtained and compared with their maximum likelihood counterpart using a numerical example and simulation study.

1. Introduction

The Weibull distribution plays an important role in life testing and reliability studies. If T is a random variable having the Weibull distribution, then its p.d.f. takes the form

$$g(t) = \theta \beta t^{\beta-1} \exp(-\theta t^{\beta}), \quad t \geq 0, \beta > 0, \theta > 0. \tag{1.1}$$

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Suppose that the lifetime of a given sample of items is Weibull and that the items do not have the same chance of being selected but each one is selected according to its length or life length, then the resulting distribution is not Weibull but Weibull length-biased. Length-biased distributions find various applications in biomedical area such as family history and disease, survival and intermediate events and latency period of AIDS due to blood transfusion (Gupta and Akman [3]), the study of human families and wildlife populations (Patill and Rao [7]). Much work was done to characterize relationships between original distributions and their length-biased versions. A table for some basic distributions and their length-biased forms is given by Patill and Rao [7] such as Lognormal, Gamma, Pareto and Beta distributions. Relationships in the context of reliability were treated by several authors such as Patill et al. [8], Gupta and Kirmani [4] and recently by Oluyede and George [6]; in these works the survival function, the failure rate, and the mean residual life functions of the length-biased distribution were expressed as functions of their counterpart of the original distribution. The density of the length-biased version of the Weibull distribution in (1.1) can be obtained by applying the following definition:

$$f(t) = \frac{tg(t)}{E(t)}. \tag{1.2}$$

Hence the Weibull length-biased density is given by

$$f(t) = \frac{\beta^2 \theta^{\left(\frac{1}{\beta}+1\right)} t^{\beta} e^{-\theta t^{\beta}}}{\Gamma\left(\frac{1}{\beta}\right)}, \quad t > 0,\ \theta,\ \beta > 0. \tag{1.3}$$

It can be noted that (1.3) is a generalized gamma as defined by Stacy [9] with parameters $\beta,\ \eta = \theta^{-\frac{1}{\beta}},\ k = \frac{1}{\beta} + 1.$ The Weibull length-biased (abbreviated through the text as WLB) distribution includes the gamma distribution $(\beta = 1)$ as special case which is the length-biased version of the Exponential distribution, tends to normal distribution for $(\beta \approx 3.448)$ and gives the length-biased version of Rayleigh distribution for $(\beta = 2)$. The reliability function of the WLB distribution is given by

$$R(t) = 1 - \frac{\gamma\left(\frac{1}{\beta} + 1,\ \theta t^{\beta}\right)}{\Gamma\left(\frac{1}{\beta} + 1\right)}. \tag{1.4}$$

The numerator represents the incomplete gamma function defined as

$$\gamma(a,\ x) = \int_0^x t^{a-1} e^{-t} dt. \tag{1.5}$$

The outline of this paper is as follows: the hazard rate function and its properties are given in Section 2. In Section 3, the MLEs of the parameters of the WLB distribution are given in Subsection 3.1, while Subsections 3.3 and 3.4 are devoted to Bayesian and approximate Bayesian estimation of the reliability function of the same distribution. In Section 4, a numerical example and simulation study are given to compare between the MLE and Bayes estimates of the reliability function alongside comments in our findings.

2. Hazard Rate Function

The hazard rate function is defined by the ratio $(f(t)/R(t))$, it takes the form

$$h(t) = \frac{\beta^2 \theta^{\frac{1}{\beta}+1} t^{\beta} e^{-\theta t^{\beta}}}{\left(\Gamma\left(\frac{1}{\beta}\right) - \beta\gamma\left(\frac{1}{\beta} + 1,\ \theta t^{\beta}\right)\right)}. \tag{2.1}$$

In order to study the behavior of this hazard rate we apply the results of Glaser [2] given in the form of Lemma 2.1.

