Far East J. Math. Sci. (FJMS) **20**(1) (2006), 41-48

LIPSCHITZ ESTIMATES FOR FRACTIONAL MULTILINEAR SINGULAR INTEGRAL WITH VARIABLE KERNELS

LAN JIACHENG

(Received August 23, 2005)

Submitted by K. K. Azad

Abstract

In this paper, the author obtained Lipschitz boundedness for a class of fractional multilinear operators on Lebesgue spaces, which is similar to the higher-order commutator for the singular integral. A simple way is obtained that is closely linked with a class of rough fractional integral operators.

1. Introduction and Results

For $0 < \alpha < n$, the fractional integral operator with variable kernel is defined by

$$T_{\Omega, \alpha} f(x) = \int_{R^n} \frac{\Omega(x, x - y)}{|x - y|^{n-\alpha}} f(y) dy,$$

and

$$\overline{T}_{\Omega, \beta} f(x) = \int_{R^n} \frac{|\Omega(x, x - y)|}{|x - y|^{n-\beta}} |f(y)| dy.$$

2000 Mathematics Subject Classification: 42B20, 42B35.

Key words and phrases: multilinear operator, variable kernels, Lipschitz spaces, fractional integral.

This material is based upon work funded by Zhejiang Provincial Natural Science Foundation of China under Grant (NO. M103069); Supported by the Education Dept. of Zhejiang Province (20021022).

When $\alpha = 0$, $T_{\Omega, \alpha}$ is much more closely related to the elliptic partial equations of second order with variable coefficients. In 1955, Calderón and Zygmund [1] proved its L^p boundedness [2]. For $0 < \alpha < n$, Muckenhoupt and Wheeden [10] proved the (L^p, L^q)-boundedness of $T_{\Omega, \alpha}$ with power weights.

Definition 1.1. Let S^{n-1} be the unit sphere in $R^n (n > 2)$, equipped with normalized Lebesgue measure $d\sigma = d\sigma(z')$. We say that a function $\Omega(x, z)$ to be in $L^\infty \times L^r(S^{n-1})$, $(r > 1)$, if $\Omega(x, z)$ satisfies the following two conditions:

(1) For any $x, z \in R^n$ and $\lambda > 0$, $\Omega(x, \lambda z) = \Omega(x, z)$;

(2) $\| \Omega \|_{L^\infty \times L^r(S^{n-1})} = \sup_{x \in R^n} \left(\int_{S^{n-1}} | \Omega(x, z') |^r d\sigma(z') \right)^{\frac{1}{r}} < \infty.$

Definition 1.2. For a function A defined on R^n, the *fractional multilinear singular integral operator* $T_{\Omega, \alpha, A}$ is defined by

$$T_{\Omega, \alpha, A} f(x) = \int_{R^n} \frac{\Omega(x, x - y)}{| x - y |^{n - \alpha + m - 1}} R_m(A; x, y) f(y) dy,$$

where $R_m(A; x, y)$ denotes the mth remainder of Taylor series of A at x about y. More precisely,

$$R_m(A; x, y) = A(x) - \sum_{| \gamma | < m} \frac{1}{\gamma!} D^\gamma A(y) (x - y)^\gamma,$$

and the corresponding fractional multilinear maximal operator,

$$M_{\Omega, \alpha, A} f(x) = \sup_{r > 0} \frac{1}{r^{n - \alpha + m - 1}} \int_{| x - y | < r} | \Omega(x, x - y) | | R_m(A; x, y) | | f(y) | dy.$$

Definition 1.3. For $\beta > 0$, the *homogeneous Lipschitz space* $\dot{\Lambda}_\beta$ is the space of functions f such that

$$\| f \|_{\dot{\Lambda}_\beta} = \sup_{x, h \in R^n, \, h \neq 0} \frac{| \Delta_h^{[\beta] + 1} f(x) |}{| h |^\beta} < \infty,$$

where $\Delta_h^1 f(x) = f(x + h) - f(x)$, $\Delta_h^{k+1} f(x) = \Delta_h^k f(x + h) - \Delta_h^k f(x)$, $k \geq 1$.

