JP Journal of Algebra, Number Theory and Applications
Volume 9, Number 2, 2007, Pages 241-275
Published online: January 7, 2008
This paper is available online at http://www.pphmj.com

COMPOSITES OF MINIMAL RING EXTENSIONS

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Abstract

Let R be a (commutative unital) ring. If S and T are distinct minimal ring extensions of R, their composite ST may not exist; i.e., there may not exist a (commutative unital) R-algebra U containing both S and T as R-subalgebras. We assume henceforth that such U exists. It seems natural to ask $(*)$: is ST necessarily a minimal ring extension of both S and T? If R is a field and S, T (as above) are splitting field extensions of R, the answer to $(*)$ is "yes"; without this "splitting field" assumption on the fields S and T, the answer is, in general, "no". If R is a field and either S or T is not a field, the answer to $(*)$ is, in general, "no". Let M and N be the so-called crucial maximal ideals of R relative to S and T, respectively. If $M \neq N$, the answer to $(*)$ is "yes". Assume henceforth that R is a ring with von Neumann regular total quotient ring and that

2000 Mathematics Subject Classification: Primary 13B99, 13A15; Secondary 13B21, 13G05, 13C13.

Keywords and phrases: minimal ring extension, composite, total quotient ring, overring, von Neumann regular ring, crucial maximal ideal, integrality, flat emimorphism, idealization, algebra, Noetherian domain.

Communicated by Marco Fontana

Received March 7, 2007; Revised April 15, 2007

S and T are overrings of R. If R is integrally closed in both S and T, the answer to $(*)$ is "yes". If $M = N$, it cannot be the case that T is integral over R while R is integrally closed in S. If $M = N$ with S (and hence T) integral over R, then the answer to $(*)$ can be "no" and we give best-possible finite upper bounds for the cardinalities of chains of rings between S and ST and chains of rings between T and ST.

1. Introduction

All rings and algebras considered below are commutative with identity; all subrings/subalgebras and ring/algebra homomorphisms are unital. Recall that a ring extension $A \subset B$ is called a *minimal ring extension* if the inclusion map $A \hookrightarrow B$ is a minimal homomorphism in the sense of [9], i.e., if there is no ring D such that $A \subset D \subset B$. (As usual, $\subset$ denotes proper inclusion.) If $A \subset B$ is a minimal ring extension, then it was shown in [9, Théorème 2.2(ii), (iii)] that either B is integral over A or $A \hookrightarrow B$ is a flat epimorphism (in the category of commutative rings). Suppose that $R \subset S$ and $R \subset T$ are minimal ring extensions such that the composite ST exists, i.e., such that there exists a (commutative unital) R-algebra U containing S and T as R-subalgebras. (See Example 2.1 for cases where S and T are minimal ring extensions of R for which no such U exists.) If $R \hookrightarrow S$ and $R \hookrightarrow T$ are also each flat epimorphisms, then $ST \cong S \otimes_R T$ and both $S \hookrightarrow ST$ and $T \hookrightarrow ST$ are flat epimorphisms. By analogy, we are led to ask the following basic question of this paper. If $R \subset S$ and $R \subset T$ are *arbitrary* minimal ring extensions such that the composite ST exists, must $S \subset ST$ and $T \subset ST$ be minimal ring extensions? In the next four paragraphs, we summarize the answers that are given in the following sections.

Section 2 begins with some simple contexts in which our basic question has a positive answer. For instance, it is shown in Proposition 2.2(b) that if $K \subset F$ and $K \subset L$ are distinct minimal field extensions inside some algebraic closure of K such that F and L are each splitting fields over K, then $F \subset FL$ and $L \subset FL$ are each minimal field extensions. However, Example 2.3 shows that the basic question has a negative answer (with base field $R = K$, a field) if F, L are replaced with any of the other kinds of minimal ring extensions S, T of a field that are catalogued in [9, Lemme 1.2].

Working over more general base rings requires the following concept from [9]. If $A \subset B$ is a minimal ring extension, then there is a unique maximal ideal M of A, called the *crucial maximal ideal* of $A \subset B$ (or of A relative to B or, simply, of B) such that if P is any prime ideal of A, then the canonical injective A-algebra homomorphism $A_P \to B_P$ is an isomorphism if $P \neq M$ and a minimal homomorphism if $P = M$. (As usual, if E is an A-module and P is a prime ideal of A, then $E_P := E_{A \setminus P}$.) Perhaps our most useful positive answer for the basic question is given in Theorem 2.5: if $R \subset S$ and $R \subset T$ are minimal ring extensions, with distinct crucial maximal ideals M and N, respectively, and if the composite ST exists in some ring extension of R, then both $S \subset ST$ and $T \subset ST$ are minimal ring extensions.

The most striking applications of Theorem 2.5 arise when S and T are overrings of R. (As usual, if A is a ring, then $\mathrm{tq}(A)$ denotes a/the total quotient ring of A; and, by an overring of A, we mean an A-subalgebra of $\mathrm{tq}(A)$, i.e., a ring B such that $A \subseteq B \subseteq \mathrm{tq}(A)$.) Perhaps the most important kinds of minimal ring extensions are overrings. Indeed, recall from [19, p. 1738] that if R is a (commutative integral) domain which is not a field and B is a domain such that $R \subset B$ is a minimal ring extension, then B is (R-algebra isomorphic to) an overring of R. (See [18, Proposition 3.9] and [4, Theorem 2.2] for some recent generalizations of this result to contexts involving nontrivial zero-divisors.) As an application of Theorem 2.5, it is shown in Corollary 2.7 that if R is a ring such that $K := \mathrm{tq}(R)$ is a von Neumann regular ring (for instance, R a domain) and $S, T \subseteq K$ are distinct minimal overrings of R such that R is integrally closed in both S and T, then both $S \subset ST$ and $T \subset ST$ are minimal ring extensions. Thus, Corollary 2.7 gives an affirmative answer to our basic question if S, T are flat epimorphic overrings of a ring with von Neumann regular total quotient ring. It is noteworthy that the proof of Corollary 2.7 makes use of the classification in [7] of the minimal ring extensions of rings R having von Neumann regular $\mathrm{tq}(R)$ and no minimax prime ideal, and hence, makes use of the generalized Kaplansky transform that was introduced in [20]. This same background is used in proving Theorem 2.8, which shows that if $\mathrm{tq}(R)$ is von Neumann regular, then R cannot have composable minimal overrings S and T such that R is integrally closed in S, T is integral over R, and the minimal ring extensions $R \subset S$ and $R \subset T$ have the same crucial maximal ideal.

In view of Example 2.3, Corollary 2.7, Theorem 2.8, and the classification of minimal ring extensions in [7, Corollary 2.5], the study of our basic question is reduced to considering composable integral minimal overrings S, T of a ring R with the same crucial maximal ideal M. Section 3 gives an extensive study of this context, showing that in most cases, the basic question has a negative answer. The relevant *cases* arise since S/M and T/M are minimal ring extensions of the field R/M and, hence, are each isomorphic to one of the three archetypes noted in [9, Lemme 1.2]. These three possibilities for integral minimal overrings were characterized by generator-and-relations in [7, Proposition 2.12], which is restated for convenience as Lemma 3.1 below. In general, any chain of rings contained between S and ST or between T and ST must be finite. An upper bound for the cardinalities of such chains is given in Proposition 3.3. Example 3.9 shows this upper bound to be best possible, even when R is a Noetherian domain. In particular, $S \subset ST$ need not be a minimal ring extension in general (cf. Examples 3.6, 3.8 and 3.9). Proposition 3.4(a) identifies cases for which $S \subset ST$ is, in general, a minimal ring extension.

In addition to the notation and terminology introduced above, it is convenient to use the following conventions. If A is a ring, then $\mathrm{Spec}(A)$ denotes the set of all prime ideals of A; $\mathrm{Max}(A)$ the set of all maximal ideals of A; and $\mathrm{Min}(A)$ the set of all minimal prime ideals of A. If I is an ideal of A, then $\mathrm{Rad}(I) := \mathrm{Rad}_A(I)$ denotes the radical of I in A. If E is an A-module, then $A(+)E$ denotes the idealization built from A and E;

a convenient reference for basic facts about the idealization construction is [12]. Also, X and Y denote commuting algebraically independent indeterminates over the ambient coefficient ring(s). As usual, char(K) denotes the characteristic of a field K; $\dim_K(E)$ denotes the vector space dimension of a K-vector space E over a field K; $|S|$ denotes the cardinal number of a set S; and $\mathbb{N}$ denotes the set of positive integers. Unexplained material is standard, as in [10] and [15].

2. Generalities and the Relatively Integrally Closed Case

We begin by showing that minimal ring extensions $R \subset S$ and $R \subset T$ need not have a composite.

Example 2.1. There exist minimal ring extensions $R \subset S$ and $R \subset T$ for which there does not exist a (commutative unital) R-algebra U containing both S and T as R-subalgebras.

Proof. Suppose that a domain R has a minimal overring S such that the crucial maximal ideal for the minimal ring extension $R \subset S$ is N and there exists a nonzero element $n \in N$ such that $n^{-1} \in S$. (We recall below one way to construct such a Bézout domain R having Krull dimension 1 and such an overring S.) Next, let T be another minimal ring extension of R having crucial maximal ideal N such that T is not R-algebra isomorphic to S. (For instance, by [3, Corollary 2.5] and [6, Theorem 2.7], take T to be either the idealization $R(+)R/N$ or the direct product $R \times R/N$.) Then one cannot form the composite ST in any sensible universe.