Lemma 2.1. *Let T be a continuous random variable with twice differentiable density function $f(t)$. Define the quantity* $\eta(t) = -\frac{f'(t)}{f(t)}$, *where*

$f'(t)$ denotes the first derivative of the density function with respect to t. Suppose that the first derivative of $\eta(t)$, named $\eta'(t)$, exists. Glaser [2] *gave the following results*:

1. *If $\eta'(t) < 0$, for all $t > 0$, then the hazard rate is monotonically decreasing failure rate* (*DFR*).

2. *If $\eta'(t) > 0$, for all $t > 0$, then the hazard rate is monotonically increasing failure rate* (*IFR*).

3. *If there exists t_0, such that $\eta'(t) > 0$ for all $(0 < t < t_0)$; $\eta'(t_0) = 0$ and $\eta'(t) < 0$ for all $(t > t_0)$; in addition to that $\lim_{t \to 0} f(t) = 0$, then the hazard rate is upside down bathtub shaped* (UBT).

Using the above lemma we prove the following theorem for the model in (1.3).

Theorem 2.1. *Let T be a non negative random variable having the Weibull length-biased* (*WLB*) *distribution. Then its hazard rate $h(t)$ is IFR for values of the shape parameter that are greater than or equal to one* $(\beta \geq 1)$, *and UBT otherwise – it means for* $(0 < \beta < 1)$.

Proof. Let $\eta(t) = \frac{\beta}{t}(\theta t^{\beta} - 1)$ and $\eta'(t) = \frac{\beta}{t^2}(1 + (\beta - 1)\theta t^{\beta})$.

According to the values of the shape parameter β:

1. For $\beta < 1$, it is easily seen that the third part of the lemma follows; where t_0 is solution of $\eta'(t_0) = 0 \Rightarrow t_0 = (\theta(1 - \beta))^{-\frac{1}{\beta}}$. It results that the hazard rate is UBT shaped.

2. For $\beta = 1$, $\eta'(t) = \frac{1}{t^2}$, this is strictly positive function for all values of t. It results from Lemma 2.1 that $h(t)$ is IFR, in this case also the WLB distribution reduces to gamma distribution with shape parameter $\left(k = \frac{1}{\beta} + 1 = 2\right)$ with an increasing hazard rate.

3. For $\beta > 1$, $\eta'(t) > 0$ for all t, then the hazard rate is monotonically increasing (IFR); this agrees with the theorem given in Gupta and Kirmani [4] which indicated that the length-biased version preserves the IFR property of the original random variable.

The shapes of the hazard rate of the WLB distribution for special values of the shape parameter β are illustrated in Figure (a); the scale parameter θ was taken to be unity since it does not influence the shape of the hazard rate.

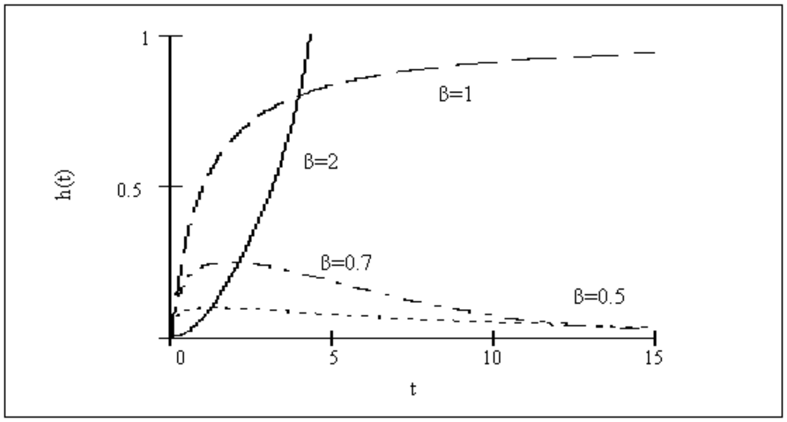

Figure (a). The Hazard rate of the WLB distribution for given values of the shape parameter β.

3. Estimation of Reliability

In this section the maximum likelihood estimates of the two parameters of the WLB distribution and their variance-covariance matrix are derived in order to use them to find the MLE of the reliability function and its variance, next Bayes and approximate Bayes estimates of the same function are obtained.