Towards the development of Calderón-Zygmund operators and their commutators with higher order remainder, it is well known that the multilinear fractional operators have been widely studied by many authors ([1], [2], [4], [10]). The purpose of this paper is to study the behaviour of $T_{\Omega, \alpha, A}$ on Lebesgue space. We prove that if $\Omega(x, z) \in L^{\infty}(R^n) \times L^r(S^{n-1})$ with variable kernel, then $T_{\Omega, \alpha, A}$ is bounded from L^p to L^q.

Now, let us formulate our results as follows:

Theorem 1.1. *Let* $0 < \alpha < n$, $0 < \beta < 1$, *and* $0 < \alpha + \beta < n$, $1 < p < \frac{n}{\alpha + \beta}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha + \beta}{n}$, $D^{\gamma} A \in \dot{\Lambda}_{\beta} (|\gamma| = m - 1)$. *If there exists an* $r > p'$ *such that* $\Omega(x, z) \in L^{\infty}(R^n) \times L^r(S^{n-1})$, *then there is a* $C > 0$, *independent of f and A, such that*

$$\| T_{\Omega, \alpha, A} f \|_{L^q} \leq C \left(\sum_{|\gamma| = m-1} \| D^{\gamma} A \|_{\dot{\Lambda}_{\beta}} \right) \| f \|_{L^p}.$$

Theorem 1.2. *Under the same conditions as in Theorem* 1.1, *there is a* $C > 0$, *independent of f and A, such that*

$$\| M_{\Omega, \alpha, A} f \|_{L^q} \leq C \left(\sum_{|\gamma| = m-1} \| D^{\gamma} A \|_{\dot{\Lambda}_{\beta}} \right) \| f \|_{L^p}.$$

Remark 1.1. For $m = 1$, $T_{\Omega, \alpha, A}$ is obviously the commutator operator,

$$[A, T_{\Omega, \alpha}] f(x) = T_{\Omega, \alpha} f(x) - T_{\Omega, \alpha}(Af)(x).$$

2. Lemmas and Proofs of Theorems

Lemma 2.1 [13]. *Let* $0 < \beta < n$, $1 < p < \frac{n}{\beta}$, $\frac{1}{q} = \frac{1}{p} - \frac{\beta}{n}$. *If there exists an* $r > p'$ *such that* $\Omega(x, z) \in L^{\infty}(R^n) \times L^r(S^{n-1})$, *then there is a* $C > 0$ *such that*

$$\| T_{\Omega, \beta} f \|_{L^q} \leq C \| f \|_{L^p}.$$

Lemma 2.2 [1]. *Let* $A(x)$ *be a function on* R^n *with mth order derivatives in* $L^l_{loc}(R^n)$ *for some* $l > n$. *Then*

$$|R_m(A;\, x,\, y)| \le C|x-y|^m \sum_{|r|=m} \left(\frac{1}{|Q_x^y|} \int_{Q_x^y} |D^\gamma A(z)|^l dz \right)^{\frac{1}{l}},$$

where Q_x^y *is the cube centered at x and having diameter* $5\sqrt{n}|x-y|$.

Lemma 2.3 [2]. *For* $0 < \beta < 1, 1 \le q < \infty$, *we have*

$$\|f\|_{\dot{\Lambda}_\beta} = \sup_Q \frac{1}{|Q|^{1+\beta/n}} \int_Q |f(x) - m_Q(f)| dx$$

$$\approx \sup_Q \frac{1}{|Q|^{\beta/n}} \left(\frac{1}{|Q|} \int_Q |f(x) - m_Q(f)|^q dx \right)^{\frac{1}{q}}.$$

Lemma 2.4 [2]. *Let* $Q^* \subset Q$, $g \in \dot{\Lambda}_\beta (0 < \beta < 1)$. *Then*

$$|m_{Q^*}(g) - m_Q(g)| \le C|Q|^{\beta/n} \|g\|_{\dot{\Lambda}_\beta}.$$

We state the following important lemma:

Lemma 2.5. *Suppose* $0 < \alpha < n, 0 < \beta < 1,$ *with* $0 < \alpha + \beta < n, \Omega(x, z) \in L^\infty(R^n) \times L^r(S^{n-1}), D^\gamma A \in \dot{\Lambda}_\beta$. *Then there exists a constant C, only depend on* $m, n, \alpha, \beta,$ *such that*

$$|T_{\Omega, \alpha, A} f(x)| \le C \sum_{|\gamma|=m-1} \|D^\gamma A\|_{\dot{\Lambda}_\beta} \overline{T}_{|\Omega|,\, \alpha+\beta}(|f|)(x).$$

Proof. For any $x \in R^n$, let the cube be centered at x and possess diameter l, where $l > 0$. Then we have

$$T_{\Omega, \alpha, A} f(x) = \left(\int_Q + \int_{Q^C} \right) \frac{\Omega(x,\, x-y)}{|x-y|^{n-\alpha+m-1}} R_m(A;\, x,\, y) f(y) dy := H_1 + H_2.$$

Below, we give the estimates of H_1. Let

$$|H_1| \leq \sum_{j=0}^{\infty} \int_{2^{-j}Q \setminus 2^{-j-1}Q} \frac{|\Omega(x, x-y)||R_m(A; x, y)|}{|x-y|^{n-\alpha+m-1}} |f(y)| dy$$

$$\leq \sum_{j=0}^{\infty} \int_{2^{-j}Q \setminus 2^{-j-1}Q} \frac{|\Omega(x, x-y)||R_m(A_{2^{-j}Q}; x, y)|}{|x-y|^{n-\alpha+m-1}} |f(y)| dy.$$

Note that $A_{2^{-j}Q}(y) = A(y) - \sum_{|\gamma|=m-1} \frac{1}{\gamma!} m_{2^{-j}Q}(D^\gamma A) y^r$. When $y \in 2^{-j}Q \setminus 2^{-j-1}Q$, by Lemmas 2.2, 2.3 and 2.4, we have

$$|R_m(A_{2^{-j}Q}; x, y)| \leq C(2^{-j}l)^\beta |x-y|^{m-1} \sum_{|\gamma|=m-1} \|D^\gamma A\|_{\dot{\Lambda}_\beta}.$$

Note that $|x-y| \geq 2^{-j-1}$. Then we have $|x-y|^\beta \geq 2^{-\beta}(2^{-j}l)^\beta$, such that

$$|H_1| \leq C \sum_{|\gamma|=m-1} \|D^\gamma A\|_{\dot{\Lambda}_\beta} \sum_{j=0}^{\infty} (2^{-j}l)^\beta \int_{2^{-j}Q \setminus 2^{-j-1}Q} \frac{|\Omega(x, x-y)||f(y)|}{|x-y|^{n-\alpha}} dy$$

$$\leq C \sum_{|\gamma|=m-1} \|D^\gamma A\|_{\dot{\Lambda}_\beta} \sum_{j=0}^{\infty} \int_{2^{-j}Q \setminus 2^{-j-1}Q} \frac{(2^{-j}l)^\beta |\Omega(x, x-y)||f(y)|}{|x-y|^{n-\alpha}} dy$$

$$\leq C \sum_{|\gamma|=m-1} \|D^\gamma A\|_{\dot{\Lambda}_\beta} \sum_{j=0}^{\infty} \int_{2^{-j}Q \setminus 2^{-j-1}Q} \frac{2^\beta |x-y|^\beta |\Omega(x, x-y)|}{|x-y|^{n-\alpha}} |f(y)| dy$$

$$\leq C \sum_{|\gamma|=m-1} \|D^\gamma A\|_{\dot{\Lambda}_\beta} \int_Q \frac{|\Omega(x, x-y)||f(y)|}{|x-y|^{n-(\alpha+\beta)}} dy$$

$$\leq C \sum_{|\gamma|=m-1} \|D^\gamma A\|_{\dot{\Lambda}_\beta} \int_{R^n} \frac{|\Omega(x, x-y)||f(y)|}{|x-y|^{n-(\alpha+\beta)}} dy$$

$$= C \sum_{|\gamma|=m-1} \|D^\gamma A\|_{\dot{\Lambda}_\beta} \overline{T}_{|\Omega|, \alpha+\beta}(|f|)(x).$$

