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HAMILTONIAN CYCLES WITH SIX 3-CONTRACTIBLE EDGES WHICH HAVE THREE CONSECUTIVE NONDEGENERATE SEGMENTS

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Abstract

We classify all pairs (G, C) of a 3-connected graph G and a hamiltonian cycle C of G such that C contains precisely six contractible edges of G, C has precisely three nondegenerate segments, and the three nondegenerate segments are consecutive on C.

1. Introduction

In this paper, we consider only finite, simple, undirected graphs with no loops and no multiple edges.

A graph G is called 3-*connected* if $|V(G)| \geq 4$ and $G - S$ is connected for any subset S of $V(G)$ having cardinality 2. An edge e of a 3-connected graph G is called *contractible* if the graph which we obtain from G by contracting e (and replacing each of the resulting pairs of parallel edges by a simple edge) is 3-connected; otherwise e is called *noncontractible*. In [2], Dean et al. proved that every longest cycle in a 3-connected graph other than K_4 or $K_2 \times K_3$ contains at least three contractible edges. Further Aldred et al. [1], Ota [6], Fujita [4], and Fujita and Kotani [5]

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classify all pairs (G, C) of a 3-connected graph G and a longest cycle C of G such that C contains at most five contractible edges of G. In these classifications, it turns out that for all such pairs (G, C), C is a hamiltonian cycle of G. Thus it is desirable that one should obtain a classification of those pairs (G, C) of a 3-connected graph G and a hamiltonian cycle C such that C contains precisely six contractible edges of G. Along this line of research, the following theorem was proved in [3] (see the paragraph following Lemma 4.3 for the definition of the term "nondegenerate"; also see [3, Section 1] for the definition of Type 1):

Theorem 1. *Let G be a* 3*-connected graph and let C be a hamiltonian cycle of G. Suppose that C contains precisely six contractible edges of G, and C has four consecutive nondegenerate segments. Then the pair (G, C) is of Type* 1.

In this paper, we consider the case where C has three consecutive nondegenerate segments. More precisely, we prove the following theorem:

Theorem 2. *Let G be a* 3*-connected graph and let C be a hamiltonian cycle of G. Suppose that C contains precisely six contractible edges of G. Suppose further that C has precisely three nondegenerate segments and they are consecutive on C. Then the pair (G, C) belongs to one of the* 7 *types*, *Types* 2 *through* 8, *which are defined in Section* 2.

The organization of this paper is as follows. In Section 2, we define the type of a pair (G, C) satisfying the assumption of Theorem 2. Section 3 contains fundamental results concerning noncontractible edges lying on a hamiltonian cycle of a 3-connected graph. In Section 4, we derive basic properties of a pair (G, C) satisfying the assumption of Theorem 2, and we complete the proof of Theorem 2 in Section 5.

Our notation and terminology are standard except possibly for the following. Let G be a graph. For $U \subseteq V(G)$, we let $\langle U \rangle = \langle U \rangle_G$ denote the graph induced by U in G. For $U, V \subseteq V(G)$, we let $E(U, V)$ denote the set of edges of G which join a vertex in U and a vertex in V; if $U = \{u\} (u \in V(G))$, then we write $E(u, V)$ for $E(\{u\}, V)$. A subset S of $V(G)$ is called a *cutset* if $G - S$ is disconnected; thus G is 3-connected if and only if $|V(G)| \geq 4$ and G has no cutset of cardinality 2. If G is 3-connected, then for $e = uv \in E(G)$, we let $K(e) = K(u, v)$ denote the set of vertices x of G such that $\{u, v, x\}$ is a cutset; thus e is contractible if and only if $K(e) = \emptyset$. If e is noncontractible, then for each $x \in K(e)$, $\{u, v, x\}$ is called a *cutset associated with* e.

2. Definition of the Type of a Pair (G, C)

In this section, we define the type of a pair (G, C) of a 3-connected graph G and a hamiltonian cycle C of G such that C contains precisely six contractible edges of G. Throughout this section, we let n_0, n_1, n_2, n_3, n_4 and n_5 be nonnegative integers, and let G denote a graph of order $n_0 + n_1 + n_2 + n_3 + n_4 + n_5 + 6$ with vertex set $V(G) = \{a_i \mid 0 \leq i \leq n_0\} \cup \{b_i \mid 0 \leq i \leq n_1\} \cup \{c_i \mid 0 \leq i \leq n_2\} \cup \{d_i \mid 0 \leq i \leq n_3\} \cup \{e_i \mid 0 \leq i \leq n_4\} \cup \{f_i \mid 0 \leq i \leq n_5\}$ such that G contains $C = a_0 a_1 \cdots a_{n_0} b_0 b_1 \cdots b_{n_1} c_0 c_1 \cdots c_{n_2} d_0 d_1 \cdots d_{n_3} e_0 e_1 \cdots e_{n_4} f_0 f_1 \cdots f_{n_5} a_0$ as a hamiltonian cycle. In the definition of each type, it is easy to verify that if G satisfies the required conditions, then G is 3-connected, and $a_{n_0} b_0$, $b_{n_1} c_0$, $c_{n_2} d_0$, $d_{n_3} e_0$, $e_{n_4} f_0$, $f_{n_5} a_0$ are the only contractible edges of G that are on C. Further if we let $C_0 = \{a_0, a_1, \ldots, a_{n_0}\}$, $C_1 = \{b_0, b_1, \ldots, b_{n_1}\}$, $C_2 = \{c_0, c_1, \ldots, c_{n_2}\}$, $C_3 = \{d_0, d_1, \ldots, d_{n_3}\}$, $C_4 = \{e_0, e_1, \ldots, e_{n_4}\}$ and $C_5 = \{f_0, f_1, \ldots, f_{n_5}\}$, then C_3, C_4 and C_5 are nondegenerate and C_0, C_1 and C_2 are degenerate (see the paragraph following Lemma 4.3 for the definition of the terms "nondegenerate" and "degenerate").