3.1. Maximum likelihood estimates of the parameters

Suppose that a sample was drawn from (1.3), then the logarithm of the likelihood function is given by

$$l = 2n \ln \beta + \beta \sum_{i=1}^{n} \ln(t_i) - \theta \sum_{i=1}^{n} (t_i)^{\beta} + n\left(\frac{1}{\beta} + 1\right) \ln \theta - n \ln\left(\Gamma\left(\frac{1}{\beta}\right)\right). \tag{3.1}$$

Differentiating (3.1) with respect to θ and β in turn and equating the derivatives to zero, gives

$$-\sum_{i=1}^{n}(t_i^{\hat{\beta}}) + \frac{n\left(\frac{1}{\hat{\beta}}+1\right)}{\hat{\theta}} = 0, \tag{3.1.a}$$

$$\frac{2n}{\hat{\beta}} + \sum_{i=1}^{n}\ln(t_i) - \hat{\theta}\sum_{i=1}^{n}\ln(t_i)(t_i^{\hat{\beta}}) + \frac{n}{\hat{\beta}^2}\left(\Psi\left(\frac{1}{\hat{\beta}}\right) - \ln\hat{\theta}\right) = 0, \tag{3.1.b}$$

where the Psi-function $\Psi(a)$ is defined as (see Abramowitz and Stegun [1]).

$$\Psi(a) = \frac{\partial}{\partial a}(\ln(\Gamma(a))) = \frac{\Gamma'(a)}{\Gamma(a)}, \quad a > 0. \tag{3.2}$$

Solving (3.1.a) for $\hat{\theta}$ gives

$$\hat{\theta} = \frac{n\left(\frac{1}{\hat{\beta}}+1\right)}{\sum_{i=1}^{n}(t_i^{\hat{\beta}})}. \tag{3.3}$$

Replacing it in equation (3.1.b), we obtain

$$\frac{2n}{\hat{\beta}} + \sum_{i=1}^{n}\ln(t_i) - \frac{n\left(\frac{1}{\hat{\beta}}+1\right)\left(\sum_{i=1}^{n}\ln(t_i)(t_i^{\hat{\beta}})\right)}{\sum_{i=1}^{n}(t_i^{\hat{\beta}})}$$

$$+\frac{n}{\hat{\beta}^2}\left(\ln\left(\sum_{i=1}^{n}t_i^{\hat{\beta}}\right) - \ln(n) - \ln\left(\frac{1}{\hat{\beta}}+1\right) + \Psi\left(\frac{1}{\hat{\beta}}\right)\right) = 0, \tag{3.4}$$

which gives an equation in $\hat{\beta}$ only. This nonlinear equation does not seem to have a closed form solution and must be solved iteratively to obtain the estimate of the shape parameter which will be replaced in (3.3) to get the

MLE of the scale parameter θ, or the system of the two equations can be solved simultaneously. The asymptotic observed variance-covariance matrix of the estimates can be obtained by inverting the information matrix with elements that are the negatives of the second derivatives of the likelihood function with respect to the parameters θ and β evaluated at the MLE of the parameters. The second derivatives of the log likelihood function are given by

$$l_{20} = \frac{\partial^2 l}{\partial \theta^2} = -\frac{n\left(\frac{1}{\beta}+1\right)}{\theta^2},$$

$$l_{11} = \frac{\partial^2 l}{\partial \theta \partial \beta} = -\sum_{i=1}^{n} \ln(t_i) t_i^{\beta} - \frac{n}{\theta \beta^2},$$

$$l_{02} = \frac{\partial^2 l}{\partial \beta^2}$$

$$= -\frac{2n}{\beta^2} - \theta \sum_{i=1}^{n} (\ln(t_i))^2 (t_i^{\beta}) - \frac{n}{\beta^3}\left(-2\ln(\theta) + 2\Psi\left(\frac{1}{\beta}\right) + \frac{\Psi'\left(\frac{1}{\beta}\right)}{\beta}\right), \tag{3.5}$$

where $\Psi'(.)$ is the derivative of the digamma function.