Below, we give the estimates of H_2. For $0 < \alpha + \beta < n$, we get

$$|H_2| \leq \sum_{j=0}^{\infty} \int_{2^{j+1}Q \setminus 2^j Q} \frac{|\Omega(x, x-y)||R_m(A; x, y)|}{|x-y|^{n-\alpha+m-1}} |f(y)| dy$$

$$\leq \sum_{j=0}^{\infty} \int_{2^{j+1}Q \setminus 2^j Q} \frac{|\Omega(x, x-y)||R_m(A_{2^{j+1}Q}; x, y)|}{|x-y|^{n-\alpha+m-1}} |f(y)| dy,$$

for any $y \in 2^{j+1}Q \setminus 2^j Q$,

$$A_{2^{j+1}Q}(y) = A(y) - \sum_{|\gamma|=m-1} \frac{1}{\gamma!} m_{2^{j+1}Q}(D^\gamma A).$$

Thus, by Lemmas 2.2 and 2.3, we obtain

$$|R_m(A_{2^{j+1}Q}; x, y)| \leq C(2^j l)^\beta |x-y|^{m-1} \sum_{|\gamma|=m-1} \| D^\gamma A \|_{\dot{\Lambda}_\beta}.$$

And for $|x-y| \geq 2^j l$, we have $|x-y|^\beta \geq (2^j l)^\beta$. Hence

$$|H_2| \leq \left(\sum_{|\gamma|=m-1} \| D^\gamma A \|_{\dot{\Lambda}_\beta} \right) \sum_{j=0}^{\infty} (2^j l)^\beta \int_{2^{j+1}Q \setminus 2^j Q} \frac{|\Omega(x, x-y)||f(y)|}{|x-y|^{n-\alpha}} dy$$

$$\leq C \sum_{|\gamma|=m-1} \| D^\gamma A \|_{\dot{\Lambda}_\beta} \sum_{j=0}^{\infty} \int_{2^{j+1}Q \setminus 2^j Q} \frac{|x-y|^\beta |\Omega(x, x-y)||f(y)|}{|x-y|^{n-\alpha}} dy$$

$$\leq C \sum_{|\gamma|=m-1} \| D^\gamma A \|_{\dot{\Lambda}_\beta} \sum_{j=0}^{\infty} \int_{2^{j+1}Q \setminus 2^j Q} \frac{|\Omega(x, x-y)||f(y)|}{|x-y|^{n-(\alpha+\beta)}} dy$$

$$\leq C \sum_{|\gamma|=m-1} \| D^\gamma A \|_{\dot{\Lambda}_\beta} \int_{R^n} \frac{|\Omega(x, x-y)||f(y)|}{|x-y|^{n-(\alpha+\beta)}} dy$$

$$\leq C \sum_{|\gamma|=m-1} \| D^\gamma A \|_{\dot{\Lambda}_\beta} \overline{T}_{|\Omega|, \alpha+\beta}(|f|)(x).$$

From the proof above, we obtain

$$|T_{\Omega,\alpha,A}f(x)| \le |H_1| + |H_2| \le C \sum_{|\gamma|=m-1} \| D^\gamma A \|_{\dot{\Lambda}_\beta} \overline{T}_{|\Omega|,\alpha+\beta}(|f|)(x).$$

Lemma 2.6. *Let* $0 < \alpha < n$, $\Omega \in L(S^{n-1})$. *Then for* $x \in R^n$,

$$\overline{T}_{|\Omega|,\alpha,A}f(x) \ge M_{\Omega,\alpha,A}f(x).$$

Proof.