To see this, suppose, on the contrary, that it makes sense to consider the composite ST inside some *universe* U. Then n is a unit of U (since the ring extension $S \subset U$ is unital). However, n is also a nontrivial zero-divisor of U, since

$$n(0, 1+N) = (n, 0)(0, 1+N) = (n, n+N)(0, 1+N) = (0, 0) = 0$$

in both $R(+)R/N$ and $R \times R/N$. The presence of this unit which is a non-trivial zero-divisor contradicts the fact that U is a nonzero ring.

We next indicate one way to construct R and S as above and offer an alternate explanation for the assertion. By taking R to be the intersection of two incomparable valuation domains of (Krull) dimension 1 having the same quotient field, we obtain R as a one-dimensional Bézout domain with exactly two maximal ideals, say M and N (cf. [15, Theorem 107]). As $\operatorname{Spec}(R) = \{0, M, N\}$ and R is a Prüfer domain, it follows from [10, Theorem 26.1(2)] that $S := R_M$ is a minimal ring extension of R. It is clear that the crucial maximal ideal for $R \subset S$ is N. Moreover, any element $n \in N \setminus M$ satisfies $n^{-1} \in S$. Taking T as above, we could argue as above to get the assertion. The following is a more ornate way to reach the same conclusion.

Suppose, on the contrary, that it makes sense to consider the composite ST inside some *universe* U. Then $ST = R_M T$ is canonically identified with $T_{R \setminus M}$. Indeed, since R_M is R-flat, the canonical R-algebra homomorphism $T_{R\setminus M} \cong T \otimes_R R_M \to U \otimes_R R_M \cong U_{R\setminus M}$ is an injection whose image clearly is $TR_M = R_M T = ST$. Therefore, we can view $ST = T_{R\setminus M}$, which is canonically identified with $R_M = S$ since the crucial maximal ideal of T is not M. Thus, $S = ST \supset T \supset R$, contradicting the minimality of the ring extension $R \subset S$. This proves that no such U exists.

Chastened by the above example, we retreat to the context of a base field, beginning there with the case of minimal field extensions. Recall from [13, Definition 3, pp. 83-84] that if F and L are field extensions of a field K, there is a classical concept of *composite* which uses the fact that some K-algebra E does contain K-algebra isomorphic copies of both F and L. (For instance, take the field $E := (F_{\otimes K} L)/N$, where $N \in \operatorname{Max}(F_{\otimes K} L)$.) However, our concept of a *composite* of minimal ring extensions $R \subset S$ and $R \subset T$ is more prosaic, requiring that the given S and T (not just copies of them) lie in some *universe* R-algebra U; and our *composite* is then just the subring of U generated by $S \cup T$. For the case of minimal field extensions $K \subset F$ and $K \subset L$, we wish to use classical field theory and so we will take U to be an algebraic closure of K.

The simplest case where our basic question has an affirmative answer concerns quadratic field extensions. Indeed, if F and L are distinct field extensions of a field K inside some algebraic closure of K such that $[F : K] = 2 = [L : K]$, then FL is minimal over F (and, similarly, minimal over L) since $1 < [FL : F] \leq [L : K] = 2$ and the conclusion follows since $[FL : F] = 2$, a prime number. We collect some generalizations of this example in Proposition 2.2. For motivation, note that if $K \subset M$ is any field extension such that $[M : K] = 2$, then M is a splitting field (i.e., normal) over K.

Consider any splitting field M over a field K. Let M_s (resp., M_i) denote the set of elements of M that are separable (resp., purely inseparable) over K. Then M_s and M_i are fields contained between K and M whose composite $M_s M_i = M$ [14, Theorem 46, p. 56]. It follows that if $K \subset M$ is also a minimal field extension, then M coincides with either M_s or M_i; i.e, M is either Galois or purely inseparable over K. We claim that in either of these cases, $[M : K]$ is a prime number. (The referee has kindly pointed out that this assertion has appeared in unpublished work of A. Philippe in 1969: see [18, Proposition 2.2].) Indeed, if M is Galois over K, the assertion follows by combining the Fundamental Theorem of Galois Theory with the existence of Sylow subgroups and the solvability of p-groups. On the other hand, if M is purely inseparable over K, then $[M : K] = p^n$, where $0 < p = \text{char}(K)$ is a prime number and $n \in \mathbb{N}$, with $M = K(u)$ and $u^{p^n} \in K$. (Of course, more generally, any minimal field extension $M \supset K$ is of the form $M = K(u)$ for some $u \in M$.) If $n > 1$, then $K(u^{p^{n-1}})$ is a field contained strictly between K and M, a contradiction. Therefore, $n = 1$, whence $[M : K] = p$, completing the proof of the claim. These observations will be used in the proof of Proposition 2.2(b).

Proposition 2.2. *Let $K \subset F$ and $K \subset L$ be distinct field extensions that are contained in some algebraic closure of K. Then*:

(a) *If* $[F : K]$ *and* $[L : K]$ *are distinct prime numbers*, *then* $K \subset F$, $K \subset L$, $F \subset FL$ *and* $L \subset FL$ *are each minimal field extensions*.

(b) *If* $K \subset F$ *and* $K \subset L$ *are minimal field extensions such that* F *and* L *are each splitting fields over* K, *then* $F \subset FL$ *and* $L \subset FL$ *are each minimal field extensions*.

Proof. (a) By hypothesis, $p := [F : K]$ and $q := [L : K]$ are prime numbers. Hence, $K \subset F$ and $K \subset L$ are minimal field extensions. Since $p \neq q$ by hypothesis, it follows from [14, Theorem 3(c), p. 6] that $[FL : F] = [L : K] = q$ and similarly that $[FL : L] = p$, whence $F \subset FL$ and $L \subset FL$ are minimal field extensions.

(b) By the above comments, F and L are each either Galois or purely inseparable over K and, furthermore, $p := [F : K]$ and $q := [L : K]$ are (possibly equal) prime numbers. By (a), we may assume that $p = q$.

Suppose first that both F and L are Galois over K. We show that $F \subset FL$ is a minimal field extension (as the proof for $L \subset FL$ is similar). Since F and L are distinct and minimal over K, it follows that $F \cap L = K$. Then by standard Galois theory (cf. [16, Theorem 1.12, p. 266]), FL is a Galois field extension of F and we have an isomorphism of Galois groups $\mathrm{Gal}(FL/F) \cong \mathrm{Gal}(L/K)$. Thus, by the Fundamental Theorem of Galois Theory, $[FL : K] = |\mathrm{Gal}(FL/F)| = |\mathrm{Gal}(L/K)| = [L : K] = p$ is prime, and so $F \subset FL$ is a minimal field extension, as asserted.

Suppose next that both F and L are purely inseparable over K. By the above comments, $F = K(u)$ and $L = K(v)$ with $u^p, v^p \in K$. Then $F \subset FL$ is a minimal field extension since $FL = F(v)$ is purely inseparable over F with $v^p \in F$. Similarly, one shows that $L \subset FL$ is a minimal field extension.

Since $FL = LF$, there is essentially only one remaining case: suppose that F is Galois over K and L is purely inseparable over K. By the above comments, $L = K(v)$ with $v^p \in K$. Then $F \subset FL$ is a minimal field extension since $FL = F(v)$ is purely inseparable over F with $v^p \in F$. In fact, $[FL : F] = p$. Consequently,

$$[FL : L] = \frac{[FL : K]}{[L : K]} = \frac{[FL : F][F : K]}{[L : K]} = \frac{p \cdot p}{p} = p,$$

which is a prime number, and so $L \subset FL$ is a minimal field extension, to complete the proof.

Continuing with a base field K, we show next by example that if S and T are composable minimal ring extensions of K such that not both S and T are fields, then, in contrast to the conclusions in Proposition 2.2, our basic question has a negative answer. Recall from the classification in [9, Lemme 1.2] that S and T are each isomorphic to either $K[X]/(X^2)$, $K \times K$, or a minimal field extension of K. As $ST = TS$, these are essentially five cases, and these are covered by the parts of Example 2.3.

Example 2.3. Let K be a field. Then:

(a) Let $S = L$ be a minimal field extension of K. Let $T := K[X]/(X^2)$, viewed as a (minimal) ring extension of K via $K \hookrightarrow T = K[x] = K \oplus Kx$, $a \mapsto a$, where $x := X + (X^2) \in T$ satisfies $x^2 = 0$. Consider the K-algebra $U := L[Y]/(Y^2) = L \oplus Ly$, where $y := Y + (Y^2) \in U$ satisfies $y^2 = 0$. View $S = L \subset U$ as above. View $T \subset U$ via $a + bx \mapsto a + by$ for all $a, b \in K$. Then $ST = U$ and $S \subset ST$ is a minimal ring extension, while $T \subset ST$ is not a minimal ring extension.

(b) Let $S = L$ be a minimal field extension of K. Let $T := K \times K$, viewed as a (minimal) ring extension of K via the diagonal map $\Delta_K : K \hookrightarrow T$, $a \mapsto (a, a)$. Consider the K-algebra $U := L \times L$. View $S = L \subset U$ via the diagonal map Δ_L. View $T \subset U$ as usual. Then $ST = U$ and $S \subset ST$ is a minimal ring extension, while $T \subset ST$ is not a minimal ring extension.