3.2. Maximum likelihood estimate of the reliability function

Using the invariance property of the maximum likelihood method, the MLE $\hat{R}$ of the reliability R can be obtained by replacing $\hat{\theta}$ and $\hat{\beta}$, the maximum likelihood estimates of θ and β, respectively, in the formula (1.4) and is given by

$$\hat{R} = \hat{R}(t) = 1 - \frac{\gamma\left(\frac{1}{\hat{\beta}}+1,\ \hat{\theta} t^{\beta}\right)}{\Gamma\left(\frac{1}{\hat{\beta}}+1\right)}. \tag{3.6}$$

Using Taylor expansion of order one of $\hat{R}$ about the parameter estimates we can write

$$\hat{R} = R + \frac{\partial R}{\partial \theta}(\hat{\theta} - \theta) + \frac{\partial R}{\partial \beta}(\hat{\beta} - \beta).$$

By taking the expectation of the above formula and from the properties of the MLE, it results that $\hat{R}$ is asymptotically unbiased estimate of R with asymptotic variance:

$$\operatorname{Var}(\hat{R}) = \left(\frac{\partial R}{\partial \theta}\right)^2 \operatorname{Var}(\hat{\theta}) + \left(\frac{\partial R}{\partial \beta}\right)^2 \operatorname{Var}(\hat{\beta}) + 2\frac{\partial R}{\partial \theta}\frac{\partial R}{\partial \beta}\operatorname{Cov}(\hat{\theta}, \hat{\beta}), \quad (3.7)$$

where the variances and the covariance of the maximum likelihood estimates of the parameters can be obtained by inverting the information matrix with elements that are the negatives of (3.5).

3.3. Bayes estimate of the reliability function

Suppose that a little information is available about the parameters, and then the appropriate prior for this case assuming independence is Jeffreys' vague prior given by

$$\pi(\theta, \beta) \propto \frac{1}{\theta\beta}. \quad (3.8)$$

$\propto$ being the sign of proportionality. Using Bayes theorem which combines the likelihood function with the prior given in (3.8), we obtain the following joint posterior:

$$\pi(\theta, \beta/t) \propto \frac{\beta^{2n-1}\theta^{n\left(\frac{1}{\beta}+1\right)-1}}{\Gamma^n\left(\frac{1}{\beta}\right)} \exp\left\{-\theta\sum_{i=1}^{n} t_i^{\beta} + \beta\sum_{i=1}^{n}\ln(t_i)\right\}. \quad (3.9)$$

The Bayes estimate of the reliability function with respect to the squared error loss function is given by

$$\tilde{R}(t) = E(R(t)/t) = \frac{\int_0^\infty\int_0^\infty R(t)\pi(\theta, \beta/t)d\theta d\beta}{\int_0^\infty\int_0^\infty \pi(\theta, \beta/t)d\theta d\beta} = \frac{C_2}{C_1}, \quad (3.10.a)$$

with

$$C_1 = \int_0^\infty \frac{\beta^{2n-1}\Gamma\left(n\left(\frac{1}{\beta}+1\right)\right)}{\Gamma^n\left(\frac{1}{\beta}\right)} \exp\left(\beta\sum_{i=1}^n \ln(t_i) - n\left(\frac{1}{\beta}+1\right)\ln\left(\sum_{i=1}^n t_i^\beta\right)\right) d\beta, \quad (3.10.b)$$

$$C_2 = \int_0^\infty\int_0^\infty \frac{\beta^{2n-1}\left(\Gamma\left(\frac{1}{\beta}\right) - \beta\gamma\left(\frac{1}{\beta}+1, \theta t^\beta\right)\right)\theta^{n\left(\frac{1}{\beta}+1\right)-1}\exp\left(-\theta\sum_{i=1}^n t_i^\beta + \beta\sum_{i=1}^n \ln(t_i)\right)}{\left(\Gamma\left(\frac{1}{\beta}\right)\right)^{n+1}} d\theta d\beta.$$

(3.10.c)