Type 2. Let $n_0 = 0$ or 2, $n_1 = 0$, $n_2 = 0$ or 2, $n_3 \geq 1$, $n_4 \geq 1$, and $n_5 \geq 1$. Let

$$X = \{d_j d_{j+2} \mid 0 \leq j \leq n_3 - 2\} \cup \{e_x e_{x+2} \mid 0 \leq x \leq n_4 - 2\} \cup \{f_y f_{y+2} \mid 0 \leq y \leq n_5 - 2\},$$

$$Y = \{b_0 d_j \mid 0 \leq j \leq n_3\} \cup \{b_0 e_x \mid 0 \leq x \leq n_4\} \cup \{b_0 f_y \mid 0 \leq y \leq n_5\},$$

$$F_1 = \begin{cases} \{a_0 f_{n_5 - 1}\} & (\text{if } n_0 = 0) \\ \{a_0 a_2,\ a_1 f_{n_5 - 1}\} & (\text{if } n_0 = 2), \end{cases} \quad F_1' = \begin{cases} \varnothing & (\text{if } n_0 = 0) \\ \{a_1 b_0,\ a_1 f_{n_5}\} & (\text{if } n_0 = 2), \end{cases}$$

$$F_2 = \begin{cases} \{c_0 d_1\} & (\text{if } n_2 = 0) \\ \{c_0 c_2,\ c_1 d_1\} & (\text{if } n_2 = 2), \end{cases} \quad F_2' = \begin{cases} \varnothing & (\text{if } n_2 = 0) \\ \{c_1 b_0,\ c_1 d_0\} & (\text{if } n_2 = 2), \end{cases}$$

$$F_3 = \{d_{n_3 - 1} e_1,\ e_{n_4 - 1} f_1\}, \quad F_3' = \{d_{n_3} e_1,\ d_{n_3 - 1} e_0,\ e_{n_4} f_1,\ e_{n_4 - 1} f_0\},$$

$$W_1 = \begin{cases} Y & (\text{if } n_0 = 0) \\ Y \cup \{a_1 b_0\} & (\text{if } n_0 = 2), \end{cases} \quad W_2 = \begin{cases} Y & (\text{if } n_2 = 0) \\ Y \cup \{b_0 c_1\} & (\text{if } n_2 = 2), \end{cases}$$

$$Z_1 = \begin{cases} \{b_0 d_0\} & (\text{if } n_2 = 0) \\ \varnothing & (\text{if } n_2 = 2), \end{cases} \quad Z_2 = \begin{cases} \{b_0 f_{n_5}\} & (\text{if } n_0 = 0) \\ \varnothing & (\text{if } n_0 = 2). \end{cases}$$

Under this notation, G is said to be of *Type* 2 if G satisfies the following four conditions:

- $X \cup F_1 \cup F_2 \cup F_3 \subseteq E(G) - E(C)$
$$\subseteq X \cup Y \cup F_1 \cup F_1' \cup F_2 \cup F_2' \cup F_3 \cup F_3',$$
- for each i with $1 \le i \le 2$, $(W_i - Z_i) \cap E(G) \ne \varnothing$,
- if $n_0 = 0$ and $n_5 = 1$, then $\{f_0 b_0,\ f_0 e_{n_4 - 1}\} \cap E(G) \ne \varnothing$,
- if $n_2 = 0$ and $n_3 = 1$, then $\{d_1 b_0,\ d_1 e_1\} \cap E(G) \ne \varnothing$.

Type 3. Let $n_0 = 0$ or 2, $n_1 = 2$, $n_2 = 0$ or 2, $n_3 \ge 1$, $n_4 \ge 1$ and $n_5 \ge 1$. Let

$$X = \{d_j d_{j+2} \mid 0 \le j \le n_3 - 2\} \cup \{e_x e_{x+2} \mid 0 \le x \le n_4 - 2\}$$
$$\cup \{f_y f_{y+2} \mid 0 \le y \le n_5 - 2\} \cup \{b_0 b_2\},$$

$$F_1 = \begin{cases} \{a_0 b_1,\ a_0 f_{n_5 - 1}\} & (\text{if } n_0 = 0) \\ \{a_0 a_2,\ a_1 b_1,\ a_1 f_{n_5 - 1}\} & (\text{if } n_0 = 2), \end{cases} \quad F_1' = \begin{cases} \varnothing & (\text{if } n_0 = 0) \\ \{a_1 f_{n_5}\} & (\text{if } n_0 = 2), \end{cases}$$

$$F_2 = \begin{cases} \{c_0 d_1\} & (\text{if } n_2 = 0) \\ \{c_0 c_2,\ c_1 d_1\} & (\text{if } n_2 = 2), \end{cases} \quad F_2' = \begin{cases} \varnothing & (\text{if } n_2 = 0) \\ \{c_1 d_0\} & (\text{if } n_2 = 2), \end{cases}$$

$F_3 = \{d_{n_3-1}e_1,\ e_{n_4-1}f_1\},\quad F_3' = \{d_{n_3}e_1,\ d_{n_3-1}e_0,\ e_{n_4}f_1,\ e_{n_4-1}f_0\},$

$$F_4 = \begin{cases} \{b_1 f_{n_5-1}\}, & (\text{if } n_0 = 0) \\ \varnothing & (\text{if } n_0 = 2), \end{cases} \quad F_4' = \begin{cases} \{b_1 f_{n_5}\} & (\text{if } n_0 = 0) \\ \{b_1 f_{n_5-1},\ b_1 f_{n_5}\} & (\text{if } n_0 = 2). \end{cases}$$

Under this notation, G is said to be of *Type* 3 if G satisfies the following three conditions:

- $X \cup F_1 \cup F_2 \cup F_3 \cup F_4 \subseteq E(G) - E(C) \subseteq X \cup F_1 \cup F_1' \cup F_2 \cup F_2' \cup F_3 \cup F_3' \cup F_4 \cup F_4',$
- if $n_0 = 2$, then $F_4' \cap E(G) \neq \varnothing$,
- if $n_2 = 0$ and $n_3 = 1$, then $d_1 e_1 \in E(G)$.

Type 4. Let $n_0 = 0$ or 2, $n_1 = 2$, $n_2 = 0$ or 2, $n_3 \geq 1$, $n_4 \geq 1$, and $n_5 \geq 2$. Let

$$X = \{d_j d_{j+2} \mid 0 \leq j \leq n_3 - 2\} \cup \{e_x e_{x+2} \mid 0 \leq x \leq n_4 - 2\} \cup \{f_y f_{y+2} \mid 0 \leq y \leq n_5 - 2\} \cup \{b_0 b_2\},$$

$$F_1 = \begin{cases} \{a_0 f_{n_5-1}\} & (\text{if } n_0 = 0) \\ \{a_0 a_2,\ a_1 f_{n_5-1}\} & (\text{if } n_0 = 2), \end{cases} \quad F_1' = \begin{cases} \varnothing & (\text{if } n_0 = 0) \\ \{a_1 f_{n_5}\} & (\text{if } n_0 = 2), \end{cases}$$

$$F_2 = \begin{cases} \{c_0 d_1\} & (\text{if } n_2 = 0) \\ \{c_0 c_2,\ c_1 d_1\} & (\text{if } n_2 = 2), \end{cases} \quad F_2' = \begin{cases} \varnothing & (\text{if } n_2 = 0) \\ \{c_1 d_0\} & (\text{if } n_2 = 2), \end{cases}$$

$$F_3 = \{d_{n_3-1}e_1,\ e_{n_4-1}f_1\},\quad F_3' = \{d_{n_3}e_1,\ d_{n_3-1}e_0,\ e_{n_4}f_1,\ e_{n_4-1}f_0\}.$$

Let p be an integer with $1 \leq p \leq n_5 - 1$, and set

$$Y = \{b_1 f_y \mid p-1 \leq y \leq p+1\} \text{ and } W = \begin{cases} Y - \{b_1 f_{n_5}\} & (\text{if } n_0 = 0) \\ Y & (\text{if } n_0 = 2). \end{cases}$$

Now G is said to be of *Type* 4 if there exists p with $1 \leq p \leq n_5 - 1$ such that G satisfies the following three conditions:

- $X \cup F_1 \cup F_2 \cup F_3 \subseteq E(G) - E(C) \subseteq X \cup Y \cup F_1 \cup F_1' \cup F_2 \cup F_2' \cup F_3 \cup F_3',$

- $W \cap E(G) \neq \emptyset$,
- if $n_2 = 0$ and $n_3 = 1$, then $d_1e_1 \in E(G)$.