(c) Let $S := K[X]/(X^2) = K \oplus Kx$ and $T := K[Y]/(Y^2) = K \oplus Ky$, viewed as minimal ring extensions of K as above. Consider the K-algebra

$U := K[X, Y]/(X^2, Y^2) = K[v, w] = K \oplus Kv \oplus Kw \oplus Kvw$ (where $v := X + (X^2, Y^2)$, $w := Y + (X^2, Y^2)$ satisfy $v^2 = 0 = w^2$). View $S \subset U$ via $a + bx \mapsto a + bv$ for $a, b \in K$; and view $T \subset U$ via $a + by \mapsto a + bw$ for $a, b \in K$. Then $ST = U$ and neither $S \subset ST$ nor $T \subset ST$ is a minimal ring extension.

(d) Let $S := K[X]/(X^2 - X) = K[x] = K \oplus Kx$, where $x := X + (X^2 - X)$ satisfies $x^2 = x$ and, similarly, $T := K[Y]/(Y^2 - Y) = K[y] = K \oplus Ky$, with $y := Y + (Y^2)$ satisfying $y^2 = y$. Consider the K-algebra $U := K[X, Y]/(X^2 - X, Y^2 - Y) = K[v, w] = K \oplus Kv \oplus Kw \oplus Kvw$, where $v := X + (X^2 - X, Y^2 - Y)$ and $w := Y + (X^2 - X, Y^2 - Y)$. View $S \subset U$ via $a + bx \mapsto a + bv$ for all $a, b \in K$. View $T \subset U$ via $a + by \mapsto a + bw$ for all $a, b \in K$. Then $S \cong K \times K \cong T$ as K-algebras, S and T are each minimal ring extensions of K, $ST = U$, and neither $S \subset ST$ nor $T \subset ST$ is a minimal ring extension.

(e) Let $S := K[X]/(X^2) = K \oplus Kx$ and $T := K \times K$, viewed as minimal ring extensions of K as above. Consider the K-algebra $V := K[X, Y]/(X^2, Y^2) = K[v, w] = K \oplus Kv \oplus Kw \oplus Kvw$, (where $v := X + (X^2, Y^2)$, $w := Y + (X^2, Y^2)$ satisfy $v^2 = 0 = w^2$). Put $U := K[v] \times K[w]$. View $S \subset U$ via $a + bx \mapsto (a + bv, a + bw)$ for all $a, b \in K$. View $T \subset U$ via $(c, d) \mapsto (c, d)$ for all $c, d \in K$. Then $ST = U$ and neither $S \subset ST$ nor $T \subset ST$ is a minimal ring extension.

Proof. (a) The specified identifications allow us to view $x = y$. It follows that $ST = U$, for the typical element of U, namely, $c + dy$ with $c \in L$ and $d \in L$, is the sum of the element $c \in L = S$ and the product of the element $d \in S$ with the element $y = x \in T$. Hence, $L = S \subset ST = U = L[Y]/(Y^2)$ is one of the kinds of minimal ring extensions that were noted in [9, Lemme 1.2]. However, $T \subset ST$ is not a minimal ring extension, in view of the proper inclusions

$$T = K \oplus Kx \subset K \otimes Lx \subset L \oplus Lx = L \oplus Ly = U = ST.$$

(b) To see that $ST = U$, note that if $c, d \in L$, then

$$(c, d) = (c, c)(1, 0) + (d, d)(0, 1) \in L \times L,$$

with $(c, c) \in L = S$, $(1, 0) \in K \times K = T$, $(d, d) \in S$, and $(0, 1) \in T$. Hence, $L = S \subset ST = U = L \times L$ is one of the kinds of minimal ring extensions that were noted in [9, Lemme 1.2]. However, $T \subset ST$ is not a minimal ring extension, since the rings $K \times L$ and $L \times K$ are each properly contained between T and ST.

(c) The specified inclusion maps allow us to view $x = v$ and $y = w$. It is then clear that $ST = U$. Note that neither $S \subset ST$ nor $T \subset ST$ is a minimal ring extension because of the proper inclusions

$$S \subset K[x, xy] = K[v, vw] \subset ST \text{ and } T \subset K[y, xy] = K[w, vw] \subset ST.$$

(d) Observe that $S = K[X]/(X(X-1)) \cong K[X]/(X) \times K[X]/(X-1) \cong K \times K$ by the Chinese Remainder Theorem. Similarly, $T \cong K \times K$. Thus (or directly by considering vector space dimensions), we can view $K \subset S$ and $K \subset T$ as minimal ring extensions. The above proof of (c) can now be repeated *verbatim*.

(e) The specified identifications allow us to view $x = (v, w)$. Also, it is clear that $ST \subseteq U$. To prove the reverse inclusion, it is enough to show that both $(v, 0)$ and $(0, w)$ lie in ST. This, in turn, follows since $(v, w) = x \in S \subset ST$ and ST is closed under multiplication by $(1, 0), (0, 1) \in T$. Next, note that $T \subset ST$ is not a minimal ring extension since the rings $K \times K[w]$ and $K[v] \times K$ are each contained properly between $K \times K = T$ and $K[v] \times K[w] = ST$.

It remains only to show that $S \subset ST$ is not a minimal ring extension. Observe that $(0, w)$ cannot be expressed as $(a + bv, a + bw)$ with $a, b \in K$. Consequently, $(0, w) \notin S$ and $S \subset A := S[(0, w)] = K[(v, w)][(0, w)]$. Hence, it suffices to find an element in $K[v] \times K[w]$ which is not in A. Note that

$(a + bv,\ a + bw)(0,\ w) = (0,\ aw)$ for all $a,\ b \in K$. It follows easily that

$$A = K[(v,\ w)][(0,\ w)] = \{(a + bv,\ a + bw) \mid a,\ b \in K\} + \{(0,\ cw) \mid c \in K\}.$$

The element $(1,\ w)$ is not in the displayed set. The proof is complete.

Remark 2.4. (a) The behavior noted in Example 2.3 depended on the fact that the base ring was a field. Indeed, the negative answers that we found there can disappear when the base ring has more than one maximal ideal. Consider, for instance, the following analogue of Example 2.3(c). (It is an analogue because $K[X]/(X^2) \cong K(+)K$ for any field K.) Let M and N be distinct maximal ideals of a ring R. As noted in the proof of [7, Corollary 2.5], $S := R(+)R/M$ and $T := R(+)R/N$ are each minimal ring extensions of R (via the canonical inclusions). Then the ring $B := R(+)(R/M \oplus R/N)$ contains both S and T as subrings (viewing R/M and R/N as subgroups of $R/M \oplus R/N$ via the canonical injections), the composite $ST = B$, and, in contrast to Example 2.3(c), one can show that ST is a minimal ring extension of both S and T. (The verification of the final assertion is immediate from [3, Remark 2.9] and the fact that R/N and R/M are simple R-modules.) Thus, the present context supports a positive answer to our basic question. It turns out that the behavior noted here is possible because the crucial maximal ideals of the minimal ring extensions $R \subset S$ and $R \subset T$ (namely, M and N, respectively) are distinct. As we will prove in Theorem 2.5, this is a general phenomenon.

(b) The point made in (a) applies as well to analogues of Example 2.3(d) in which the base ring R is not quasilocal and the given minimal ring extensions have distinct crucial maximal ideals. To see this, consider distinct fields K and L which are sub fields of a field F which is a minimal field extension of both K and L. (For instance, take K and L each to be algebraic of different prime degrees over some field k, working in an algebraic closure of k, take $F := KL$, and apply Proposition 2.2(a).) Put $R := K \times L$, and consider the maximal ideals of R, namely, $M := K \times \{0\}$ and $N := \{0\} \times L$. As was noted in the proof of [7, Corollary 2.5], $R \times R/M$

and $R \times R/N$ are each minimal ring extensions of R (via the canonical inclusions). Observe that $S := (K \times L) \times L \cong R \times R/M$ and $T := (K \times L) \times K \cong R \times R/N$. Moreover, the composite ST exists inside, and in fact coincides with, $(K \times L) \times F$. It is easy to see, in contrast to Example 2.3(d), that ST is a minimal ring extension of both S and T (the point being, for instance, that any ring A such that $S \subseteq A \subseteq ST$ must be of the form $(K \times L) \times B$, where B is a ring such that $L \subseteq B \subseteq F$).

(c) The chains $S \subset K[x, xy] \subset ST$ and $T \subset K[y, xy] \subset ST$ that were noted in Example 2.3(c) are saturated, in the sense that each of $S \subset K[x, xy]$, $K[x, xy] \subset ST$, $T \subset K[y, xy]$ and $K[y, xy] \subset ST$ is a minimal ring extension. One can say in this example that $S \subset ST$ and $T \subset ST$ each *factored* into two minimal ring extensions. This leads one to wonder if, when one relaxes the condition that the base ring is a field, one can find examples of minimal ring extensions $R \subset S$ and $R \subset T$ such that $S \subset ST$ or $T \subset ST$ is not a minimal ring extension but, similarly, *factors* as a product of *infinitely many* minimal ring extensions. This question has a negative answer when the base ring is an arbitrary domain. This can be seen by combining the later results in this section and Proposition 3.3 with the classification result in [6, Theorem 2.7].

We now present an important class of pairs of minimal ring extensions for which our basic question has an affirmative answer, and thereby present the promised generalization of parts (a) and (b) of Remark 2.4.