To obtain the variance of $\tilde{R}(t)$ we use the following formula:

$$\mathrm{Var}(\tilde{R}/t) = E(\tilde{R}^2/t) - E^2(\tilde{R}/t), \quad (3.10.d)$$

where

$$E(R^2(t)/t) = \frac{\int_0^\infty\int_0^\infty R^2(t)\pi(\theta,\ \beta/t)d\theta d\beta}{\int_0^\infty\int_0^\infty \pi(\theta,\ \beta/t)d\theta d\beta} = \frac{C_3}{C_1}, \quad (3.10.e)$$

$$C_3 = \int_0^\infty\int_0^\infty \frac{\beta^{2n-1}\left(\Gamma\left(\frac{1}{\beta}\right) - \beta\gamma\left(\frac{1}{\beta}+1, \theta t^\beta\right)\right)^2 \theta^{n\left(\frac{1}{\beta}+1\right)-1}}{\left(\Gamma\left(\frac{1}{\beta}\right)\right)^{n+2}} \exp\left(-\theta\sum_{i=1}^n t_i^\beta + \beta\sum_{i=1}^n \ln(t_i)\right) d\theta d\beta.$$

(3.10.f)

No closed form solutions exist for the integrals in (3.10.a) and (3.10.e) even if one of the two parameters is known. Numerical integration and approximate methods may be used in this case.

3.4. Approximate Bayes estimate of reliability

Lindley [5] gave an alternative method to approximate the integrals that occur in Bayesian statistics when the analytical method is not attractable. The form of ratio of integrals considered by Lindley [5] is as follows:

$$\frac{\int w(\eta)\exp(l(\eta))d\eta}{\int v(\eta)\exp(l(\eta))d\eta}, \tag{3.11}$$

where $\eta = (\eta_1, \eta_2, \ldots, \eta_m)$ is a vector parameter, $l(\eta)$ is the logarithm of the likelihood function and $w(.)$ and $v(.)$ are arbitrary functions of η. Let $v(\eta) = \pi(\eta)$ the prior density of the parameter η, $w(\eta) = u(\eta)\pi(\eta)$ and $\rho(\eta) = \ln(\pi(\eta))$, the ratio in (3.11) will be the posterior expectation of the function $u(\eta)$ under the squared error loss function and it is given by

$$E(u(\eta)) = \frac{\int u(\eta)\exp(l(\eta)+\rho(\eta))d\eta}{\int \exp(l(\eta)+\rho(\eta))d\eta}. \tag{3.12}$$

The basic idea to evaluate this ratio is to expand on Taylor series the functions involved in it about the maximum likelihood $\hat{\eta}$ of η, this leads to the following formula, where the first term omitted is $O(n^{-2})$:

$$E(u(\eta)/t) \sim u + \frac{1}{2}\sum_{i,j}(u_{ij} + 2u_i\rho_j)\sigma_{ij} + \frac{1}{2}\sum_{i,j,k} l_{ijk}u_l\sigma_{ij}\sigma_{kl}, \tag{3.13}$$

where each suffix denotes differentiation once with respect to the variable having that suffix; this means

$l_{ijk} = \frac{\partial^3 l(\eta)}{\partial\eta_i\partial\eta_j\partial\eta_k}$, $u_{ij} = \frac{\partial^2 u(\eta)}{\partial\eta_i\partial\eta_j}$, $u_i = \frac{\partial u(\eta)}{\partial\eta_i}$, $\rho_i = \frac{\partial\rho(\eta)}{\partial\eta_i}$, etc. σ_{ij} is the (i, j) element of the observed variance-covariance matrix. All the quantities in (3.13) are to be evaluated at the MLE of η and the summation run over all suffixes from one to m (the dimensionality of η). Lindley [5] gave the one-parameter and two-parameter versions of (3.13).