Type 5. Let $n_0 = 0$ or 2, $n_1 = 2$, $n_2 = 0$ or 2, $n_3 \geq 1$, $n_4 \geq 1$, and $n_5 = 1$. Let

$$X = \{d_jd_{j+2} \mid 0 \leq j \leq n_3 - 2\} \cup \{e_xe_{x+2} \mid 0 \leq x \leq n_4 - 2\} \cup \{b_0b_2\},$$

$$F_1 = \begin{cases} \{a_0f_{n_5-1}\} & (\text{if } n_0 = 0) \\ \{a_0a_2,\, a_1f_{n_5-1}\} & (\text{if } n_0 = 2), \end{cases} \quad F_1' = \begin{cases} \emptyset & (\text{if } n_0 = 0) \\ \{a_1f_{n_5}\} & (\text{if } n_0 = 2), \end{cases}$$

$$F_2 = \begin{cases} \{c_0d_1\} & (\text{if } n_2 = 0) \\ \{c_0c_2,\, c_1d_1\} & (\text{if } n_2 = 2), \end{cases} \quad F_2' = \begin{cases} \emptyset & (\text{if } n_2 = 0) \\ \{c_1d_0\} & (\text{if } n_2 = 2), \end{cases}$$

$$F_3 = \{d_{n_3-1}e_1,\, e_{n_4-1}f_1\}, \quad F_3' = \{d_{n_3}e_1,\, d_{n_3-1}e_0,\, e_{n_4}f_1,\, e_{n_4-1}f_0\},$$

$$F_4 = \begin{cases} \{b_1f_0\}, & (\text{if } n_0 = 0) \\ \emptyset & (\text{if } n_0 = 2), \end{cases} \quad F_4' = \begin{cases} \{b_1f_1\} & (\text{if } n_0 = 0) \\ \{b_1f_0,\, b_1f_1\} & (\text{if } n_0 = 2). \end{cases}$$

Under this notation, G is said to be of *Type* 5 if G satisfies the following three conditions:

- $X \cup F_1 \cup F_2 \cup F_3 \cup F_4 \subseteq E(G) - E(C) \subseteq X \cup F_1 \cup F_1'$
$$\cup\, F_2 \cup F_2' \cup F_3 \cup F_3' \cup F_4 \cup F_4',$$
- if $n_0 = 2$, then $F_4' \cap E(G) \neq \emptyset$,
- if $n_2 = 0$ and $n_3 = 1$, then $d_1e_1 \in E(G)$.

Type 6. Let $n_0 = 0$ or 2, $n_1 = 2$, $n_2 = 0$ or 2, $n_3 \geq 1$, $n_4 \geq 2$, and $n_5 \geq 1$. Let

$$X = \{d_jd_{j+2} \mid 0 \leq j \leq n_3 - 2\} \cup \{e_xe_{x+2} \mid 0 \leq x \leq n_4 - 2\}$$
$$\cup\, \{f_yf_{y+2} \mid 0 \leq y \leq n_5 - 2\} \cup \{b_0b_2\},$$

$$F_1 = \begin{cases} \{a_0f_{n_5-1}\} & (\text{if } n_0 = 0) \\ \{a_0a_2,\, a_1f_{n_5-1}\} & (\text{if } n_0 = 2), \end{cases} \quad F_1' = \begin{cases} \emptyset & (\text{if } n_0 = 0) \\ \{a_1f_{n_5}\} & (\text{if } n_0 = 2), \end{cases}$$

$$F_2 = \begin{cases} \{c_0d_1\} & (\text{if } n_2 = 0) \\ \{c_0c_2,\, c_1d_1\} & (\text{if } n_2 = 2), \end{cases} \quad F_2' = \begin{cases} \varnothing & (\text{if } n_2 = 0) \\ \{c_1d_0\} & (\text{if } n_2 = 2), \end{cases}$$

$$F_3 = \{d_{n_3-1}e_1,\, e_{n_4-1}f_1\}, \quad F_3' = \{d_{n_3}e_1,\, d_{n_3-1}e_0,\, e_{n_4}f_1,\, e_{n_4-1}f_0\}.$$

Let p be an integer with $1 \leq p \leq n_4 - 1$, and set

$$Y = \{b_1e_x \mid p - 1 \leq x \leq p + 1\}.$$

Now G is said to be of *Type* 6 if there exists p with $1 \leq p \leq n_4 - 1$ such that G satisfies the following four conditions:

- $X \cup F_1 \cup F_2 \cup F_3 \subseteq E(G) - E(C)$
$$\subseteq X \cup Y \cup F_1 \cup F_1' \cup F_2 \cup F_2' \cup F_3 \cup F_3',$$
- $Y \cap E(G) \neq \varnothing$,
- if $n_0 = 0$ and $n_5 = 1$, then $e_{n_4-1}f_0 \in E(G)$.
- if $n_2 = 0$ and $n_3 = 1$, then $d_1e_1 \in E(G)$.

Type 7. Let $n_0 = 0$ or 2, $n_1 = 2$, $n_2 = 0$ or 2, $n_3 \geq 1$, $n_4 = 1$, and $n_5 \geq 1$. Let

$$X = \{d_jd_{j+2} \mid 0 \leq j \leq n_3 - 2\} \cup \{f_yf_{y+2} \mid 0 \leq y \leq n_5 - 2\} \cup \{b_0b_2\},$$

$$F_1 = \begin{cases} \{a_0f_{n_5-1}\} & (\text{if } n_0 = 0) \\ \{a_0a_2,\, a_1f_{n_5-1}\} & (\text{if } n_0 = 2), \end{cases} \quad F_1' = \begin{cases} \varnothing & (\text{if } n_0 = 0) \\ \{a_1f_{n_5}\} & (\text{if } n_0 = 2), \end{cases}$$

$$F_2 = \begin{cases} \{c_0d_1\} & (\text{if } n_2 = 0) \\ \{c_0c_2,\, c_1d_1\} & (\text{if } n_2 = 2), \end{cases} \quad F_2' = \begin{cases} \varnothing & (\text{if } n_2 = 0) \\ \{c_1d_0\} & (\text{if } n_2 = 2), \end{cases}$$

$$F_3 = \{d_{n_3-1}e_1,\, e_{n_4-1}f_1\}, \quad F_3' = \{d_{n_3}e_1,\, d_{n_3-1}e_0,\, e_{n_4}f_1,\, e_{n_4-1}f_0\},$$

$$Y = \{b_1e_0,\, b_1e_1\}.$$

Under this notation, G is said to be of *Type* 7 if G satisfies the following four conditions:

- $X \cup F_1 \cup F_2 \cup F_3 \subseteq E(G) - E(C)$
$$\subseteq X \cup F_1 \cup F_1' \cup F_2 \cup F_2' \cup F_3 \cup F_3' \cup Y,$$

- $Y \cap E(G) \neq \emptyset$,
- if $n_0 = 0$ and $n_5 = 1$, then $e_{n_4-1}f_0 \in E(G)$.
- if $n_2 = 0$ and $n_3 = 1$, then $d_1e_1 \in E(G)$.

Type 8. Let $n_0 = 0$ or 2, $n_1 = 2$, $n_2 = 0$ or 2, $n_3 \geq 1$, $n_4 \geq 1$, and $n_5 \geq 1$. Let

$$X = \{d_jd_{j+2} \mid 0 \leq j \leq n_3 - 2\} \cup \{e_xe_{x+2} \mid 0 \leq x \leq n_4 - 2\}$$
$$\cup \{f_yf_{y+2} \mid 0 \leq y \leq n_5 - 2\} \cup \{b_0b_2\},$$

$$F_1 = \begin{cases} \{a_0f_{n_5-1}\} & (\text{if } n_0 = 0) \\ \{a_0a_2,\, a_1f_{n_5-1}\} & (\text{if } n_0 = 2), \end{cases} \quad F_1' = \begin{cases} \emptyset & (\text{if } n_0 = 0) \\ \{a_1f_{n_5}\} & (\text{if } n_0 = 2), \end{cases}$$

$$F_2 = \begin{cases} \{c_0d_1\} & (\text{if } n_2 = 0) \\ \{c_0c_2,\, c_1d_1\} & (\text{if } n_2 = 2), \end{cases} \quad F_2' = \begin{cases} \emptyset & (\text{if } n_2 = 0) \\ \{c_1d_0\} & (\text{if } n_2 = 2), \end{cases}$$

$$F_3 = \{d_{n_3-1}e_1\}, \quad F_3' = \{d_{n_3}e_1,\, d_{n_3-1}e_0,\, e_{n_4}f_1,\, e_{n_4-1}f_0,\, e_{n_4-1}f_1\},$$

$$F_4' = \{b_1e_{n_4-1},\, b_1e_{n_4},\, b_1f_0,\, b_1f_1\},$$

$$W_1 = \{b_1e_{n_4-1},\, b_1e_{n_4}\}, \quad W_2 = \{b_1f_1,\, b_1f_0\}, \quad W_3 = \{e_{n_4-1}f_1,\, e_{n_4-1}f_0\},$$

$$W_4 = \{f_1e_{n_4-1},\, f_1e_{n_4}\}, \quad W_5 = \{e_{n_4-1}f_1,\, e_{n_4-1}b_1\}, \quad W_6 = \{f_1e_{n_4-1},\, f_1b_1\}.$$

Under this notation, G is said to be of *Type* 8 if G satisfies the following four conditions:

- $X \cup F_1 \cup F_2 \cup F_3 \subseteq E(G) - E(C)$
$$\subseteq X \cup F_1 \cup F_1' \cup F_2 \cup F_2' \cup F_3 \cup F_3' \cup F_4',$$
- for each i with $1 \leq i \leq 6$, $W_i \cap E(G) \neq \emptyset$,
- if $n_0 = 0$ and $n_5 = 1$, then $\{f_0b_1,\, f_0e_{n_4-1}\} \cap E(G) \neq \emptyset$.
- if $n_2 = 0$ and $n_3 = 1$, then $d_1e_1 \in E(G)$.

3. Preliminaries

In this section, we prove fundamental results concerning noncontractible edges lying on a hamiltonian cycle of a 3-connected graph.

Throughout this section, we let G denote a 3-connected graph of order $n+1$ $(n \geq 4)$, and let $C = v_0 v_1 \cdots v_n v_0$ denote a hamiltonian cycle of G. Lemmas 3.1 through 3.8 are proved in Section 3 of [4] (and also in Ota [6]) and Lemma 3.9 is proved in Section 2 of [3], so we omit their proofs (in Lemmas 3.1 through 3.8, we assume that the edge $v_n v_0$ is noncontractible, and let $\{v_n, v_0, v_a\}$ be a cutset associated with it).

Lemma 3.1. (i) *No edge of G joins a vertex in $\{v_k \mid 1 \leq k \leq a-1\}$ and a vertex in $\{v_k \mid a+1 \leq k \leq n-1\}$.*

(ii) *There exists k with $1 \leq k \leq a-1$ such that $v_n v_k \in E(G)$.*

Lemma 3.2. *If $a = 2$, then $E(v_1, V(G)) - E(C) = \{v_1 v_n\}$.*

Lemma 3.3. *Suppose that $v_0 v_1$ is noncontractible and $v_a \in K(v_0, v_1)$. Then $v_n v_1 \in E(G)$.*

Lemma 3.4. *Suppose that $v_a v_{a+1}$ is noncontractible, and let $\{v_a, v_{a+1}, v_j\}$ be a cutest associated with it. Then $a+3 \leq j \leq n$ (and hence $a \leq n-3$). Further, if $j = n$, then $v_0 v_{a+1} \in E(G)$.*

Lemma 3.5. *Let $1 \leq j \leq a-2$. Suppose that $v_j v_{j+1}$ is noncontractible, and let $\{v_j, v_{j+1}, v_l\}$ be a cutset associated with it, and suppose that $a+1 \leq l \leq n-1$. Then $l = a+1$, $v_a v_l$ is contractible and, unless $l = n-1$, we have $v_l \in K(v_n, v_0)$.*

Lemma 3.6. *Suppose that $v_0 v_1$ is noncontractible, and let $\{v_0, v_1, v_j\}$ be a cutset associated with it, and suppose that $a+1 \leq j \leq n-2$. Then $v_j \in K(v_n, v_0)$.*

Lemma 3.7. *Suppose that* $K(v_n, v_0) = \{v_2\}$, *and that* v_0v_1 *is noncontractible. Then* $K(v_0, v_1) = \{v_{n-1}\}$.