Theorem 2.5. *Let $R \subset S$ and $R \subset T$ be minimal ring extensions, with crucial maximal ideals M and N, respectively. If $M \neq N$ and if the composite ST exists in some ring extension U of R, then both $S \subset ST$ and $T \subset ST$ are minimal ring extensions.*

Proof. All the calculations and identifications given below are to be interpreted canonically inside the appropriate rings of fractions of U. Note that $S \neq T$ since $M \neq N$. Now, since M is the crucial maximal ideal of the minimal ring extension $R \subset S$, we have that $R_M \subset S_M := S_{R \setminus M}$ is a minimal ring extension and $R_P = S_P$ for all $P \in \operatorname{Spec}(R)\setminus\{M\}$. Similarly, $R_N \subset T_N$ is a minimal ring extension and $R_P = T_P$ for all $P \in \operatorname{Spec}(R)\setminus\{N\}$. We show next that $S \subset ST$ is a minimal ring extension, leaving the similar verification for $T \subset ST$ to the reader.

Note that $(ST)_M = S_M T_M = S_M R_M = S_M$ and, similarly, $(ST)_N = T_N$. In particular, if $P \in \operatorname{Spec}(R)\setminus\{M, N\}$, then

$$(ST)_P = S_P T_P = R_P R_P = R_P = S_P = T_P.$$

Now, consider any ring A such that $S \subseteq A \subseteq ST$. It suffices to show that A is either S or ST, equivalently by globalization, that A agrees locally with either S or ST. Of course, $S_M \subseteq A_M \subseteq (ST)_M = S_M$, and so $S_M = A_M = (ST)_M$. Similarly, if $P \in \operatorname{Spec}(R)\setminus\{M, N\}$, then $S_P = A_P = (ST)_P (= R_P)$.

On the other hand, since $R_N = S_N \subseteq A_N \subseteq (ST)_N = T_N$ and $R_N \subset T_N$ is a minimal ring extension, we have that A_N is either S_N or T_N. If $A_N = S_N$ (resp., $A_N = T_N$), it follows that A agrees locally with S (resp., with ST), which completes the proof.

Corollary 2.6. *Let R be a ring such that $tq(R)$ is a von Neumann regular ring* (*for instance, take R to be a domain*). *Let $R \subset S$ and $R \subset T$ be distinct minimal ring extensions such that R is integrally closed in both S and T. If the composite ST exists in some ring extension U of R, then both $S \subset ST$ and $T \subset ST$ are minimal ring extensions.*

Proof. Let M be the crucial maximal ideal of the minimal ring extension $R \subset S$ and let K_1 be a total quotient ring of R such that $R \subset S \subseteq K_1$. By [7, Theorem 3.7], $S = \Omega_{K_1}(M)$, the generalized Kaplansky transform of M inside K_1, in the sense of [20]. In other words,

$$S = \{q \in K_1 \mid \operatorname{Rad}(R :_{K_1} q) = M\} \cup R.$$

Similarly, if N denotes the crucial maximal ideal of the minimal ring extension $R \subset T$ and K_2 is a total quotient ring of R such that $R \subset T \subseteq K_2$, then

$$T = \{q \in K_2 \mid \mathrm{Rad}(R :_{K_2} q) = N\} \cup R.$$

By Theorem 2.5, we are done if $M \neq N$. Thus, without loss of generality, $M = N$. Now, let f denote the unique R-algebra isomorphism $K_1 \to K_2$. It is easy to see from the above description of the generalized Kaplansky transform that $f(\Omega_{K_1}(M)) = \Omega_{K_2}(M)$, and so $f(S) = T$. In particular, $S \cong T$ as R-algebras. If $K_1 = K_2$, we have $S = T$, the desired contradiction, as it was shown in the first paragraph of [6, Remark 2.8(a)] that distinct overrings of a given ring A inside the *same* total quotient ring of A cannot be isomorphic as A-algebras. To handle the general case, we proceed to adapt the reasoning from [6].

Let $\mathfrak{S}$ denote the multiplicatively closed set consisting of all the non-zero-divisors of R. All the calculations and identifications given below are to be interpreted canonically inside $U_{\mathfrak{S}}$. In particular, we view $(K_1)_{\mathfrak{S}} = K_1 = S_{\mathfrak{S}} \subseteq U_{\mathfrak{S}}$ and, similarly, $(K_2)_{\mathfrak{S}} = K_2 = T_{\mathfrak{S}} \subseteq U_{\mathfrak{S}}$.

To obtain the desired contradiction, it is enough to show that $S = T$. We will show that $S \subseteq T$, leaving the similar proof of the reverse inclusion for the reader. Consider an arbitrary element $s \in S$. Since $f(s) \in T$, it is enough to prove that $f(s) = s$. As K_1 is a total quotient ring of R, there exist $r \in R$ and $z \in \mathfrak{S}$ such that $s = rz^{-1} \in K_1$ (viewed inside $U_{\mathfrak{S}}$). Hence, $sz = r$. Recall that f restricts to an R-algebra isomorphism $S \to T$. This induces an R-algebra isomorphism $h : S_{\mathfrak{S}} \to T_{\mathfrak{S}}$, which can be viewed as the unique R-algebra isomorphism $f : K_1 \to K_2$. Given the above identifications, we thus have that $f(s)z = f(s)f(z) = f(sz) = f(r) = r$, since $z \in R$, and so $f(s) = rz^{-1} = s$. The proof is complete.

Corollary 2.7. *Let R be a ring such that $K := tq(R)$ is a von Neumann regular ring* (*for instance*, *take R to be a domain*). *Let S and T be distinct* *minimal overrings of R, with $S \subseteq K$ and $T \subseteq K$, such that R is integrally closed in both S and T. Then both $S \subset ST$ and $T \subset ST$ are minimal ring extensions.*

Proof. Apply Corollary 2.6, with $U := K$.

Recall from [7, Corollary 2.5] that if $tq(R)$ is a von Neumann regular ring such that $\mathrm{Max}(R) \cap \mathrm{Min}(R) = \varnothing$, then up to R-algebra isomorphism, the minimal rings extensions of R take one of the forms $R(+)R/M$ (with $M \in \mathrm{Max}(R)$), $R \times R/M$ (with $M \in \mathrm{Max}(R)$), and a (minimal) overring of R. Given the above material, we therefore focus our study of the basic question for the rest of this paper on the case in which S and T are minimal overrings (inside the same total quotient ring) of R. In view of Corollary 2.7 and our next result, Section 3 need only focus on S and T being integral minimal overrings of R.

Theorem 2.8. *Let R be a ring such that a total quotient ring of R is a von Neumann regular ring (for instance, take R to be a domain). Then there do not exist minimal overrings S and T of R (possibly inside different total quotient rings of R) such that the composite ST exists in some ring extension U of R with R being integrally closed in S, T being integral over R, and the minimal ring extensions $R \subset S$ and $R \subset T$ having the same crucial maximal ideal.*

Proof. Assume, on the contrary, that S and T exist with the stated properties. As in the proof of Corollary 2.6, we may show, by working with canonical identifications inside $U_{\mathfrak{S}}$, that we can assume that S and T are contained in the same total quotient ring of R, say K. Since [9, Théorème 2.2(iii)] ensures that S is a flat (epimorphic) extension of R, we obtain the exact sequence

$$0 \to S \otimes_R R \to S \otimes_R T \to S \otimes_R T/R \to 0,$$

or equivalently, $0 \to S \to S \otimes_R T \to S \otimes_R T/R \to 0$. We claim that $S \otimes_R T/R = 0$. It is enough to prove this locally, i.e., that $(S \otimes_R T/R)_P = 0$ for all $P \in \mathrm{Spec}(R)$.

Let M denote the common crucial maximal ideal of the minimal ring extensions $R \subset S$ and $R \subset T$. Now, if $P \neq M$, we have

$$(S \otimes_R T/R)_P \cong S \otimes_R (T_P/R_P) = S \otimes_R (R_P/R_P) = S \otimes_R 0 = 0,$$

since $R_P = T_P$ canonically (as P is not the crucial maximal ideal relative to T). Thus, the above claim reduces to showing that $(S \otimes_R T/R)_M = 0$, or equivalently, that $S_M \otimes_{R_M} T_M/R_M = 0$.

Since K is von Neumann regular, S_M and T_M are minimal overrings of R_M inside $K_M = tq(R_M)$ (cf. [17, Lemme 2.5], [7, Lemma 3.5]). As R_M is integrally closed in S_M, it follows from [7, Theorem 3.1] that $S_M \cong (R_M)_{PR_M} \cong R_P$ for some divided prime ideal PR_M of R_M (in the sense of [1]) such that R_M/PR_M is a (valuation) domain of Krull dimension 1.

In particular, $P \subset M$. As $R_P = T_P$ canonically since P is not the crucial maximal ideal relative to T, we therefore have that $S_M \otimes_{R_M} T_M/R_M$ is isomorphic to

$$(R_M)_{PR_M} \otimes_{R_M} T_M/R_M \cong (T_M)_{PR_M}/(R_M)_{PR_M} \cong T_P/R_P = R_P/R_P = 0.$$

This completes the proof of the above claim.

By the exactness of the displayed sequence, we can now conclude that the canonical ring homomorphism $S \to S \otimes_R T$, $s \mapsto s \otimes 1$, is an isomorphism. However, since S is R-flat, we have that the multiplication map $S \otimes_R T \to ST\left(\sum s_i \otimes t_i \mapsto \sum s_i t_i \text{ for } s_i \in S, t_i \in T\right)$ is an R-module isomorphism. Hence, $S = ST \supseteq T \supset R$. The minimality of $R \subset S$ entails $S = T$, a contradiction since R cannot be integrally closed in a proper integral extension. The proof is complete.