To find the approximate Bayes estimate of the reliability function we need the following:

1. The third derivatives of the log likelihood function which are given by

$$l_{30} = \frac{\partial^3 l}{\partial \theta^3} = 2n \frac{\left(\frac{1}{\beta} + 1\right)}{\theta^3},$$

$$l_{12} = \frac{\partial^3 l}{\partial \theta \partial \beta^2} = \frac{2n}{\theta \beta^3} - \sum_{i=1}^{n} (\ln t_i)^2 t_i^{\beta},$$

$$l_{21} = \frac{\partial^3 l}{\partial \theta^2 \partial \beta} = \frac{n}{\beta^2 \theta^2},$$

$$l_{03} = \frac{\partial^3 l}{\partial \beta^3} = \frac{4n}{\beta^3} - \theta \sum_{i=1}^{n} (\ln(t_i))^3 t_i^{\beta} + \frac{n}{\beta^4} \left(\frac{\Psi''\left(\frac{1}{\beta}\right)}{\beta^2} + \frac{6\Psi'\left(\frac{1}{\beta}\right)}{\beta} + 6\Psi\left(\frac{1}{\beta}\right) - 6 \ln \theta \right).$$

2. The derivatives of the logarithm of the prior function given by

$$\rho(\theta, \beta) = \ln(\pi(\theta, \beta)) = -\ln \theta - \ln \beta.$$

Differentiating this function with respect to each parameter in turn yields:

$$\rho_1 = \frac{\partial \rho(\theta, \beta)}{\partial \theta} = -\frac{1}{\theta}, \qquad \rho_2 = \frac{\partial \rho(\theta, \beta)}{\partial \beta} = -\frac{1}{\beta}.$$

3. The derivatives of the reliability function:

Differentiating the reliability function given in (1.4) with respect to θ and β in turn gives:

- The first derivatives:

$$R_1 = \frac{\partial R}{\partial \theta} = -\frac{\beta t^{\beta+1} \theta^{\frac{1}{\beta}} \exp(-\theta t^{\beta})}{\Gamma\left(\frac{1}{\beta}\right)} = -\frac{t f(t)}{\beta \theta},$$

$$R_2 = \frac{\partial R}{\partial \beta} = -\left(\frac{t \ln(t)}{\beta}\right) f(t) + \frac{I_1}{\beta \Gamma\left(\frac{1}{\beta}\right)} - \left(\frac{\Psi\left(\frac{1}{\beta}\right)}{\beta^2} + \frac{1}{\beta} \right) (1 - R).$$

- The second derivatives:

$$R_{11} = \frac{\partial^2 R}{\partial\theta^2} = -\frac{tf(t)}{\beta\theta}\left(-t^\beta + \frac{1}{\beta\theta}\right),$$

$$R_{12} = \frac{\partial^2 R}{\partial\theta\partial\beta} = -\frac{tf(t)}{\beta\theta}\left(\ln(t)(1-\theta t^\beta) - \frac{\ln\theta}{\beta^2} + \frac{\Psi\left(\frac{1}{\beta}\right)}{\beta^2} + \frac{1}{\beta}\right),$$

$$R_{22} = \frac{\partial^2 R}{\partial\beta^2} = -\frac{t\ln tf(t)}{\beta}\left(\ln t\left(1-\frac{1}{\beta}-\theta t^\beta\right) + \frac{2}{\beta^2}\left(\Psi\left(\frac{1}{\beta}\right) - \ln\theta + \beta\right)\right)$$

$$+\frac{\left(2I_1\Psi\left(\frac{1}{\beta}\right) - I_2\right)}{\beta^3\Gamma\left(\frac{1}{\beta}\right)} + \frac{(1-R)}{\beta^2}\left(\frac{\Psi'\left(\frac{1}{\beta}\right)}{\beta^2} - \left(\frac{\Psi\left(\frac{1}{\beta}\right)}{\beta}\right)^2\right),$$

where

$$I_1 = \int_0^{\theta t^\beta} \ln(x)x^{\frac{1}{\beta}}e^{-x}dx, \quad \text{and} \quad I_2 = \int_0^{\theta t^\beta} (\ln(x))^2 x^{\frac{1}{\beta}}e^{-x}dx.$$