Lemma 3.8. *If* $a = 2$, *then* v_1v_2 *is contractible*; *if* $a \geq 3$, *then there exists j with* $0 \leq j \leq a-1$ *such that* v_jv_{j+1} *is contractible*; *if* $a \geq 3$ *and there exists only one j with* $0 \leq j \leq a-1$ *such that* v_jv_{j+1} *is contractible*; *then* v_av_{a+1} *is contractible.*

Lemma 3.9. *Let l be an integer with* $3 \leq l \leq n-1$.

(i) *Suppose that for each j with* $l+1 \leq j \leq n$, $v_{j-1}v_j$ *is noncontractible and* $K(v_{j-1}, v_j) \cap \{v_i \mid 1 \leq i \leq l-2\} \neq \emptyset$. *Then G has no edge* $v_{j_1}v_{j_2}$ *such that* $l \leq j_1 < j_1 + 3 \leq j_2 \leq n$.

(ii) *Suppose that* $l \leq n-3$, *let h be an integer with* $l+2 \leq h \leq n-1$, *and suppose that for each j with* $l+1 \leq j \leq n$ *and* $j \neq h$, $v_{j-1}v_j$ *is noncontractible and* $K(v_{j-1}, v_j) \cap \{v_i \mid 1 \leq i \leq l-2\} \neq \emptyset$. *Further let* $v_{j_1}v_{j_2} \in E(G)$ *be an edge such that* $l \leq j_1 < j_1 + 3 \leq j_2 \leq n$. *Then* $j_1 = h-2$ *and* $j_2 = h+1$.

Lemma 3.10. *Let* $1 \leq i_1$ *and* $i_1 + 2 \leq i_2 < i_3 \leq n-1$. *Suppose that* v_iv_{i+1} *is noncontractible for all* $0 \leq i \leq i_1 - 1$, $K(v_i, v_{i+1}) \cap \{v_j \mid i_2 \leq j \leq i_3\} \neq \emptyset$ *for all* $0 \leq i \leq i_1 - 1$, *and* $v_{i_3} \notin K(v_0, v_1)$. *Then* $v_{i_3} \notin K(v_i, v_{i+1})$ *for each* $0 \leq i \leq i_1 - 1$.

Proof. Take $v_k \in K(v_0, v_1) \cap \{v_j \mid i_2 \leq j \leq i_3\}$. We have $k \neq i_3$ by assumption. Let $1 \leq i \leq i_1 - 1$, and suppose that $v_{i_3} \in K(v_i, v_{i+1})$. Then applying Lemma 3.5 or 3.6 to $\{v_0, v_1, v_k\}$ and $\{v_i, v_{i+1}, v_{i_3}\}$, we get $v_{i_3} \in K(v_0, v_1)$, a contradiction.

4. Initial Reduction

Throughout the rest of this paper, we let G and C be as in Theorem 2, and write $C = a_0a_1 \cdots a_{n_0}b_0b_1 \cdots b_{n_1}c_0c_1 \cdots c_{n_2}d_0d_1 \cdots d_{n_3}e_0e_1 \cdots e_{n_4}f_0f_1 \cdots$

$f_{n_5}a_0$, where $a_{n_0}b_0$, $b_{n_1}c_0$, $c_{n_2}d_0$, $d_{n_3}e_0$, $e_{n_4}f_0$ and $f_{n_5}a_0$ are the six contractible edges contained in C. Note that C is a hamiltonian cycle, thus $|V(G)| = n_0 + n_1 + n_2 + n_3 + n_4 + n_5 + 6$. Let $C_0 = \{a_0, a_1, ..., a_{n_0}\}$, $C_1 = \{b_0, b_1, ..., b_{n_1}\}$, $C_2 = \{c_0, c_1, ..., c_{n_2}\}$, $C_3 = \{d_0, d_1, ..., d_{n_3}\}$, $C_4 = \{e_0, e_1, ..., e_{n_4}\}$ and $C_5 = \{f_0, f_1, ..., f_{n_5}\}$.

In this section, we derive some basic properties of (G, C). Lemmas 4.1 and 4.5 are proved in Section 4 of [4]; we can prove Lemmas 4.2 through 4.4, 4.6 and 4.7 by arguing exactly as in the corresponding lemmas in Section 4 of [4], and Lemma 4.8 by arguing exactly as in Lemma 3.8 of [3].

Lemma 4.1. *Suppose that* $n_1 = 2$. *Then one of the following holds*:

(i) $K(b_0, b_1) = \{c_0\}$ *and* $K(b_1, b_2) = \{a_{n_0}\}$; *or*

(ii) $K(b_0, b_1) \neq \{c_0\}$ *and* $K(b_1, b_2) \neq \{a_{n_0}\}$.

Lemma 4.2. *Suppose that* $n_1 \geq 1$.

(i) *If* $n_1 \neq 2$, *then* $K(b_0, b_1) \subseteq C_3 \cup C_4 \cup C_5 \cup \{c_{n_2}, a_0\}$.

(ii) *If* $n_1 = 2$, *then* $K(b_0, b_1) \subseteq C_3 \cup C_4 \cup C_5 \cup \{c_0, c_{n_2}, a_0\}$.

Lemma 4.3. *One of the following holds*:

(i) $n_1 = 0$;

(ii) $n_1 = 2$ *and* $K(b_0, b_1) = \{c_0\}$ *and* $K(b_1, b_2) = \{a_{n_0}\}$; *or*

(iii) $n_1 \geq 1$ *and* $K(b_i, b_{i+1}) \cap (C_3 \cup C_4 \cup C_5) \neq \emptyset$ *for all* $0 \leq i \leq n_1 - 1$.

With Lemma 4.3 in mind, we define the terms *degenerate* and *nondegenerate* as follows: for each $0 \leq l \leq 5$, C_l is said to be *nondegenerate* if $n_l \geq 1$ and $K(u, v) \cap (C_{l+2} \cup C_{l+3} \cup C_{l+4}) \neq \emptyset$ for all $uv \in E(\langle C_l \rangle_C)$ (indices of the letter C are to be read modulo 6); otherwise C_l is said to be *degenerate*. Thus, for example, C_1 is nondegenerate if and only if (iii) of Lemma 4.3 holds, and it is degenerate if and only if (i) or (ii) of Lemma 4.3 holds.

Lemma 4.4. *At most four of the* C_l $(0 \leq l \leq 5)$ *are nondegenerate.*

Lemma 4.5. *Suppose that* C_0 *is degenerate and* $n_0 = 2$. *Then the following hold*:

(i) $E(a_0, V(G)) - E(C) = \{a_0 a_2\}$, *and* $E(a_2, V(G)) - E(C) = \{a_0 a_2\}$.

(ii) $E(\{a_0, a_2\}, V(G)) - E(C) = \{a_0 a_2\}$.

Lemma 4.6. *Suppose that* C_0 *is degenerate, and that* C_5 *is nondegenerate and* $b_0 \in K(f_{n_5 - 1}, f_{n_5})$.