We next record an important special case of Theorem 2.8.

Corollary 2.9. *Let R be a ring such that $K := tq(R)$ is a von Neumann regular ring* (*for instance*, *take R to be a domain*). *Then there do* *not exist minimal overrings S and T of R, with $S \subseteq K$ and $T \subseteq K$, such that R is integrally closed in S, T is integral over R, and the minimal ring extensions $R \subset S$ and $R \subset T$ have the same crucial maximal ideal.*

We close the section with an example showing that one cannot delete the hypothesis in Theorem 2.8 and Corollary 2.9 that $R \subset S$ and $R \subset T$ have the same crucial maximal ideal.

Remark 2.10. There exist a ring R and minimal overrings S, T of R inside $K := \mathrm{tq}(R)$ such that K is von Neumann regular, R is integrally closed in S, and T is integral over R. (Also, necessarily by Corollary 2.9, the minimal ring extensions $R \subset S$ and $R \subset T$ do not have the same crucial maximal ideal.) For instance, take a quasilocal domain (A, M) (resp., B, N)) with a minimal overring C inside $\mathrm{tq}(A)$ (resp., D inside $\mathrm{tq}(B)$) such that A is integrally closed in C (resp., D is integral over B). Put $R := A \times B$. Then $\mathrm{tq}(R) \cong \mathrm{tq}(A) \times \mathrm{tq}(B) =: K$ is von Neumann regular. Moreover, $C \times B$ is a minimal overring of R with crucial maximal ideal $M \times B$ such that R is integrally closed in $C \times B$; and $A \times D$ is an integral minimal overring of R with crucial maximal ideal $A \times N$.

3. The Case of Integral Overrings

As noted prior to the statement of Theorem 2.8, this section is devoted to the case of our basic question in which S and T are integral minimal overrings of a ring R, with S and T both contained in $K := \mathrm{tq}(R)$. As in Section 2, the role of the crucial maximal ideals is central in studying this context. Proposition 3.3(a) establishes that in general (for S, T as above), there is a finite upper bound on the cardinality of any chain of rings between S and ST. Moreover, we show that in some instances (with S, T as above), ST is actually a minimal ring extension of S. However, the main emphasis of this section is a comprehensive presentation of examples in which S and T are integral minimal overrings of a Noetherian domain R, with S and T each inside $K := \mathrm{tq}(R)$ and ST not a minimal ring extension of S.

To organize that study, we begin by stating [7, Proposition 2.12], which was built in part on results from [9].

Lemma 3.1. *Let R be a ring with total quotient ring K. Let $B \subseteq K$ be an integral overring of R. Then B is a minimal ring extension of R if and only if there exists $M \in \mathrm{Max}(R)$ such that one of the following three conditions holds*:

(1) *M is a maximal ideal of B and B/M is a minimal field extension of R/M*;

(2) *there exists $q \in B \setminus R$ such that $B = R[q]$, $q^2 - q \in M$, and $Mq \subseteq R$*;

(3) *there exists $q \in B \setminus R$ such that $B = R[q]$, $q^2 \in R$, $q^3 \in R$, and $Mq \subseteq R$.*

If any of the above three conditions holds, then M is uniquely determined as $(R : B)$, the crucial maximal ideal of the extension $R \subset B$. Furthermore, conditions (1), (2), (3) *are mutually exclusive. Indeed, if B is an integral minimal ring extension of R, then* (2) (*resp.*, (3)) *is equivalent to B/M being isomorphic as an R/M-algebra to $R/M \times R/M$* (*resp.*, $(R/M)[X]/(X^2)$).

It will be convenient to say that an integral minimal overring B of R (inside $K = \mathrm{tq}(R)$) is of *type* 1, *type* 2, or *type* 3 according as to whether B satisfies condition (1), (2), or (3) of Lemma 3.1. Given two distinct integral minimal overrings $S, T \subset K$ of R, we will say that S, T are a *type* (a, b) example if S is of type a and T is of type b, while $R \subset S$ and $R \subset T$ have the same crucial maximal ideal M. Occasionally, we will relax this terminology by not requiring the ring extensions S, T to be overrings of R.

We next collect some examples of type $(1, 1)$. Without seeking maximum generality, Remark 3.2 indicates how the study of this type of example is often equivalent to the corresponding field-theoretic considerations.

Remark 3.2. (a) We assume familiarity with the lore of the ideals and overrings of the classical $D + M$ construction, as summarized in [2, Theorems 2.1 and 3.1]. Let F and L be distinct proper field extensions of a field k which are contained in the same algebraic closure $\overline{k}$ of k. Let $V = \overline{k} + M$ be any valuation domain with maximal ideal $M \neq 0$ such that V contains a copy of its residue field. Put $R := k + M$, $S := F + M$, and $T := L + M$. Then S, T are integral overrings of R that are contained in V (which is actually the integral closure of R); and S, T are each quasilocal, with common maximal ideal M. Moreover, S and T are minimal overrings of R if and only if F and L are minimal field extensions of k (the point being that, cf. [18, Corollary 1.4], the rings E between R and S are in one-to-one correspondence with the fields D between K and F via $D \mapsto E := D + M$). If these equivalent conditions hold, then S, T give an example of type $(1, 1)$. Observe that the composite $ST = FL + M$ (viewed inside V). By reasoning as above, we see that $S \subset ST$ (resp., $T \subset ST$) is a minimal ring extension if and only if $F \subset FL$ (resp., $L \subset FL$) is a minimal field extension.

(b) We next slightly generalize the considerations in (a). Let S, T be distinct integral proper overrings of a domain R inside some quotient field K of R. Let $M \in \mathrm{Max}(R)$ also be an ideal of both S and T. Consider the field $k := R/M$ and the rings $F := S/M$, $L := T/M$, together with the composite ST (viewed inside $V := (M :_K M)$). Then $FL = (S/M)(T/M) = (ST)/M \subseteq V/M$. By a standard homomorphism theorem, S and T are minimal overrings of R if and only if F and L are minimal ring extensions of k. If these equivalent conditions hold with $R \subset S$ and $R \subset T$ having the same crucial maximal ideal M which is a maximal ideal of both S and T, then S, T give an example of type $(1, 1)$. Suppose next that $M \in \mathrm{Spec}(V)$. (For instance, take R to be a pseudo-valuation domain, in the sense of [11]; note that the R in (a) is a pseudo-valuation domain and that the V in (a) is both $(M : M)$ and the canonically associated valuation overring of R.) Then F, L, and the composite FL are fields, and $S \subset ST$ (resp., $T \subset ST$) is a minimal ring extension if and only if $F \subset FL$ (resp., $L \subset FL$) is a minimal field extension.

The considerations involving $(M : M)/M$ in the preceding remark are relevant more generally, as we illustrate in the proof of part (c) of the next result. Proposition 3.3(a) provides the final step to a finiteness result that was promised in Remark 2.4(c) in Section 2. Example 3.9 shows that the inequalities in Proposition 3.3(d) are best possible.

Proposition 3.3. *Let R be a ring. Let $R \subset S$ and $R \subset T$ be distinct integral minimal ring extensions with the same crucial maximal ideal M. Suppose also that S and T are each R-algebra isomorphic to overrings of R (possibly inside different total quotient rings of R) and that S and T are not isomorphic as R-algebras. Suppose that the composite ST exists in some ring extension of R. Let k denote the field R/M. Put $d := \dim_k(S/M)$ and $n := \dim_k(T/M)$. Let $\mathcal{C}$ be any chain of rings contained between S and ST, and let $\mathcal{D}$ be any chain of rings contained between T and ST. Then*:

(a) *$|\mathcal{C}|$ and $|\mathcal{D}|$ are each finite. In fact, $|\mathcal{C}| \leq (n-1)d+1$ and $|\mathcal{D}| \leq (d-1)n+1$.*

(b) *Suppose also that S/M (resp., T/M) is a field. Then $|\mathcal{C}| \leq n$ (resp., $|\mathcal{D}| \leq d$).*

(c) *Suppose also that S/M and T/M are each fields and that S, T are each contained in $K = \mathrm{tq}(R)$. Then $|\mathcal{C}| \leq n$ and $|\mathcal{D}| \leq d$. Moreover, if $M \in \mathrm{Spec}(M :_K M)$, then $|\mathcal{C}| \leq \log_2(n)$ and $|\mathcal{D}| \leq \log_2(d)$.*

(d) *If neither S/M nor T/M is a field, then $|\mathcal{C}| \leq 3$ and $|\mathcal{D}| \leq 3$.*

Proof. By reasoning as in the proof of Corollary 2.6, we may suppose that S, T are each contained in $K = \mathrm{tq}(R)$; and then, since S and T are not isomorphic as R-algebras, that $S \neq T$.