Using all the above results evaluated at the MLE of the parameters and setting $u(\eta) = u(\beta, \theta) = R(t)$ in (3.13) such that: $(u_1 = R_1, \ u_2 = R_2, \ u_{11} = R_{11}, \ u_{12} = u_{21} = R_{12}, \ u_{22} = R_{22})$ yield

$$\tilde{\tilde{R}}(t) \approx \hat{R}(t) + \Delta\hat{R}(t). \tag{3.14.a}$$

To get the second posterior moment of the reliability function we put $u(\eta) = u(\beta, \theta) = R^2(t)$ the first and the second derivatives of this function are: $u_1 = 2RR_1$, $u_2 = 2RR_2$, $u_{11} = 2(R_1^2 + R_{11}R)$, $u_{12} = u_{21} = 2(R_1R_2 + R_{12}R)$ and $u_{22} = 2(R_2^2 + R_{22}R)$ we get

$$E(R^2(t)/t) \approx \hat{R}^2(t) + \Delta\hat{R}^2(t). \tag{3.14.b}$$

- If we suppose that the shape parameter is known, we get

$$\Delta\hat{R}(t) = \frac{t\hat{f}(t)}{2n(\hat{\beta}+1)}\left(\hat{\theta}t^{\hat{\beta}} - \frac{1}{\hat{\beta}}\right), \tag{3.14.c}$$

$$\Delta\hat{R}^2(t) = \frac{t\hat{f}(t)}{n(\hat{\beta}+1)}\left(\frac{t\hat{f}(t)}{\hat{\beta}} + \left(\hat{\theta}t^{\hat{\beta}} + \frac{1}{\hat{\beta}}\right)\hat{R}(t)\right), \tag{3.14.d}$$

where $\hat{f}(t)$ is the WLB density evaluated at the MLE of the unknown parameters. Note that $\Delta\hat{R}(t)$ and $\Delta\hat{R}^2(t)$ tend to zero as the sample size n tends to infinity, and the Bayes estimate approaches the MLE.

- If the scale parameter is assumed to be known, we get

$$\Delta\hat{R}(t) = \frac{1}{2}\left(D + \frac{nA}{\hat{\beta}^3}\right)^{-1}\left(\hat{R}_{22} + \hat{R}_2\left(\left(C + \frac{nB}{\hat{\beta}^4}\right)\left(D + \frac{nA}{\hat{\beta}^3}\right)^{-1} - \frac{2}{\hat{\beta}}\right)\right), \tag{3.14.e}$$

$$\Delta\hat{R}^2(t) = \left(D + \frac{nA}{\hat{\beta}^3}\right)^{-1}\left(\hat{R}_2^2 + \hat{R}\left(\hat{R}_{22} + \hat{R}_2\left[\left(C + \frac{nB}{\hat{\beta}^4}\right)\left(D + \frac{nA}{\hat{\beta}^3}\right)^{-1} - \frac{2}{\hat{\beta}}\right]\right)\right). \tag{3.14.f}$$

4. Numerical Example and Simulation Results

In this section a numerical example is given to illustrate the above methods and simulation is carried out to study the behavior of the three estimates of the reliability function of the WLB distribution.

4.1. Numerical example

To illustrate the above formulas and methods the following data were taken from Gupta and Akman [3], they represent millions of revolutions to failure for 23 ball bearings in fatigue test:

17.880	28.920	33.000	41.520	42.120	45.600	48.480	51.840
51.960	54.120	55.560	67.800	68.640	68.640	68.880	84.120
93.120	98.640	105.120	105.840	127.920	128.040	173.400	

These data have been previously fitted assuming several distributions such as lognormal, Inverse Gaussian, length-biased inverse Gaussian and Weibull distribution. Fitting the above data with the original Weibull distribution gives the following results:

$$\hat{\beta} = 2.094 \text{ and } \hat{\theta} = 9.864 \times 10^{-5}$$

$$V = \begin{bmatrix} 2.279 \times 10^{-8} & -4.903 \times 10^{-5} \\ -4.903 \times 10^{-5} & 0.107 \end{bmatrix},$$

which are close to the results obtained by Thoman et al. [10] ($\hat{\beta} = 2.102$ and $\hat{\theta} = 9.490 \times 10^{-5}$).