$$\begin{aligned}
&\overline{T}_{|\Omega|,\alpha,A}f(x)\\
&= \int_{R^n} \frac{|\Omega(x, x-y)|}{|x-y|^{n-\alpha+m-1}} |R_m(A; x, y)||f(y)|dy\\
&\ge \int_{|x-y|<r} \frac{|\Omega(x, x-y)|}{|x-y|^{n-\alpha+m-1}} |R_m(A; x, y)||f(y)|dy\\
&\ge \frac{1}{r^{n-\alpha+m-1}} \int_{|x-y|<r} |\Omega(x, x-y)||R_m(A; x, y)||f(y)|dy\\
&\overline{T}_{|\Omega|,\alpha,A}f(x) \ge M_{\Omega,\alpha,A}f(x).
\end{aligned}$$

Proof of Theorem 1.1. The proof depends on the weighted boundedness of the fractional integral operator $T_{\Omega,\beta}$. By Lemma 2.2, we have

$$\begin{aligned}
\| T_{\Omega,\alpha,A}f \|_{L^q} &\le C \sum_{|\gamma|=m-1} \| D^\gamma A \|_{\dot{\Lambda}_\beta} \| \overline{T}_{\Omega,\beta} \|_{L^q}\\
&\le C \sum_{|\gamma|=m-1} \| D^\gamma A \|_{\dot{\Lambda}_\beta} \| f \|_{L^p}.
\end{aligned}$$

This completes the proof of Theorem 1.1.

The proof of Theorem 1.2 follows from Lemma 2.6 and Theorem 1.1.

References

[1] A. P. Calderón and A. Zygmund, On a problem of Mihlin, Trans. Amer. Math. Soc. 78 (1955), 209-224.

[2] A. -P. Calderón and A. Zygmund, On singular integral with variable kernels, Appl. Anal. 7 (1978), 221-238.

[3] J. Cohen and J. Gosselin, A BMO estimate for multilinear singular integrals, Illinois J. Math. 30 (1986), 445-464.

[4] Y. Ding, J. C. Chen and D. S. Fan, A class of integral operators with variable kernels on Hardy spaces, Chinese Ann. Math. Ser. A 23(3) (2002), 289-296.

[5] J. C. Lan, A note on multilinear singular integrals with rough kernel, Anal. Theory Appl. 20(2) (2004), 182-188.

[6] J. C. Lan, The weighted Lipschitz estimates for multilinear integrals, Acta Math. Sci. Ser. A 25(3) (2005), 357-361.

[7] J. C. Lan and S. Z. Lu, Weighted Lipschitz estimates for the multilinear fractional integrals, J. Beijing Normal Univ. (Natur. Sci.) 40(3) (2004), 285-291.

[8] J. C. Lan and S. Z. Lu, Boundedness of some fractional integral operators on weighted Herz type Hardy spaces, J. Math. Res. Exposition 25(2) (2005), 311-318.

[9] Shanzhen Lu and Pu Zhang, Lipschitz estimates for generalized commutators of fractional integrals with rough kernel, Math. Nachr. 252(2) (2003), 70-85.

[10] B. Muckenhoupt and R. L. Wheeden, Weighted norm inequalities for singular and fractional integrals, Trans. Amer. Math. Soc. 161 (1971), 249-258.

[11] M. Paluszynski, Characterization of the Besov space via the commutator operator of Coifman, Rochberg and Weiss, Indiana Univ. Math. J. 44(1) (1995), 1-17.

[12] D. Y. Yan, G. E. Hu and J. C. Lan, Weak type endpoint estimates for multilinear singular integrals operators, Acta Math. Sin. (Engl. Ser.) 21(1) (2005), 209-214.

[13] P. Zhang and J. C. Chen, A class of integral operators with variable kernels on the Herz spaces, Chinese Ann. Math. Ser. A 25(5) (2004), 561-570.

Department of Mathematics
Lishui University, Lishui 323000
Zhejiang, P. R. China