(I) *If* $n_0 = 0$, *then* $E(C_0, V(G)) - E(C) = \{a_0 f_{n_5 - 1}\}$.

(II) *Suppose that* $n_0 = 2$. *Then the following hold*:

(i) $\{a_0 a_2, a_1 f_{n_5 - 1}\} \subseteq E(C_0, V(G)) - E(C)$

$$\subseteq \{a_0 a_2, a_1 b_0, a_1 f_{n_5 - 1}, a_1 f_{n_5}\}.$$

(ii) *Suppose further that* C_1 *is degenerate, and that either* $n_1 = 2$, *or* $n_1 = 0$ *and* $n_2 \geq 1$ *and* $a_2 \in K(c_0, c_1)$. *Then* $\{a_0 a_2, a_1 f_{n_5 - 1}\} \subseteq E(C_0, V(G)) - E(C) \subseteq \{a_0 a_2, a_1 f_{n_5 - 1}, a_1 f_{n_5}\}$.

Lemma 4.7. *Suppose that* C_5 *is nondegenerate. Then* $f_i f_j \notin E(G)$ *for any* i, j *with* $i + 3 \leq j$.

Lemma 4.8. *Suppose that* C_3, C_4 *and* C_5 *are nondegenerate. Then* $K(d_j, d_{j+1}) \cap C_1 \neq \varnothing$ *for all* $0 \leq j \leq n_3 - 1$, $K(e_x, e_{x+1}) \cap C_1 \neq \varnothing$ *for all* $0 \leq x \leq n_4 - 1$, *and* $K(f_y, f_{y+1}) \cap C_1 \neq \varnothing$ *for all* $0 \leq y \leq n_5 - 1$.

5. Proof of Theorem 2

We continue with the notation of the preceding section, and complete the proof of Theorem 2. Theorem 2 follows from the following proposition:

Proposition 1. *Suppose that* C_3, C_4 *and* C_5 *are nondegenerate, and* C_0, C_1 *and* C_2 *are degenerate. Then* (G, C) *is of Type* 2, 3, 4, 5, 6, 7 *or* 8.

Proof. By Lemma 4.8, we have

$$K(d_j, d_{j+1}) \cap C_1 \neq \emptyset \text{ for all } 0 \leq j \leq n_3 - 1, \tag{5.1}$$

$$K(e_x, e_{x+1}) \cap C_1 \neq \emptyset \text{ for all } 0 \leq x \leq n_4 - 1 \tag{5.2}$$

and

$$K(f_y, f_{y+1}) \cap C_1 \neq \emptyset \text{ for all } 0 \leq y \leq n_4 - 1. \tag{5.3}$$

Hence it follows from Lemma 3.9 that

$$E(C_3, C_4) - E(C) \subseteq \{d_{n_3-1}e_0, d_{n_3-1}e_1, d_{n_3}e_1\} \tag{5.4}$$

and

$$E(C_4, C_5) - E(C) \subseteq \{e_{n_4-1}f_0, e_{n_4-1}f_1, e_{n_4}f_1\}. \tag{5.5}$$

Claim 5.1. $E(C_3, C_5) = \emptyset$.

Proof. Suppose that $E(C_3, C_5) \neq \emptyset$. Then there exist j and y with $0 \leq j \leq n_3 - 1$ and $0 \leq y \leq n_5 - 1$ such that $d_j f_y \in E(G)$. But then, since it follows from (5.2) that $\{e_0, e_1, b_t\}$ is a cutset for some t with $0 \leq t \leq n_1$, we get a contradiction by Lemma 3.1(i).

If $n_1 = 0$ (so $C_1 = \{b_0\}$), then in view of (5.4), (5.5) and Claim 5.1, combining the proof of Proposition 3 of [4] for the case $n_1 = 0$, and the argument used in the proof of (5-6) and Claim 5.11 in Proposition 2 of [4], we see that (G, C) is of Type 2. Thus we henceforth assume that $n_1 = 2$ (so $C_1 = \{b_0, b_2\}$). Applying Lemma 4.5(ii) to C_1, we get

$$E(\{b_0, b_2\}, V(G)) - E(C) = \{b_0 b_2\}. \tag{5.6}$$

Hence it follows from Lemma 3.1(i) that $b_1 \notin K(d_j, d_{j+1})$ for all $0 \leq j \leq n_3 - 1$, $b_1 \notin K(e_x, e_{x+1})$ for all $0 \leq x \leq n_4 - 1$, and $b_1 \notin K(d_y, d_{y+1})$ for all $0 \leq y \leq n_5 - 1$. Thus by (5.1) through (5.3), we obtain

$$K(d_j, d_{j+1}) \cap \{b_0, b_2\} \neq \emptyset \text{ for all } 0 \leq j \leq n_3 - 1, \tag{5.7}$$

$$K(e_x, e_{x+1}) \cap \{b_0, b_2\} \neq \emptyset \text{ for all } 0 \leq x \leq n_4 - 1 \tag{5.8}$$

and

$$K(f_y,\, f_{y+1}) \cap \{b_0,\, b_2\} \neq \varnothing \text{ for all } 0 \leq y \leq n_5 - 1. \tag{5.9}$$

Claim 5.2. One of the following holds:

(i) $K(f_y, f_{y+1}) \cap \{b_0, b_2\} = \{b_2\}$ for all $0 \leq y \leq n_5 - 1$, and $K(e_x, e_{x+1}) \cap \{b_0, b_2\} = \{b_2\}$ for all $0 \leq x \leq n_4 - 1$, and $K(d_j, d_{j+1}) \cap \{b_0, b_2\} = \{b_2\}$ for all $0 \leq j \leq n_3 - 1$;

(ii) $n_5 \geq 2$, and there exists p with $1 \leq p \leq n_5 - 1$ such that $b_0 \in K(f_y, f_{y+1})$ for all $p \leq y \leq n_5 - 1$ and $b_2 \in K(f_y, f_{y+1})$ for all $0 \leq y \leq p - 1$ and $K(e_x, e_{x+1}) \cap \{b_0, b_2\} = \{b_2\}$ for all $0 \leq x \leq n_4 - 1$ and $K(d_j, d_{j+1}) \cap \{b_0, b_2\} = \{b_2\}$ for all $0 \leq j \leq n_3 - 1$;

(iii) $n_5 = 1$ and $b_0, b_2 \in K(f_0, f_1)$, and $K(e_x, e_{x+1}) \cap \{b_0, b_2\} = \{b_2\}$ for all $0 \leq x \leq n_4 - 1$, and $K(d_j, d_{j+1}) \cap \{b_0, b_2\} = \{b_2\}$ for all $0 \leq j \leq n_3 - 1$;