In the present work the Ball bearings data were also fitted using the length-biased Weibull. The parameters were estimated by the maximum likelihood method via iterating the system (3.1.a) and (3.1.b) using the package Mathcad 2001, the following results were found

$$\hat{\beta} = 1.571 \text{ and } \hat{\theta} = 1.768 \times 10^{-3}.$$

The observed variance-covariance matrix was also evaluated and it is given by

$$V = \begin{bmatrix} 6.883 \times 10^{-6} & -8.058 \times 10^{-4} \\ -8.058 \times 10^{-4} & 0.095 \end{bmatrix}.$$

At $\alpha = 5\%$ level of significance the Kolmogorov-Smirnov test does not reject that this data come from a WLB distribution since the computed statistic $D_{23} = 0.136$ is less than the theoretical statistic $D_{23}(0.05) = 0.247$.

The MLE, the Bayes and approximate Bayes estimates of the reliability function of the WLB distribution were obtained for certain values of time; for comparison the same estimate using the original distribution are also given in the following table:

Table (a). MLE, Bayes and approximate Bayes estimates of the reliability function with variances

Estimates / Time t	$\hat{R}(t)$ (MLE)		$\tilde{R}(t)$ (Bayes estimate)		$\tilde{\tilde{R}}(t)$ (Approximate Bayes estimate)	
	Length-biased	Original	Length-biased	Original	Length-biased	Original
10	0.992 (2.487(–4))*	0.988 (2.671(–4))	0.955 (6.314(–9))	0.904 (4.118(–5))	0.990 (3.246(–5))	0.984 (7.694(–5))
15	0.979 (1.834(–3))	0.972 (1.462(–3))	0.921 (7.928(–9))	0.859 (5.732(–5))	0.974 (1.812(–4))	0.966 (3.049(–4))
20	0.958 (7.171(–3))	0.949 (4.829(–3))	0.884 (8.703(–9))	0.816 (6.906(–5))	0.952 (5.336(–4))	0.942 (7.551(–4))
50	0.695 (0.253)	0.700 (0.141)	0.664 (6.543(–9))	0.6 (8.246(–5))	0.693 (6.224(–3))	0.697 (6.104(–3))
80	0.370 (0.502)	0.385 (0.310)	0.487 (3.128(–9))	0.441 (6.085(–5))	0.380 (6.464(–3))	0.395 (6.634(–3))
100	0.209 (0.384)	0.218 (0.236)	0.397 (1.697(–9))	0.358 (4.511(–5))	0.223 (4.443(–3))	0.232 (4.604(–3)
173	0.011 (8.534(–3))	8.24(–3) (1.462(–3))	0.185 (4.314(–11))	0.169 1.244(–5)	0.019 (1.185(–4))	0.015 (6.473(–5))

$^*2.487(-4) = 2.487 \times 10^{-4}$.

From Table (a) we observe that $\hat{R}(t)$ and $\tilde{\tilde{R}}(t)$ are very close, while $\tilde{R}(t)$ presents a slight difference for the original and the length-biased distribution. The Bayesian method gives the smallest variances for all values of t for the two distributions.

4.2. Simulation Results

In order to study the behavior of the three estimates of the reliability function, 1000 samples were generated from the WLB distribution with sample sizes: $n = 10, 20, 30, 50, 100$ for certain values of the shape and scale parameters, and values of time corresponding the following true values of reliability: 0.99, 0.95, 0.85, 0.75, 0.5, 0.25, 0.05, respectively. (Tables for this study are available from authors upon request.) From this study we concluded that

1. The variance, bias and mean square error (MSE) decrease as increasing the sample size n for almost all values of time t and all the given combinations of parameters for the three methods of estimation.

2. Increasing the time t for a fixed sample size n the MSE increases for values of reliability $R(t) \geq 0.5$ and decreases for the other values of reliability for the three methods except for the approximate Bayesian method, when both parameters are less than unity the MSE start to decrease from $R(t) < 0.75$.

3. The Bayesian method does not behave well when both parameters are less than unity.

4. The maximum likelihood gives the best estimates with the smallest MSE.

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