(a) Since $ST = TS$, it suffices to prove the assertion concerning $\mathcal{C}$. Let $a_1, \ldots, a_n \in T$ be such that their cosets modulo M form a k-vector space basis of T/M. In other words, $\{a_i\}$ is a minimal generating set of T as an R-module. In the same way, there are elements $b_1, \ldots, b_d \in S$

forming a mimimal generating set for the R-module S (i.e., whose cosets modulo M form a k-basis of S/M). Now, since $ST = S\left(\sum_{i=1}^{n} Ra_i\right) = \sum_{i=1}^{n} Sa_i$, we have that $(ST)/M$ can be written as

$$\sum_{i=1}^{n}(S/M)(a_i + M) = \sum_{i=1}^{n}\left(\sum_{j=1}^{d} k(b_j + M)\right)(a_i + M) = \sum_{(i,\, j)} k(a_i b_j + M).$$

So, $\dim_k((ST/M)) \le nd$. Note that the given chain $\mathcal{C}$ induces a chain of the same cardinality of k-vector spaces from S/M to $(ST)/M$ (via $A \mapsto A/M$). The cardinality of any such chain of subspaces from (the d-dimensional k-vector space) S/M to (the at most (nd)-dimensional k-vector space) $(ST)/M$ is at most $nd - (d-1) = (n-1)d + 1$.

(b) It is enough to prove that if T/M is a field, then the given chain $\mathcal{D}$ has cardinality at most d. (By (a), we know that $\mathcal{D}$ is finite, of cardinality at most $(d-1)n + 1$, but this general bound exceeds d because $n \ge 1$.) For a proof, recall that $\dim_k((ST)/M) \le nd$. Consider $k \subset T/M \subset (ST)/M$. Since T/M is a field, we have (cf. [13, page 66]) that $\dim_k((ST)/M) = \dim_k(T/M) \cdot \dim_{T/M}((ST)/M) = n \cdot \dim_{T/M}((ST)/M)$. Thus, $n \cdot \dim_{T/M}((ST)/M) \le nd$. Cancelling the factor n, we find that $\dim_{T/M}((ST)/M) \le d$. To finish the proof of (b), consider $\dim_{T/M}$ for the members of the chain $\{E/M \mid E \in \mathcal{D}\}$.

(c) Since $ST = TS$, it suffices to prove the assertions concerning $\mathcal{C}$. The first assertion is immediate from (b). Consider the fields $k := R/M$, $F := S/M$, and $L := T/M$. We have $FL = (ST)/M = (S/M)(T/M) \subseteq (M :_K M)/M$. By a standard homomorphism theorem, there is an order isomorphism between the poset of rings E such that $S \subseteq E \subseteq ST$ and the poset of rings D such that $F \subset D \subset FL$, given by $E \mapsto D := E/M$. The first assertion concerning $\mathcal{C}$ (which we have already established) is therefore equivalent to the statement that if $\mathcal{K}$ is any chain of rings (necessarily fields) contained between F and FL, then $|\mathcal{K}| \leq n$. The final assertion follows from the multiplicativity of field degrees in towers of fields, as in the comment following the proof of [5, Proposition 2.1].

(d) It follows from [9, Lemme 1.2] that S/M and T/M are each k-algebra isomorphic to either the ring of dual numbers over k or the direct product of two copies of k. Thus, $n = d = 2$, and so the conclusion follows from (a).

Some useful upper bounds that are weaker than those given in Proposition 3.3(c) can be found as follows. Assume the notation and hypotheses of Proposition 3.3(c). Let λ (resp., μ) be the minimal degree of a monic polynomial $f \in R[X]$ (resp., $g \in R[X]$) such that $f(t) = 0$ for some element $t \in T$ satisfying $R[t] = T$ (resp., such that $g(s) = 0$ for some element $s \in S$ satisfying $R[s] = S$). Then $\lambda \geq n$ and $\mu \geq d$, so that Proposition 3.3(c) (or its proof) gives, for instance, that $|\mathcal{C}| \leq \lambda$ and $|\mathcal{D}| \leq \mu$.

Despite Proposition 3.3, we will show in Corollary 3.7 that for all nine types (a, b) of examples S, T of integral minimal overrings of a ring R (with the same crucial maximal ideal), our basic question (whether both $S \subset ST$ and $T \subset ST$ must be minimal ring extensions) has, in general, a negative answer, even if R is a Noetherian ring. First, we present in Proposition 3.4(a) the two situations where ST, if it exists, must be a minimal ring extension of S. Proposition 3.4(c) gives the first of the Noetherian examples.

Proposition 3.4. *Let R be any ring. Suppose that (S, T) are a type $(1, b)$ example of integral minimal ring extensions of R, each with crucial maximal ideal M. Suppose also that the composite ST exists. Then*:

(a) *Suppose that $b \in \{2, 3\}$. Then ST is a minimal ring extension of S of type b.*

(b) *Suppose that $b = 1$ and, in addition, that S and T are overrings of R that are each contained in $K := \mathrm{tq}(R)$. Then ST is a minimal ring extension of S if and only if $S/M \subset (S/M)(T/M)$ is a minimal extension of fields.*

(c) *For* $b = 1$ *it is possible to choose* R, S *and* T *as above so that* R *is a Noetherian domain*, S *and* T *are integral minimal overrings of* R *each of type* 1 *and with the same crucial maximal ideal*, *and* $S \subset ST$ *is not a minimal ring extension*.

Proof. (a) Proposition 3.3 can be used to show that ST is a minimal ring extension of S. Indeed, with $k := R/M$, we have that $b \in \{2, 3\}$ gives T/M isomorphic, as a k-algebra, to either $k \times k$ or $k[X]/(X^2)$. In particular, $n := \dim_k(T/M) = 2$. Hence, the minimality of $S \subset ST$ follows from Proposition 3.3(b). It is possible to extend the reasoning in Example 2.3 to show that this extension is of type b, but instead, we next show how to use Lemma 3.1 to obtain all the assertions at once.

Since T is of type b, with $b \in \{2, 3\}$, there exists $q \in T \setminus R$ such that $T = R[q]$, $Mq \subseteq R$, and either $q^2 - q \in M$ (if $b = 2$) or $q^2, q^3 \in R$ (if $b = 3$). Note that $M \in \mathrm{Max}(S)$ since S is of type 1. Moreover, $ST = SR[q] = S[q]$, $q \in ST \setminus S$ (since $S \neq T$), and $Mq \subseteq S$; and either $q^2 - q \in M$ (if $b = 2$) or $q^2, q^3 \in S$ (if $b = 3$). All the assertions now follow from Lemma 3.1.

(b) We rework some of the ideas and notation in Remark 3.2(b). Since S, T are a type $(1, 1)$ example, we can consider the fields $k := R/M$, $F := S/M$ and $L := T/M$, together with the composite ST (viewed inside $V := (M :_K M)$). By a standard homomorphism theorem, S and T are minimal overrings of R since (in fact, if and only if) F and L are minimal field extensions of k. Note that $FL = (S/M)(T/M) = ST/M \subseteq V/M$. Another application of the homomorphism theorem yields the assertion.

(c) Let K be a field with algebraic closure $\overline{K}$, and let F, L be distinct minimal field extensions of K inside $\overline{K}$. Let $V = FL + M$ be a DVR with nonzero maximal ideal M (for instance, $V = FL[[X]]$). By [10, Exercise 8(3), pp. 270-271], the domain $R := K + M$ is Noetherian, as are its integral overrings $S := F + M$ and $T := L + M$. Note that S, T (as overring extensions of R) give an example of type $(1, 1)$. It follows from each of Remark 3.2(b) and part (b) of this proposition that $S \subset ST = V$ is a minimal ring extension if and only if $F \subset FL$ is a minimal field extension. Accordingly, it suffices to find K, F, L as above such that $F \subset FL$ is not a minimal field extension. The following example of such data may be known but is included here for the sake of completeness.

Let $p \geq 5$ be a prime number, ω a primitive pth root of unity (in the complex numbers) and α the real pth root of 2. Consider the fields $K := \mathbb{Q}$, $F := \mathbb{Q}(\alpha)$ and $L := \mathbb{Q}(\omega\alpha)$. Of course, $[F : K] = p = [L : K]$, since $X^p - 2$ is irreducible over Q (by, for instance, Eisenstein's Criterion). Also, the standard theory of cyclotomic field extensions tells us that $[\mathbb{Q}(\omega) : \mathbb{Q}] = \varphi(p) = p - 1$ (where φ denotes the Euler phi-function). Since p and $p - 1$ are relatively prime, [14, Theorem 3(c), p. 6] ensures that the degree of $\mathbb{Q}(\omega)F = \mathbb{Q}(\alpha, \omega)$ over F is $p - 1$. Note that $\mathbb{Q}(\alpha, \omega)$ is the splitting field of $X^p - 2$ over $\mathbb{Q}$, and so is a Galois field extension of $\mathbb{Q}$. By standard Galois Theory, it follows that $FL = \mathbb{Q}(\alpha, \omega)$ is Galois over F. Therefore, by the comments preceding Proposition 2.2, $F \subset FL$ is a minimal field extension if and only if $p - 1$ is a prime number. But $p - 1$ is not a prime number because it is even and greater than 2. This completes the proof.

Remark 3.5. Despite Proposition 3.4(c), it is possible for integral minimal overrings S, T giving an example of type $(1, 1)$ (and having the same crucial maximal ideal) to be such that $S \subset ST$ is a minimal ring extension. For instance, let us work inside the valuation domain $V := \mathbb{Q}[\sqrt{2}, \sqrt{3}][[X]]$, with maximal ideal $M = XV$. As in the proof of Proposition 3.4(c), the domain $R := \mathbb{Q} + M$ is Noetherian, as are its integral overrings $S := \mathbb{Q}[\sqrt{2}] + M$ and $T := \mathbb{Q}[\sqrt{3}] + M$. Since $\mathbb{Q}[\sqrt{2}]$ and $\mathbb{Q}[\sqrt{3}]$ are each minimal field extensions of $\mathbb{Q}$, it follows from any of Remark 3.2(a), Remark 3.2(b) or Proposition 3.4(b) that $S \subset ST = V$ and $T \subset ST$ are each minimal ring extensions of type 1.