(iv) $K(f_y, f_{y+1}) \cap \{b_0, b_2\} = \{b_0\}$ for all $0 \leq y \leq n_5 - 1$, and $K(e_x, e_{x+1}) \cap \{b_0, b_2\} = \{b_2\}$ for all $0 \leq x \leq n_4 - 1$, and $K(d_j, d_{j+1}) \cap \{b_0, b_2\} = \{b_2\}$ for all $0 \leq j \leq n_3 - 1$;

(v) $K(f_y, f_{y+1}) \cap \{b_0, b_2\} = \{b_0\}$ for all $0 \leq y \leq n_5 - 1$, and $n_4 \geq 2$, and there exists p with $1 \leq p \leq n_4 - 1$ such that $b_0 \in K(e_x, e_{x+1})$ for all $p \leq x \leq n_4 - 1$ and $b_2 \in K(e_x, e_{x+1})$ for all $0 \leq x \leq p - 1$ and $K(d_j, d_{j+1}) \cap \{b_0, b_2\} = \{b_2\}$ for all $0 \leq j \leq n_3 - 1$;

(vi) $K(f_y, f_{y+1}) \cap \{b_0, b_2\} = \{b_0\}$ for all $0 \leq y \leq n_5 - 1$, and $n_4 = 1$ and $b_0, b_2 \in K(e_0, e_1)$, and $K(d_j, d_{j+1}) \cap \{b_0, b_2\} = \{b_2\}$ for all $0 \leq j \leq n_3 - 1$;

(vii) $K(f_y, f_{y+1}) \cap \{b_0, b_2\} = \{b_0\}$ for all $0 \leq y \leq n_5 - 1$, and $K(e_x, e_{x+1}) \cap \{b_0, b_2\} = \{b_0\}$ for all $0 \leq x \leq n_4 - 1$, and $K(d_j, d_{j+1}) \cap \{b_0, b_2\} = \{b_2\}$ for all $0 \leq j \leq n_3 - 1$;

(viii) $K(f_y, f_{y+1}) \cap \{b_0, b_2\} = \{b_0\}$ for all $0 \le y \le n_5 - 1$, and $K(e_x, e_{x+1}) \cap \{b_0, b_2\} = \{b_0\}$ for all $0 \le x \le n_4 - 1$, and $n_3 \ge 2$, and there exists p with $1 \le p \le n_3 - 1$ such that $b_0 \in K(d_j, d_{j+1})$ for all $p \le j \le n_3 - 1$ and $b_2 \in K(d_j, d_{j+1})$ for all $0 \le j \le p - 1$;

(ix) $K(f_y, f_{y+1}) \cap \{b_0, b_2\} = \{b_0\}$ for all $0 \le y \le n_5 - 1$, and $K(e_x, e_{x+1}) \cap \{b_0, b_2\} = \{b_0\}$ for all $0 \le x \le n_4 - 1$, and $n_3 = 1$ and $b_0, b_2 \in K(d_0, d_1)$; or

(x) $K(f_y, f_{y+1}) \cap \{b_0, b_2\} = \{b_0\}$ for all $0 \le y \le n_5 - 1$, and $K(e_x, e_{x+1}) \cap \{b_0, b_2\} = \{b_0\}$ for all $0 \le x \le n_4 - 1$, and $K(d_j, d_{j+1}) \cap \{b_0, b_2\} = \{b_0\}$ for all $0 \le j \le n_3 - 1$.

Proof. We first prove the following subclaim.

Subclaim 5.1. If $b_2 \in K(f_0, f_1)$, then (i), (ii) or (iii) holds; if $b_0 \in K(d_{n_3 - 1}, d_{n_3})$, then (viii), (ix) or (x) holds.

Proof. Suppose that

$$b_2 \in K(f_0, f_1). \tag{5.10}$$

Then by Lemma 3.5, it follows from (5.7) and (5.8) that

$$K(d_j, d_{j+1}) \cap \{b_0, b_2\} = \{b_2\} \text{ for all } 0 \le j \le n_3 - 1 \tag{5.11}$$

and

$$K(e_x, e_{x+1}) \cap \{b_0, b_2\} = \{b_2\} \text{ for all } 0 \le x \le n_4 - 1. \tag{5.12}$$

If $K(f_{n_5 - 1}, f_{n_5}) \cap \{b_0, b_2\} = \{b_2\}$, then by Lemma 3.10 and (5.9), $K(f_y, f_{y+1}) \cap \{b_0, b_2\} = \{b_2\}$ for all $0 \le y \le n_5 - 1$, and hence it follows from (5.11) and (5.12) that (i) holds. Thus by (5.9), we may assume

$$b_0 \in K(f_{n_5 - 1}, f_{n_5}). \tag{5.13}$$

Now if $n_5 = 1$, then it follows from (5.13), (5.10), (5.11) and (5.12) that (iii) holds; and if $n_5 > 1$, then in view of (5.13), (5.10), (5.11) and (5.12), arguing as in Claim 5.16 of [4], we see that (ii) holds. Thus it is proved that (i), (ii), or (iii) holds if $b_2 \in K(f_0, f_1)$. By symmetry, we see that (viii), (ix) or (x) holds if $b_0 \in K(d_{n_3-1}, d_{n_3})$.

We return to the proof of the claim. By Subclaim 5.1, we may assume $b_2 \notin K(f_0, f_1)$, and hence

$$K(f_y, f_{y+1}) \cap \{b_0, b_2\} = \{b_0\} \text{ for all } 0 \leq y \leq n_5 - 1 \tag{5.14}$$

by Lemma 3.10 and (5.9), and we may also assume $b_0 \notin K(d_{n_3-1}, d_{n_3})$, and hence

$$K(d_j, d_{j+1}) \cap \{b_0, b_2\} = \{b_2\} \text{ for all } 0 \leq j \leq n_3 - 1 \tag{5.15}$$

by Lemma 3.10 and (5.7). Now, assume that

$$b_2 \in K(e_0, e_1). \tag{5.16}$$

If $K(e_{n_4-1}, e_{n_4}) \cap \{b_0, b_2\} = \{b_2\}$, then by Lemma 3.10 and (5.8), $K(e_x, e_{x+1}) \cap \{b_0, b_2\} = \{b_2\}$ for all $0 \leq x \leq n_4 - 1$, and hence it follows from (5.15) and (5.14) that (iv) holds. Thus by (5.8), we may assume

$$b_0 \in K(e_{n_4-1}, e_{n_4}). \tag{5.17}$$

Now if $n_4 = 1$, then it follows from (5.17), (5.16), (5.15) and (5.14) that (vi) holds; and if $n_4 > 1$, then in view of (5.17), (5.16), (5.15) and (5.14), arguing as in Claim 5.16 of [4], we see that (v) holds. Thus we may assume $b_2 \notin K(e_0, e_1)$, and hence by Lemma 3.10 and (5.8), $K(e_x, e_{x+1}) \cap \{b_0, b_2\} = \{b_0\}$ for all $0 \leq x \leq n_4 - 1$, and this together with (5.15) and (5.14) implies that (vii) holds. Consequently the claim is proved.

Returning to the proof of the proposition, if (i), (ii), (iii), (v) or (vi) of Claim 5.2 holds, then in view of (5.4) through (5.6) and Claim 5.1, combining the proof of Proposition 3 of [4] (in the case where Claim 5.2(iii) holds, we apply the argument in the proof of Claim 5.17 of [4] with $Y = \{b_1f_0, b_1f_1\}$; in the case where Claim 5.2(vi) holds, we apply the argument in the proof of Claim 5.17 of [4] with $Y = \{b_1e_0, b_1e_1\}$) and the proof of (5-6) and Claim 5.11 of [4], we see that (G, C) is of Type 3, 4, 5, 6 or 7. Thus by symmetry, we may assume Claim 5.2(iv) holds. Applying Lemmas 3.3 and 4.7 to C_3, C_4 and C_5, we have the following claim:

Claim 5.3. $E(\langle C_3\rangle) - E(C) = \{d_j d_{j+2} \mid 0 \le j \le n_3 - 2\}$, $E(\langle C_4\rangle) - E(C) = \{e_x e_{x+2} \mid 0 \le x \le n_4 - 2\}$, and $E(\langle C_5\rangle) - E(C) = \{f_y f_{y+2} \mid 0 \le y \le n_5 - 2\}$.

Applying (I) and (II)(ii) of Lemma 4.6 to C_0 and C_2, we get the following two claims:

Claim 5.4. (i) If $n_0 = 0$, then $E(C_0, V(G)) - E(C) = \{a_0 f_{n_5 - 1}\}$.

(ii) If $n_0 = 2$, then

$$\{a_0 a_2, a_1 f_{n_5-1}\} \subseteq E(C_0, V(G)) - E(C) \subseteq \{a_0 a_2, a_1 f_{n_5-1} a_1 f_{n_5}\}.$$

Claim 5.5. (i) If $n_2 = 0$, then $E(C_2, V(G)) - E(C) = \{c_0 d_1\}$.

(ii) If $n_2 = 2$, then

$$\{c_0 c_2, c_1 d_1\} \subseteq E(C_2, V(G)) - E(C) \subseteq \{c_0 c_2, c_1 d_1, c_1 d_0\}.$$

Further applying Lemma 3.1(i) to $\{e_{n_4-1}, e_{n_4}, b_2\}$ and $\{f_0, f_1, b_0\}$, we get

$$E(b_1, V(G)) - E(C) \subseteq \{b_1 e_{n_4-1}, b_1 e_{n_4}, b_1 f_0, b_1 f_1\}. \tag{5.18}$$

Claim 5.6. $\{b_1 e_{n_4-1}, b_1 e_{n_4}\} \cap E(G) \ne \varnothing$ and $\{b_1 f_1, b_1 f_0\} \cap E(G) \ne \varnothing$.

Proof. By the assumption that Claim 5.2(iv) holds, $\{f_0, f_1, b_2\}$ is not a cutset, and hence $E(C_0 \cup (C_1 - \{b_2\}) \cup (C_5 - \{f_0, f_1\}), C_2 \cup C_3 \cup C_4) \ne \varnothing$. Since $E(C_0 \cup (C_1 - \{b_2\}) \cup (C_5 - \{f_0, f_1\}), C_2) = \varnothing$ by Claim 5.5, and since $E(C_0 \cup \{b_0\} \cup (C_5 - \{f_0, f_1\}), C_3 \cup C_4) = \varnothing$ by Claim 5.4, (5.6), Claim 5.1 and (5.5), this means $E(b_1, C_3 \cup C_4) \ne \varnothing$. Hence it follows from (5.18) that $\{b_1 e_{n_4-1}, b_1 e_{n_4}\} \cap E(G) \ne \varnothing$ and, in a similar way, we can verify $\{b_1 f_1, b_1 f_0\} \cap E(G) \ne \varnothing$.

Claim 5.7. $\{e_{n_4-1} f_1, e_{n_4-1} f_0\} \cap E(G) \ne \varnothing$ and $\{f_1 e_{n_4-1}, f_1 e_{n_4}\} \cap E(G) \ne \varnothing$.

Proof. Since C_1 is degenerate by the assumption of Proposition

3, $\{b_1, b_2, e_{n_4}\}$ is not a cutset, and hence $E(C_0 \cup \{b_0\} \cup C_5, C_2 \cup C_3 \cup (C_4 - \{e_{n_4}\})) \neq \emptyset$. Since $E(C_0 \cup \{b_0\}, C_2 \cup C_3 \cup (C_4 - \{e_{n_4}\})) = \emptyset$ by Claim 5.4 and (5.6), and since $E(C_5, C_2 \cup C_3) = \emptyset$ by Claims 5.5 and 5.1, this means $E(C_5, C_4 - \{e_{n_4}\}) \neq \emptyset$. Hence it follows from (5.5) that $\{e_{n_4-1}f_1, e_{n_4-1}f_0\} \cap E(G) \neq \emptyset$ and, in a similar way, we can verify $\{f_1e_{n_4-1}, f_1e_{n_4}\} \cap E(G) \neq \emptyset$.

Claim 5.8. $\{e_{n_4-1}f_1, e_{n_4-1}b_1\} \cap E(G) \neq \emptyset$ and $\{f_1e_{n_4-1}, f_1b_1\} \cap E(G) \neq \emptyset$.

Proof. Since $e_{n_4}f_0$ is contractible, $\{e_{n_4}, f_0, b_2\}$ is not a cutset, and hence $E(C_0 \cup (C_1 - \{b_2\}) \cup (C_5 - \{f_0\}), C_2 \cup C_3 \cup (C_4 - \{e_{n_4}\}) \neq \emptyset$. In view of Claim 5.5, (5.6) and Claim 5.4, we have $E(C_0 \cup (C_1 - \{b_2\}) \cup (C_5 - \{f_0\}), C_2) = \emptyset$, $E(b_0, C_3 \cup (C_4 - \{e_{n_4}\})) = \emptyset$, and $E(C_0, C_3 \cup (C_4 - \{e_{n_4}\})) = \emptyset$. Consequently $E(\{b_1\} \cup (C_5 - \{f_0\}), C_3 \cup (C_4 - \{e_{n_4}\}) \neq \emptyset$. Hence it follows from (5.18), Claim 5.1 and (5.5) that $\{e_{n_4-1}f_1, e_{n_4-1}b_1\} \cap E(G) \neq \emptyset$ and, in a similar way, we can verify $\{f_1e_{n_4-1}, f_1b_1\} \cap E(G) \neq \emptyset$.

Now arguing as in the proof of (5-6) and Claim 5.11 of [4], we see from (5.4), (5.5), (5.6), (5.18) and Claims 5.3 through 5.8 that (G, C) is of Type 8.

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