We next show how to use Example 2.3 to produce negative answers to our basic question for the six types (a, b) that were not addressed in Proposition 3.4, namely, those types where $a \neq 1$.

Example 3.6. Let $(a, b) \in \{2, 3\} \times \{1, 2, 3\}$. Then there exist a Noetherian ring R and distinct integral minimal overrings S, T of R that are inside the same total quotient ring of R, have the same crucial maximal ideal, and give an example of type (a, b) such that $S \subset ST$ is not a minimal ring extension.

Proof. As in Example 2.3, let K be a field. For the various parts of Example 2.3, let the symbols S, T and U have the meanings given to them earlier in Example 2.3. Let W be an indeterminate that is algebraically independent from the variables X, Y that appeared in Example 2.3. By the Hilbert Basis Theorem, the ring U is a Noetherian ring (for all parts of Example 2.3) and, hence, so is the polynomial ring $U[W]$. Consider its subring $R := K + WU[W]$. As we can see from the description of U in the various parts of Example 2.3 that U is a finite-dimensional vector space over K, it follows that $U[W] = U + WU[W]$ is a finitely generated module over $K + WU[W] = R$. Therefore, by Eakin's Theorem [8], the ring R is Noetherian. Moreover, since S and T are integral ring extensions of K, we have that $S_1 := S + WU[W]$ and $T_1 := T + WU[W]$ are each integral ring extensions of R. In fact, S_1 and T_1 are actually overrings of R, the point being that if $\lambda \in U$, then $\lambda = \dfrac{\lambda W}{W}$ in the total quotient ring of R since W is a non-zero-divisor of R. Also, since S and T are distinct integral minimal ring extensions of K (necessarily having the same crucial maximal ideal, $\{0\}$), it follows easily by analyzing the cases in Lemma 3.1 that S_1 and T_1 are distinct integral minimal overring extensions of R with the same crucial maximal ideal, $WU[W]$. As $ST = U$, we have that the composite $S_1T_1 = U + WU[W] = U[W]$. We proceed to determine the possible minimality of the extensions $S_1 \subset S_1T_1$ and $T_1 \subset S_1T_1$ for the various parts of Example 2.3.

In part (a) of Example 2.3, we saw that $T \subset K \oplus Lx \subset ST$, and so $T_1 \subset (K \oplus Lx) + WU[W] \subset ST + WU[W] = S_1T_1$. In particular, T_1, S_1 give a type (3, 1) example of integral minimal overrings of the Noetherian ring R (which have the same crucial maximal ideal $WU[W]$ and) for which $T_1 \subset T_1S_1$ is not a minimal ring extension.

In part (b) of Example 2.3, we saw that $T \subset K \times L \subset ST$, and so $T_1 \subset (K \times L) + WU[W] \subset ST + WU[W] = S_1T_1$. In particular, T_1, S_1 give a type (2, 1) example of integral minimal overrings of the Noetherian ring R (which have the same crucial maximal ideal $WU[W]$ and) for which $T_1 \subset T_1S_1$ is not a minimal ring extension.

By reasoning as above, we see similarly that Example 2.3(c) leads to S_1, T_1 giving a type $(3, 3)$ example for which neither $S_1 \subset S_1T_1$ nor $T_1 \subset T_1S_1$ is a minimal ring extension; Example 2.3(d) leads to S_1, T_1 giving a type $(2, 2)$ example for which neither $S_1 \subset S_1T_1$ nor $T_1 \subset T_1S_1$ is a minimal ring extension; and Example 2.3(e) leads to S_1, T_1 giving a type (3, 2) example (and T_1, S_1 giving a type $(2, 3)$ example) for which neither $S_1 \subset S_1T_1$ nor $T_1 \subset T_1S_1$ is a minimal ring extension.

In summary, part (a) of Example 2.3 leads to an example of type $(3, 1)$ with the asserted properties; part (b), type $(2, 1)$, part (c), type $(3, 3)$; part (d), type $(2, 2)$; and part (e), types $(3, 2)$ and $(2, 3)$.

The referee has kindly noted that our reasoning in the second, third and fourth paragraphs of the proof of Example 3.6 leads to the following enhancement of [18, Corollary 1.4]: if $A \subseteq B$ are rings with a common ideal $I \neq 0$, then $A \subseteq B$ is a minimal ring extension if and only if $A/I \subseteq B/I$ is a minimal ring extension; and if these conditions hold with B (resp., B/I) integral over A (resp., A/I), then the integral minimal ring extensions $A \subseteq B$ and $A/I \subseteq B/I$ are of the same type.

Corollary 3.7. *Let* $(a, b) \in \{1, 2, 3\} \times \{1, 2, 3\}$. *Then there exist a Noetherian ring R and distinct integral minimal overrings* S, T *of R that* *are inside the same total quotient ring of R, have the same crucial maximal ideal, and give an example of type (a, b) such that either $S \subset ST$ is not a minimal ring extension or $T \subset ST$ is not a minimal ring extension (or both).*

Proof. Example 3.6 takes care of eight of the nine cases, namely, the cases where $(a, b) \neq (1, 1)$. Finally, for the case $(a, b) = (1, 1)$, apply Proposition 3.4(c).

Much of the literature on minimal overrings concerns the case of base rings that are domains. For that reason, we next present an alternate set of examples providing a negative answer to our basic question in which the base ring is not only Noetherian but also a domain.

Example 3.8. Let $a \in \{1, 2, 3\}$. Then there exist a Noetherian domain R and distinct integral minimal overrings S, T of R that are inside the same quotient field of R, have the same crucial maximal ideal, and give an example of type $(a, 1)$ such that $S \subset ST$ is not a minimal ring extension.

Proof. The case $a = 1$ was handled in Proposition 3.4(c). We proceed to provide examples of types $(2, 1)$ and $(3, 1)$ with the asserted properties. Let X, Y be algebraically independent indeterminates over $\mathbb{Q}$. All rings considered henceforth will be subrings of the polynomial ring $\mathbb{Q}[X, Y]$. It will be easy to see that $\mathbb{Q}[X, Y]$ is module-finite, and hence integral, over each of these subrings. Hence, as in the proof of Example 3.6, each of these subrings is Noetherian. Moreover, it will also be clear that all the rings under consideration have quotient field $\mathbb{Q}(X, Y)$.

Type (2, 1). Let R be the subring of $\mathbb{Q}[X, Y]$ defined by

$$R := \mathbb{Q}[\{X^n(X^2 - X)Y^m, X^s(Y^2 - 2)Y^t \mid m, n, s, t \geq 0\}].$$

It is easy to see that the quotient field of R is $\mathbb{Q}(X, Y)$; and that $\mathbb{Q}[X, Y]$ is a module-finite R-algebra. Note that $X \notin R$. (Indeed, if X were a polynomial over $\mathbb{Q}$ in $\{X^n(X^2 - X)Y^m, X^s(Y^2 - 2)Y^t\}$, then the substitution $Y \mapsto \sqrt{2}$ would show that X is a polynomial over $\mathbb{R}$ in

$\{X^n(X^2 - X)\}$. However, X cannot be the sum of a real number and a real polynomial of degree at least 2.) Similarly, one sees by considering the substitution $X \mapsto 0$ that $Y \notin R$. Thus, the rings

$$S := R[X] = \mathbb{Q}[\{X, (X^2 - X)Y^m, (Y^2 - 2)Y^t \mid m, t \geq 0\}]$$

and $T := R[Y] = \mathbb{Q}[\{X^n(X^2 - X), Y, X^s(Y^2 - 2) \mid n, s \geq 0\}]$ are proper integral overrings of R. Moreover, we claim that $S \neq T$. The easiest way to see this is to observe that $XY \notin S$, and this can be shown by reasoning as above. (In detail, if $XY \in S$, apply the substitution $X \mapsto 1$ and infer the contradiction that Y is the sum of a rational number and a rational polynomial of degree at least 2.) On the other hand, by using the substitution $X \mapsto 0$, we see similarly that $Y \notin S[XY]$. One consequence is that $S \subset S[XY] \subset \mathbb{Q}[X, Y] = ST$. It remains only to show that S, T are minimal ring extensions of R having the same crucial maximal ideal such that S is of type 2 and T is of type 1.

Let M be the ideal of R defined by

$$M := (\{X^n(X^2 - X)Y^m, X^s(Y^2 - 2)Y^t \mid m, n, s, t \geq 0\}).$$

It is clear that $R/M \cong \mathbb{Q}$, and so $M \in \operatorname{Max}(R)$. Observe that the product of X with each member of the given generating set of M is again in M, and so $XM \subseteq M \subset R$. As $X^2 - X \in M$, Lemma 3.1 yields that S is an integral minimal overring of R of type 2 with crucial maximal ideal M.

Finally, observe (by checking the product of Y with each member of the given generating set of M) that $YM \subseteq M \subset R \subset T$, and so M is a proper ideal of T. Furthermore, there is a canonical surjective ring-homomorphism $\mathbb{Q}[Y]/(Y^2 - 2) \to T/M$, and this must be an isomorphism since $\mathbb{Q}[Y]/(Y^2 - 2) \cong \mathbb{Q}(\sqrt{2})$ is a field. It is of degree 2 and, hence, minimal over $\mathbb{Q} \cong R/M$. Therefore, Lemma 3.1 yields that T is an integral minimal overring of R of type 1 with crucial maximal ideal M. Thus, S, T give an example with the asserted properties.

Type (3, 1). Let R be the subring of $\mathbb{Q}[X, Y]$ defined by

$$R := \mathbb{Q}[\{X^2Y^m,\ X^3Y^t,\ (Y^2-2)Y^n,\ X(Y^2-2)Y^n \mid m, n, t \geq 0\}].$$

It is easy to see that the quotient field of R is $\mathbb{Q}(X, Y)$; and that $\mathbb{Q}[X, Y]$ is a module-finite R-algebra. Note that $X \notin R$. (Indeed, if X were a polynomial over $\mathbb{Q}$ in $\{X^2Y^m, X^3Y^t, (Y^2-2)Y^n, X(Y^2-2)Y^n \mid m, n, t \geq 0\}$, then the substitution $Y \mapsto \sqrt{2}$ would show that $X \in \mathbb{R}[X^2, X^3]$, which is absurd.) As above, the substitution $X \mapsto 0$ can be used to show that $Y \notin R$. Thus, the rings

$$S := R[X] = \mathbb{Q}[\{X,\ X^2Y^m,\ (Y^2-2)Y^n \mid m, n, t \geq 0\}]$$

and $T := R[Y] = \mathbb{Q}[X^2, X^3, Y, X(Y^2-2)]$ are proper integral overrings of R. We leave it to the reader to check that $XY \notin S$. Moreover, the substitution $X \mapsto 0$ can be used to show that $Y \notin S[XY]$. Consequently, $S \subset S[XY] \subset \mathbb{Q}[X, Y] = ST$. It remains only to show that S, T are minimal ring extensions of R having the same crucial maximal ideal such that S is of type 3 and T is of type 1.

Let M be the ideal of R defined by

$$M := (\{X^2Y^m,\ X^3Y^t,\ (Y^2-2)Y^n,\ X(Y^2-2)Y^n \mid m, n, t \geq 0\}).$$

It is clear that $R/M \cong \mathbb{Q}$, and so $M \in \operatorname{Max}(R)$. Observe that $MX \subseteq M \subset R$. Since $X^2, X^3 \in R$ by definition, it follows from Lemma 3.1 that S is an integral minimal overring of R of type 3 with crucial maximal ideal M.

Observe that $MY \subseteq M \subset R$. In particular, M is a proper ideal of T. One can show that $T/M \cong \mathbb{Q}[Y]/(Y^2-2) \cong \mathbb{Q}(\sqrt{2})$, which is a minimal field extension of $\mathbb{Q} \cong R/M$. Therefore, Lemma 3.1 yields that T is an integral minimal overring of R of type 1 with crucial maximal ideal M. The proof is complete.

It follows from Proposition 3.3(d) that if S and T are minimal integral overrings of a domain R with the same crucial maximal ideal such that neither S nor T is of type 1, then any chain of rings between S and ST has cardinality at most 3. We next show that this is best possible, by presenting examples in all possible cases (where neither S nor T is of type 1) where the maximal cardinality 3 can be achieved, i.e., where there is a ring properly between S and ST, with R a suitable Noetherian domain.

Example 3.9. Let $(a, b) \in \{2, 3\} \times \{2, 3\}$. Then there exist a Noetherian domain R and distinct integral minimal overrings S, T of R that are inside the same quotient field of R, have the same crucial maximal ideal, and give an example of type (a, b) such that $S \subset ST$ is not a minimal ring extension.

Proof. Let F be an arbitrary field, and let X, Y be algebraically independent indeterminates over F. All rings considered henceforth will be subrings of the polynomial ring $F[X, Y]$. It will be easy to see that $F[X, Y]$ is module-finite, and hence integral, over each of these subrings. Hence, as in the proofs of Examples 3.6 and 3.8, each of these subrings is Noetherian. Moreover, it will also be clear that all the rings under consideration have quotient field $F(X, Y)$. The verifications have much the same tempo as those in Example 3.8 and so we leave some of the details to the reader.

Type (2, 2). Let R be the subring of $F[X, Y]$ defined by

$$R := F = [\{X^n(X^2 - X)Y^m,\ X^s(Y^2 - Y)Y^t \mid n, m, s, t \geq 0\}].$$

By using the substitution $Y \mapsto 0$, one can show that $X \notin R$. Similarly, $Y \notin R$. Let M be the ideal of R defined by

$$M := (\{X^n(X^2 - X)Y^m,\ X^s(Y^2 - Y)Y^t \mid n, m, s, t \geq 0\}).$$

Evidently, $R/M \cong F$, and so $M \in \mathrm{Max}(R)$. Since $MX, MY \subseteq M$, it follows via Lemma 3.1 that the rings

$$S := R[X] = F[\{X, (X^2 - X)Y^m, (Y^2 - Y)Y^m \mid m \geq 0\}]$$

and $T := R[Y]$ are integral minimal overrings of R, each of type 2 and having crucial maximal ideal M.

Observe that $XY \notin S$ (for, otherwise, the substitution $X \mapsto 1$ would lead to a contradiction). Consequently, $S \neq T$ and $S \subset S[XY]$. It remains only to check that $S[XY] \subset ST = F[X, Y]$. To this end, it suffices to show that $Y \notin S[XY]$. This, in turn, follows by considering the substitution $X \mapsto 0$. Thus, S, T give an example with the asserted properties.

Type (3, 3). Let R be the subring of $F[X, Y]$ defined by

$$R := F[X^2, X^3, Y^2, Y^3, X^2Y, X^3Y, Y^2X, Y^3X].$$

The definition of the data S, T, M and their analysis can proceed almost exactly as for the preceding example of type $(2, 2)$ with the following exception. The verification that $XY \notin S$ is somewhat tedious but straight forward. Details are left to the reader.

Types (2, 3) and (3, 2). Let R be the subring of $F[X, Y]$ defined by

$$R := F[\{X^n(X^2 - X), X^n(X^2 - X)Y, X^sY^2, X^tY^3 \mid n, s, t \geq 0\}].$$

Note that $X \notin R$ (for, otherwise, the substitution $Y \mapsto 0$ would lead to X being the sum of a constant in F and a polynomial of degree at least 2, a contradiction). Also, $Y \notin R$ (for, otherwise, the substitution $X \mapsto 1$ would lead to $Y \in F[Y^2, Y^3]$, a contradiction).

Consider the subrings of $F[X, Y]$ defined by

$$S := R[X] = F[X, Y^2, Y^3, (X^2 - X)Y]$$

and

$$T := R[Y] = F[\{Y, X^nY^2, X^m(X^2 - X) \mid n, m \geq 0\}].$$

Let M be the ideal of R defined by

$$M := (\{X^n(X^2 - X), X^n(X^2 - X)Y, Y^2X^s, Y^3X^t \mid n, s, t \geq 0\}).$$

As $R/M \cong F$, we have that $M \in \mathrm{Max(R)}$. Since $MX, MY \subseteq M \subset R$, it follows via Lemma 3.1 that S and T are integral minimal overrings of R which each have crucial maximal ideal M and are of types 2 and 3, respectively.

Since S and T are of different types, it must be the case that $S \neq T$. Another way to see this is to show that $XY \notin S$. The latter fact can be proved by considering the substitution $X \mapsto 1$. An important consequence is that $S \subset S[XY]$. To show that S, T are a type (2, 3) example with the asserted properties, it suffices (since $ST = F[X, Y]$) to prove that $Y \notin S[XY]$. This, in turn, follows by considering the substitution $X \mapsto 0$.

It remains only to show that T, S are a type (3, 2) example such that $T \subset T[XY] \subset TS = F[X, Y]$. Now, the substitution $Y \mapsto 0$ easily leads to the fact that $X \notin T[XY]$, and so $T[XY] \subset TS$. We turn, finally, to the non-routine part of this verification, namely, the proof that $XY \notin T$ (and, hence, that $T \subset T[XY]$).

Suppose, on the contrary, that $XY \in T$. Then XY is a polynomial f (with coefficients in F) in the terms $\{Y, X^nY^2, X^{n+1} - X^n \mid n \geq 1\}$. The only way for the monomial XY to appear as a term in f would be for f to contain the expression $-(X^2 - X)Y$ (when written as a polynomial in the terms $\{Y, X^nY^2, X^{n+1} - X^n \mid n \geq 1\}$). However, in that case, the monomial $-X^2Y$ would have to be cancelled without cancelling the term XY. The only way for this to occur would be for $-(X^3 - X^2)Y$ to appear in f. However, in that case, we would need to explain the cancellation of $-X^3Y$. In fact, we would need to repeat the argument indefinitely, in order to explain the cancellation of $-X^nY$ for all $n \geq 2$. As a polynomial can have only finitely many terms, it cannot be the case that $XY \in T$. The proof is complete.

Remark 3.10. The upshot of combining Proposition 3.4(a), Example 3.8 and Example 3.9 is that we can sharpen the "Noetherian ring R" assertion in Corollary 3.7 to "Noetherian domain R".

Acknowledgments

The authors wish to thank the referee for an extensive report. Dobbs was supported in part by a University of Tennessee Professional Leave Award.

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